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An Algorithm for the Numerical Solution of System of Fractional Differential Equations

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ABSTRACT
In this paper, we present and discuss an algorithm for the numerical solution of system of the initial value problems of the form $D^{\alpha}u = f(t,v)$, $D^{\beta}v = g(t,u)$, $u(0) = u_0$, $v(0) = v_0$, $0 < \alpha, \beta < 1$, where $D^{\alpha}u$ is the derivate of u of order α , $D^{\beta}v$ is the derivative of v of order β in the sense of Caputo. The algorithm is based on the fractional Euler's method which can be seen as a generalization of the classical Euler's method.

General Terms
Algorithms, numerical solutions, fractional.

Keywords
Generalized Taylor's formula - fractional Euler's method - fractional differential equation - Caputo fractional derivative.

1. INTRODUCTION
In this we introduce an algorithm for the numerical solution of initial value problems of the form

$$D^{\alpha}u = f(t,v),\ D^{\beta}v = g(t,u), \qquad u(0) = u_0,\ v(0) = v_0,\ 0 < \alpha, \beta < 1. \qquad (1)$$

Where $D^{\alpha}u$, $D^{\beta}v$ denote the Caputo fractional differential operators.

Fractional order differential equations are generalizations of classical integer order differential equation These are increasingly used to model problems in the fluid flow, mechanics, viscoelasticity, biology, physics, engineering and other applications. Brownian motion and fractional diffusion-wave equations and many physical phenomena [1-5].

Most nonlinear fractional differential equations do not have analytic solutions, so approximations and numerical techniques must be used [9-12]. The decomposition method [13-18] and vartional iteration method [16-18] are relatively new approaches to provide an analytical approximation solution to linear and non linear problems. A comparison between the variational iteration method and Adomian decomposition method for solving fractional differential equations is given in [13]. The fact that the variational iteration method solves non linear equations without using Adomian polynomials can be considered as an advantage of this metho over Adomian decomposition method.

A few numerical methods for fractional differential equations have been presented in the literature [8-12]. In this paper. We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In sections 3 and 4, we introduce the modified trapezoidal rule and a new generalizing of taylor's formula that involves Caputo derivatives, respectively. In section 5, we derive the fractional Euler's method that is generalizing of the classical Euler's method for the numerical solution of ordinary differential equations. The algorithm itself is presented in details in section 6. In section 7, we present three examples to show the efficiency and the simplicify of the algorithm.

2. Basic definitions

Definition 2.1
A real function $f(x)$, $x > 0$, is said to be in the space C_{μ}, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^{p} f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_{μ}^{m} iff $f^{(m)} \in C_{\mu}$, $m \in N$.

Definition 2.2
The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}$, $\mu \geq -1$, is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,\ \alpha > 0,\ x > 0,$$

$$J^{0} f(x) = f(x).$$

Properties of the operator J^{α} :

For $f \in C_{\mu}, \mu \geq -1, \alpha, \beta$ and $\gamma > -1$

(1) $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$

(2) $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$

(3) $J^{\alpha} x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Definition 2.3
The fractional derivative of □□(□) in the Caputo sense is defined as

$$D^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt$$

For $m-1 < \alpha \leq m, m \in N, f \in C_{-1}^{m}$.

Definition 2.4

A two-parameter function of the Mittag-Leffer type is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\alpha>0,\beta>0.$$

Definition 2.5

The Laplace transform of the function f(t) is defined by:

$$F(s)=L\{f(t);s\}=\int_0^{\infty}e^{-st}f(t)dt,$$

Definition 2.6

$$\frac{1}{k!}\ell\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}\left(\pm at^{\alpha}\right)\}=\frac{s^{\alpha-\beta}}{\left(s^{\alpha}\mp a\right)^{k+1}}$$

Lemma 2.7

If $m-1<\alpha\le m,\quad m\in N,\quad f\in C_{-1}^{m},\quad \mu\ge -1,$ then

$$D^{\alpha}J^{\alpha}f(x)=f(x) \tag{2}$$

$$J^{\alpha}D^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}\left(0^{+}\right)\frac{x^k}{k!} \tag{3}$$

