Odd Graceful Labeling of the Revised Friendship Graphs

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ABSTRACT

The aim of this paper is to present some odd graceful graphs. In particular we show that the revised friendship graphs $F(kC_4)$, $F(kC_8)$, $F(kC_{12})$, $F(kC_{16})$ and $F(kC_{20})$ are odd graceful where k is any positive integer. Finally, we introduce a new conjecture " The revised friendship graph $F(kC_n)$ is odd graceful where k is any positive integer and $n = 0$ (*mod* 4).

Keywords

Graph Theory, odd graceful labeling, friendship graphs.

1. INTRODUCTION

A graph G of size q is odd-graceful, if there is an injection ϕ from *V(G)* to {0, 1, 2, …, 2q-1} such that, when each edge xy is assigned the label or weight | ϕ (x) - ϕ (y)|, the resulting edge labels are {1, 3, 5, …, 2q-1}. This definition was introduced in 1991 by Gnanajothi [1] who proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs. Gnanajothi [1] proved that every cycle C_n is odd graceful if n is even. It is known that the graphs which contain odd cycles are not odd graceful so Badr [2] used the subdivision notation for odd cycle in order to prove that the subdivision of linear triangular snakes are odd graceful. Badr et al [3] proved that the subdivision of ladders $S(L_n)$ is odd graceful.

Rosa [4] proved that the cycle C_n is graceful if and only if $n \equiv$ 0 or 3 (mod 4). Solairaju and Muruganantham [5] proved that the revised friendship graphs $F(kC_3)$, $F(kC_5)$ and $F(2kC_3)$ are all even vertex graceful, where k is any positive integer.

Definition 1.1: A revised friendship graph $F(kC_n)$, $n \geq 3$ is defined as a connected graph containing k copies of C_n with a vertex in common.

Example 1.2:

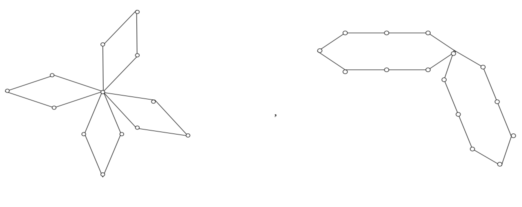

Figure 1: the revised friendship graphs $F(4C_4)$ & $F(2C_8)$

In this paper we show that the revised friendship graphs $F(kC_4)$, $F(kC_8)$, $F(kC_{12})$, $F(kC_{16})$ and $F(kC_{20})$ are odd graceful where k is any positive integer. Finally, we introduce a new conjecture " The revised friendship graph $F(kC_n)$ is odd graceful where k is any positive integer and $n = 0$ (*mod* 4).

2. THE MAIN RESULTS

Theorem 2.1: The revised friendship graph $F(kC_4)$ is odd graceful, where k is any positive integer.

Proof:

Let $G = F(kC_4)$ has q edges and p vertices. The graph G consists of the vertices u, $u_1u_2u_3\ldots u_{2k}$ and $v_1v_2v_3\ldots v_k$, where the graph G consisting k copies of C_4 with a vertex u in common, such that u_i is put between u and v_j where i = 1, 2, 3..2k and j = 1, 2, 3…k. The graph G has $q = 4k$ and $p = 3k + 1$, as shown in Figure 2.

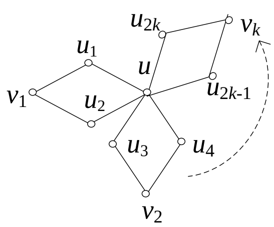

Figure 2: the revised friendship graph F(kC_4)

Define $\phi : V(G) \to \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u) = 0$