3 Results and Theorems

3.1 Modified trapezoidal rule :

We present a review of the modified trapezoidal rule, which is introduced in [20]. This is used to approximate the fractional integral $J^{\alpha}f(t)$ by a weighted sum of function values at specified points. Suppose that the interval $[0,a)$ is subdivided into k subintervals $\left[t_j,t_{j+1}\right]$ of equal width $h=\frac{a}{k}$ by using the nodes $t_j=jh$, for j=0,1,……,k . The modified trapezoidal rule

$$T(f,h,\alpha)=\left((k-1)^{\alpha+1}-(k-\alpha-1)k^{\alpha}\right)\frac{h^{\alpha}f(0)}{\Gamma(\alpha+2)}+\frac{h^{\alpha}f(a)}{\Gamma(\alpha+2)}+\sum_{j=1}^{k-1}\left((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\right)\frac{h^{\alpha}f\left(t_j\right)}{\Gamma(\alpha+2)}$$
(4)

is an approximation to the fractional integral

$$\left(J^{\alpha}f(t)\right)(a)=T(f,h,\alpha)-E_T(f,h,\alpha),\ a>0,\ \alpha>0.$$
(5)

Furthermore, if $f(t)\in C^2[0,a]$. There is a constant C_{α} depending only on α so that the error term $E_T(f,h,\alpha)$

(6)

$$E_T(f,h,\alpha)\le C_{\alpha}\|f''\|_{\infty}a^{\alpha}h^2=o\left(h^2\right)$$

3.2 Generalized Taylor's rule

In this section we introduce a new generalization of Taylor's formula that involves Caputo fractional derivative. This generalization is presented in [20]. We begin by the generalized mean value theorem.

Theorem (Generalized mean value theorem)

Suppose that $f(x)\in C[0,a]$ and $D^{\alpha}f(x)\in C(0,a]$, for $0<\alpha\le 1$. Then we have

$$f(x)=f(0+)+\frac{1}{\Gamma(\alpha)}\left(D^{\alpha}f\right)(\xi)\cdot x^{\alpha} \tag{7}$$

with $0\le\xi\le x$, $\forall x\in(0,a]$.

Proof: in [20].

Theorem :

Suppose that $D^{n\alpha}f(x)$, $D^{(n+1)\alpha}f(x)\in C(0,a]$, for $0<\alpha\le 1$. Then we have

$$\left(J^{n\alpha}D^{n\alpha}f\right)(x)-\left(J^{(n+1)\alpha}D^{(n+1)\alpha}f\right)(x)=\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}\left(D^{n\alpha}f\right)(0+) \tag{8}$$

where $D^{n\alpha}=D^{\alpha}D^{\alpha}\cdots D^{\alpha}$ (n-times).

Proof :

The proof can be obtained by using the properties of the Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator and the relation :

$$\left(J^{n\alpha}D^{n\alpha}f\right)(x)-\left(J^{(n+1)\alpha}D^{(n+1)\alpha}f\right)(x)=J^{n\alpha}\left(\left(D^{n\alpha}f\right)(x)-\left(J^{\alpha}D^{\alpha}\right)\left(D^{n\alpha}f\right)(x)\right)$$

$$=J^{n\alpha}\left(D^{n\alpha}f\right)(0+).$$

Theorem: (Generalized Taylor's rule)

Suppose that $D^{k\alpha}f(x)\in C(0,a]$ for $k=0,1,\cdots,n+1$, where $0<\alpha\le 1$. Then we have

$$f(x)=\sum_{i=0}^{n}\frac{x^{in}}{\Gamma(i\alpha+1)}\left(D^{i\alpha}f\right)(0+)+\frac{\left(D^{(n+1)\alpha}f\right)(\xi)}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha},$$

with $0\le\xi\le x$, $\forall x\in(0,a]$. (9)

Proof : From (7), we have

$$\sum_{i=0}^{n}\left(J^{i\alpha}D^{i\alpha}f\right)(x)-\left(J^{(i+1)\alpha}D^{(i+1)\alpha}f\right)(x)= \sum_{i=0}^{n}\frac{x^{in}}{\Gamma(i\alpha+1)}\left(D^{i\alpha}f\right)(0+) \quad \textbf{(10)}$$

that is,

$$f(x)-\left(J^{(n+1)\alpha}D^{(n+1)\alpha}f\right)(x)=\sum_{i=0}^{n}\frac{x^{in}}{\Gamma(i\alpha+1)}\left(D^{i\alpha}f\right)(0+). \quad (11)$$

Applying the integral mean value theorem to (12) yields

$$\left(J^{(n+1)\alpha}D^{(n+1)\alpha}f\right)(x)=\frac{\left(D^{(n+1)\alpha}f\right)(\xi)}{\Gamma((n+1)\alpha+1)}\int_0^x(x-t)^{(n+1)\alpha}dt$$

$$=\frac{\left(D^{(n+1)\alpha}f\right)(\xi)}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha}$$

(12)

3.3 The algorithm for one equation:

In this section we shall derive the fundamental algorithm for the numerical solution of the initial value problem

$$D^{\alpha}y(t)=f(t,y(t)),\ y(0)=y_0,\ 0<\alpha\le 1$$

$$t>0. \quad (13)$$

The new algorithm is based on the modified trapezoidal rule and the fractional Euler's method. Our approach depends on the analytical property that the initial value problem (13) is equivalent to the integral equation

$$y(t)=J^{\alpha}f(t,y(t))+y(0). \quad (14)$$

Let $[0,a]$ be the interval over which we want to find the approximation the solution. Suppose that the $[0,a]$ is subdivided into k subintervals $[t_j,t_{j+1}]$ of

equal width $h=\frac{a}{k}$ by using the nodes $t_j=jh$, for $j=0,1,\cdots,k$. To obtain the solution point $(t_1,y(t_1))$, we substitute $t=t_1$ into (14) and we get

$$y(t_1)=\left(J^{\alpha}f(t,y(t))\right)(t_1)+y(0) \quad (15)$$

Now if the modified trapezoidal rule (4) is used to approximate $\left(J^{\alpha}f(t,y(t))\right)(t_1)$ with step size $h=t_1-t_0$, then the result is

$$y(t_1)=\alpha\frac{h^{\alpha}f(t_0,y(t_0))}{\Gamma(\alpha+2)}+\frac{h^{\alpha}f(t_1,y(t_1))}{\Gamma(\alpha+2)}+y(0)$$

(16)

Notice that the formula on the right-hand side of (16) involves the term $y(t_1)$. So, we use an estimate for $y(t_1)$. Fractional Euler's method will suffice for this purpose.

From

$$y(t_1)=y(t_0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_0,y(t_0)).$$

into (16) yields

$$y(t_1)=\alpha\frac{h^{\alpha}f(t_0,y(t_0))}{\Gamma(\alpha+2)}+\frac{h^{\alpha}f\left(t_1,y(t_0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_0,y(t_0))\right)}{\Gamma(\alpha+2)}+y(0)$$

(17)

The process is repeated to generate a sequence of points that approximate the solution $y(t)$. The general formula for our algorithm is:

$$y(t_j)=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left((j-1)^{\alpha+1}-(j-\alpha-1)j^{\alpha}\right)f(t_0,y(t_0))+y(0)+ \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{i=1}^{j-1}\left((j-i+1)^{\alpha+1}-2(j-i)^{\alpha+1}+(j-i-1)^{\alpha+1}\right) f(t_0,y(t_0))+ \frac{h^{\alpha}}{\Gamma(\alpha+2)}f\left(t_j,y(t_{j-1})+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_{j-1},y(t_{j-1}))\right)$$

3.4 The algorithm for system of two equations:

In this paper we get a numerical solution of system of fractional differential equations:

$$D^{\alpha}u=f(t,v)$$

$$D^{\beta}v=f(t,u),\ u(t_0)=u_0,v(t_0)=v_0$$

$$0<\alpha,\beta<1$$

We will use the same way to obtain the algorithm for this system the we have

$$u(t_j)=\frac{h^\alpha}{\Gamma(\alpha+2)}\left((j-1)^{\alpha+1}-(j-\alpha-1)j^\alpha\right)f(t_0,v(t_0))+u(t_0)+$$

$$\frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{i=1}^{j-1}\left((j-i+1)^{\alpha+1}-2(j-i)^{\alpha+1}+(j-i-1)^{\alpha+1}\right)f(t_i,v(t_i))+$$
$$\frac{h^\alpha}{\Gamma(\alpha+2)}f\left(t_j,v(t_{j-1})+\frac{h^\alpha}{\Gamma(\alpha+1)}f(t_{j-1},v(t_{j-1}))\right)$$