$\phi(u_i) = 2q - 2i + 1$, $i = 1, 2, 3\ldots 2k$

$\phi(v_i) = 2q - 8i + 4$, $i = 1, 2, 3\ldots k$

a) $\max\limits_{v\in V(G)} \phi(v) = \max\{0, \max\limits_{1\le i\le 2k} 2q - 2i + 1, \max\limits_{1\le i\le k} (2q - 8i + 4) \}$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

b) Clearly, The function ϕ is one-to-one mapping from the vertex set of G to the set $\{0,1,2\ldots 2q - 1\}$

c) It remains to show that the labels of the edges of G are all the odd integers of the interval [1,2q-1] The range of $| \phi(u_i) - \phi(u) | = \{2q - 2i + 1 : i = 1, 2 \ldots 2k\}$

$= \{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}$

The range of $|\phi(v_i) - \phi(u_{2i-1})| = \{4i - 1 \quad : i = 1, 2 \ldots k\}$

$= \{3, 7 \ldots 4k - 1\}$

The range of $|\phi(v_i) - \phi(u_{2i})| = \{4i - 3 \quad : i = 1, 2 \ldots k\}$

$= \{1, 5 \ldots 4k - 3\}$

Hence, $\{|\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the revised friendship graph $F(kC_4)$ is odd graceful.

Example 2.2

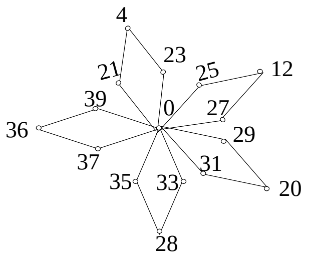

Figure 3: the odd graceful labeling of the revised friendship graph $F(5C_4)$.

Theorem 2.3: The revised friendship graph $F(kC_6)$ is odd graceful, where k is any positive integer.

Proof:

Let $G = F(kC_6)$ has q edges and p vertices. The graph G consists of the vertices u, $u_1u_2u_3\ldots u_{2k}$, $x_1x_2x_3\ldots x_{2k}$ and $v_1v_2v_3\ldots v_k$, where the graph G consisting k copies of C_6 with a vertex u in common, such that the vertex u is the common vertex, x_i is put between u and v_j, u_i is put between u and x_i, such that $i = 1,2,3..2k$, $j = 1, 2, 3\ldots k$. The graph G has $q = 6k$ and $p = 5k + 1$, as shown in Figure 4.

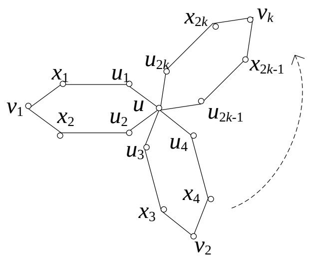

Figure 4: the revised friendship graph $F(kC_6)$

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u) = 0$

$\phi(u_i) = 2q - 2i + 1 \qquad , i = 1, 2, 3\ldots 2k$

$\phi(x_i) = (4/3)\, q - 4i + 2 \qquad , i = 1, 2, 3\ldots 2k$

$\phi(v_i) = (2/3)\, q - 4i + 3 \qquad , i\text{–odd } (i = 1, 3, 5\ldots)$

$\phi(v_i) = (2/3)\, q - 4i + 1 \qquad , i\text{–even } (i = 2, 4, 6\ldots)$

a)

$$\max_{v\in V(G)} \phi(v) = \max\{0, \max_{1\le i\le 2k}(2q - 2i + 1), \max_{1\le i\le 2k}((4/3)\, q - 4i + 2), \max^{\text{i-odd}}_{i=1,3,\ldots,k}((2/3)\, q - 4i + 3), \max^{\text{i-even}}_{i=2,4,\ldots,k}((2/3)\, q - 4i + 1)\}$$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

b) Clearly, The function ϕ is one-to-one mapping from the vertex set of G to the set $\{0,1,2\ldots 2q - 1\}$

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1,2q\text{-}1]$ and that's as following:

The range of $|\phi(u_i) - \phi(u)| = \{2q - 2i + 1 \quad , i = 1, 2 \ldots 2k\}$