and

$$v(t_j)=\frac{h^\beta}{\Gamma(\beta+2)}\left((j-1)^{\beta+1}-(j-\beta-1)j^\beta\right)g(t_0,u(t_0))+v(t_0)+$$
$$\frac{h^\beta}{\Gamma(\beta+2)}\sum_{i=1}^{j-1}\left((j-i+1)^{\beta+1}-2(j-i)^{\beta+1}+(j-i-1)^{\beta+1}\right)g(t_i,u(t_i))+$$
$$\frac{h^\beta}{\Gamma(\beta+2)}g\left(t_j,u(t_{j-1})+\frac{h^\beta}{\Gamma(\beta+1)}g(t_{j-1},u(t_{j-1}))\right)$$

3.5 Results:

Example

The homogeneous linear system :

$$D^{.95}u=v$$

$$D^{.9}v=u$$

$$u(.1)=u_0=.194,\ v(.1)=v_0=1.17$$

where $t_0=.1$

The exact solution

$$u(t)=t^{-.1}E_{1.85,.9}(t^{1.85}),\ v(t)=t^{.85}E_{1.85,1.85}(t^{1.85})$$

from definition (4)

$$u(t)=\sum_{k=0}^{\infty}\frac{t^{1.85k+.85}}{\Gamma(1.85k+1.85)},\ v(t)=\sum_{k=0}^{\infty}\frac{t^{1.85k-.1}}{\Gamma(1.85k+.9)}$$

Table 1. Numerical values for example with h=.1, $t_0=.1$

t	u_{Appro}	u_{exact}	v_{Appro}	v_{exact}
.2	0.37967	0.27236314	1.429247	1.136372
.3	0.4520032	0.3894135	1.300924	1.132096
.4	0.5786627	0.50580609	1.354413	1.153449
.5	0.7209318	0.62435337	1.425796	1.194041
.6	0.8703125	0.74715702	1.513180	1.251261

Example

The homogeneous linear system :

$$D^{.99}u=v$$

$$D^{.98}v=u$$

$u(.1)=u_0=.10848$ $v(.1)=v_0=1.0347$ where $t_0=.1$

The exact solution

$$u(t)=t^{.97}E_{1.97,1.97}(t^{1.97}),\ v(t)=t^{-.02}E_{1.97,.98}(t^{1.97})$$

and we have

$$u(t)=\sum_{k=0}^{\infty}\frac{t^{1.97k+.97}}{\Gamma(1.97k+1.97)},$$
$$v(t)=\sum_{k=0}^{\infty}\frac{t^{1.97k-.02}}{\Gamma(1.97k+.98)}$$

Table 2. Numerical values for example with h=.1, $t_0=.1$

t	u_{Appro}	u_{exact}	v_{Appro}	v_{exact}
.2	0.3256916	0.21409327	1.058076	1.0432
.3	0.3271078	0.32015133	1.138966	1.06272
.4	0.4466322	0.4285283	1.1719626	1.095433
.5	0.5663590	0.5406025	1.2242022	1.14057
.6	0.6921711	0.6576661	1.2885656	1.198047

Example:

$$D^{.99}u=v$$

$$D^{.991}v=u$$

$u(.1)=u_0=.1053023,$
$v(.1)=v_0=1.015581$ where $t_0=.1$

The exact solution

$$u(t) = t^{981} E_{1.981,1.981}\left(t^{1.981}\right),\ v(t) = t^{-.009} E_{1.981,.991}\left(t^{1.97}\right)$$

then

$$u(t) = \sum_{k=0}^{\infty} \frac{t^{1.981k+.981}}{\Gamma(1.981k + 1.981)},$$
$$v(t) = \sum_{k=0}^{\infty} \frac{t^{1.981k-.009}}{\Gamma(1.981k + .991)}$$

Table 3. Numerical values for example with h=.1, $t_0 = .1$

t	u_{Appro}	u_{exact}	v_{Appro}	v_{exact}
.2	0.31848807	0.20933559	1.0376338	1.0307319
.3	0.32068403	0.31434498	1.1165844	1.053726
.4	0.43783472	0.42194125	1.1488661	1.0884054
.5	0.5551949	0.53338573	1.1995302	1.1346551
.6	0.67843837	0.64990082	1.2620044	1.1926908

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