$= \{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}$

The range of $|\phi(u_i) - \phi(x_i)| = \{(2/3)\, q + 2i - 1, i = 1, 2 \ldots 2k\}$

$= \{(2/3)\, q + 1, (2/3)\, q + 3 \ldots (2/3)\, q + 4k - 1\}$

The range of $|\phi(v_i) - \phi(x_{2i-1})| = \{(2/3)\, q - 4i + 3$, i-odd ($i = 1, 3, 5\ldots, k$ }

$= \{(2/3)\, q - 1, (2/3)\, q - 9 \ldots\}$

The range of $|\phi(v_i) - \phi(x_{2i})| = \{(2/3)\, q - 4i - 1$, i-odd ($i = 1, 3, 5\ldots,k$)}

$= \{(2/3)\, q - 5, (2/3)\, q - 13 \ldots\}$

The range of $|\phi(v_i) - \phi(x_{2i-1})| = \{(2/3)\, q - 4i + 5$, i-even ($i = 2, 4, 6\ldots, k\text{-}1$)}

$= \{(2/3)\, q - 3, (2/3)\, q - 11 \ldots\}$

The range of $|\phi(v_i) - \phi(x_{2i})| = \{(2/3)\, q - 4i + 1$, i-even ($i = 2, 4, 6\ldots, k\text{-}1$)}

$= \{(2/3)\, q - 7, (2/3)\, q - 15 \ldots\}$

Hence, $\{|\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the revised friendship graph $F(kC_6)$ is odd graceful.

Theorem 2.4: The revised friendship graph $F(kC_8)$ is odd graceful, where k is any positive integer.

Proof:

Let $G = F(kC_8)$ has q edges and p vertices. The graph G consists of the vertices u, $u_1u_2u_3\ldots u_{2k}$, $x_1x_2x_3\ldots x_{2k}$, $h_1h_2h_3\ldots h_{2k}$ and $v_1v_2v_3\ldots v_k$, where the graph G consisting k copies of C_8 with a vertex u in common, such that the vertex u is the common vertex, h_i is put between u and v_j, x_i is put between u and h_i, u_i is put between u and x_i such that $i = 1,2,3..2k$ and $j = 1, 2, 3\ldots k$. The graph G has $q = 8k$ and $p = 7k + 1$, as shown in the next figure.

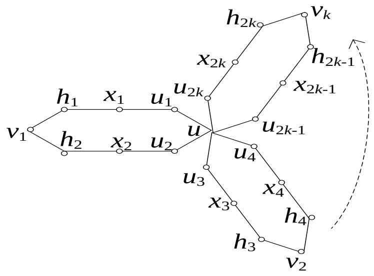

Figure 5: the revised friendship graph $F(kC_8)$

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u) = 0$

$\phi(u_i) = 2q - 2i + 1$, $i = 1, 2, 3\ldots 2k$

$\phi(x_i) = q - 4i + 2$, $i = 1, 2, 3\ldots 2k$

$\phi(h_i) = 2q - 2i - 4k + 1$, $i = 1, 2, 3\ldots 2k$

$\phi(v_i) = 2q - 8i + 4$, $i = 1, 2, 3\ldots k$

a)

$$\operatorname*{Max}_{v\in V(G)} \phi(v) = \max\{0, \max_{1\le i\le 2k}(2q - 2i + 1), \max_{1\le i\le 2k}(q - 4i + 2), \max_{1\le i\le 2k}(2q - 2i - 4k + 1), \max_{1\le i\le k}(2q - 8i + 4)\}$$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

b) Clearly, The function ϕ is one-to-one mapping from the vertex set of G to the set $\{0,1,2\ldots 2q - 1\}$

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1,2q\text{-}1]$ and that's as following:

The range of $|\phi(u_i) - \phi(u)| = \{2q - 2i + 1$, $i = 1, 2 \ldots 2k\}$

$= \{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}$

The range of $|\phi(u_i) - \phi(x_i)| = \{q + 2i - 1$, $i = 1, 2 \ldots 2k\}$

$= \{q + 1, q + 3 \ldots q + 4k - 1\}$

The range of $|\phi(h_i) - \phi(x_i)| = \{q + 2i - 4k - 1$, $i = 1, 2 \ldots 2k\}$

$= \{q - 4k + 1, q - 4k + 3 \ldots q - 1\}$

The range of $|\phi(v_i) - \phi(h_{2i-1})| = \{4k - 4i + 1$, $i = 1, 2, 3 \ldots k\}$

$= \{4k - 3, 4k - 7 \ldots 1\}$

The range of $|\phi(v_i) - \phi(h_{2i})| = \{4k - 4i + 3$, $i = 1, 2, 3 \ldots k\}$

$= \{4k - 1, 4k - 5 \ldots 3\}$

Hence, $\{|\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the revised friendship graph $F(kC_8)$ is odd graceful.

Theorem 2.5: The revised friendship graph $F(kC_{12})$ is odd graceful, where k is any positive integer.

Proof:

Let $G = F(kC_{12})$ has q edges and p vertices. The graph G consists of the vertices u, $u_1u_2u_3\ldots u_{2k}$, $x_1x_2x_3\ldots x_{2k}$, $h_1h_2h_3\ldots h_{2k}$, $a_1a_2a_3\ldots a_{2k}$, $b_1b_2b_3\ldots b_{2k}$ and $v_1v_2v_3\ldots v_k$, where the graph G consisting k copies of C_{12} with a vertex u in common, such that the vertex u is the common vertex, b_i is put between u and v_j, a_i is put between u and b_i, h_i is put between u and a_i, x_i is put between u and h_i, u_i is put between u and x_i where $i = 1, 2, 3\ldots 2k$ and $j = 1, 2, 3\ldots k$. The graph G has $q = 12k$ and $p = 11k + 1$, as shown in the next figure.

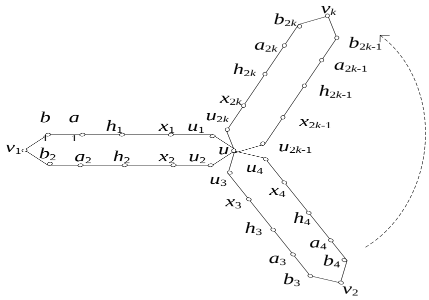

Figure 6: the revised friendship graph $F(kC_{12})$

Define $\phi : V(G) \rightarrow \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u) = 0$

$\phi(u_i) = 2q - 2i + 1$, $i = 1, 2, 3\ldots 2k$

$\phi(x_i) = (4/6)\, q - 4i + 2$, $i = 1, 2, 3\ldots 2k$

$\phi(h_i) = 2q - 2i - 4k + 1$, $i = 1, 2, 3\ldots 2k$

$\phi(a_i) = (8/6)\, q - 4i + 2$, $i = 1, 2, 3\ldots 2k$

$\phi(b_i) = (2/6)\, q - 2i + 1$, $i = 1, 2, 3\ldots 2k$

$\phi(v_i) = (4/6)\, q - 8i + 4$, $i = 1, 2, 3\ldots k$

a)

$$\operatorname*{Max}_{v\in V(G)} \phi(v) = \max\{0, \max_{1\le i\le 2k}(2q - 2i + 1), \max_{1\le i\le 2k}((4/6)q - 4i + 2), \max_{1\le i\le 2k}(2q - 2i - 4k + 1), \max_{1\le i\le 2k}((8/6)q - 4i + 2), \max_{1\le i\le 2k}((2/6)q - 2i + 1), \max_{1\le i\le k}((4/6)q - 8i + 4)\}$$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

b) Clearly, The function ϕ is one-to-one mapping from the vertex set of G to the set $\{0,1,2\ldots 2q - 1\}$

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1,2q\text{-}1]$ and that's as following:

The range of $|\phi(u_i) - \phi(u)| = \{2q - 2i + 1$, $i = 1, 2 \ldots 2k\}$

$= \{2q - 1, 2q – 3 \ldots 2q – 4k + 1\}$

The range of $| \phi(u_i) - \phi(x_i) | = \{(4/3)\, q + 2i – 1, i = 1, 2 \ldots 2k\}$

$= \{(4/3)\, q + 1, (4/3)\, q + 3 \ldots (4/3)\, q + 4k - 1\}$

The range of $| \phi(h_i) - \phi(x_i) | = \{(4/3)q + 2i – 4k - 1, i = 1,2\ldots 2k\}$

$= \{(4/3)\, q – 4k + 1, (4/3)\, q – 4k + 3 \ldots (4/3)\, q - 1\}$

The range of $| \phi(h_i) - \phi(a_i) | = \{(2/3)q + 2i – 4k - 1, i = 1,2\ldots 2k\}$

$= \{(2/3)\, q – 4k + 1, (2/3)\, q – 4k + 3, \ldots (2/3)\, q – 1\}$

The range of $| \phi(a_i) - \phi(b_i) | = \{q - 2i + 1 \ , i = 1, 2 \ldots 2k\}$

$= \{q - 1, q - 3 \ldots q – 4k + 1\}$

The range of $| \phi(v_i) - \phi(b_{2i-1}) | = \{(1/3)\, q - 4i + 1, i = 1,2,3 \ldots k\}$

$= \{(1/3)\, q - 3, (1/3)\, q - 7 \ldots 1\}$

The range of $| \phi(v_i) - \phi(b_{2i}) | = \{(1/3)\, q - 4i + 3, i = 1,2, 3 \ldots k\}$

$= \{(1/3)\, q - 1, (1/3)\, q - 5 \ldots 3\}$

Hence, $\{| \phi(u) - \phi(v) | : u\, v \in E(G)\} = \{1, 3, 5 \ldots 2q - 1\}$.

So the revised friendship graph $F(kC_{12})$ is odd graceful.

Theorem 2.6: The revised friendship graph $F(kC_{16})$ is odd graceful, where k is any positive integer.

Proof:

Let $G = F(kC_{16})$ has q edges and p vertices. The graph G consists of the vertices u, $u_1u_2u_3\ldots u_{2k}$, $x_1x_2x_3\ldots x_{2k}$, $h_1h_2h_3\ldots h_{2k}$, $a_1a_2a_3\ldots a_{2k}$, $b_1b_2b_3\ldots b_{2k}$, $c_1c_2c_3\ldots c_{2k}$, $d_1d_2d_3\ldots d_{2k}$ and $v_1v_2v_3\ldots v_k$, where the graph G consisting k copies of C_{16} with a vertex u in common, such that the vertex u is the common vertex, d_i is put between u & v_j, c_i is put between u and d_i, b_i is put between u and c_i, a_i is put between u and b_i, h_i is put between u and a_i, x_i is put between u and h_i, u_i is put between u and x_i where $i = 1,2,3..2k$ and $j = 1, 2, 3\ldots k$. The graph G has $q = 16k$ and $p = 15k + 1$, as shown in the next figure.

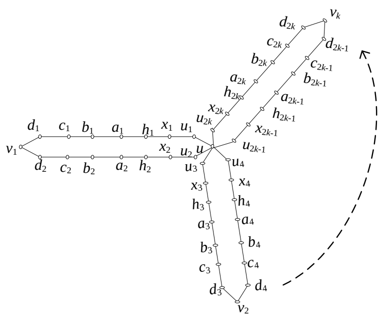

Figure 7: the revised friendship graph $F(kC_{16})$

Define $\phi : V(G) \to \{0, 1, 2\ldots 2q\text{-}1\}$ as following:

$\phi(u) = 0$

$\phi(u_i) = 2q – 2i + 1 \quad , i = 1, 2, 3\ldots 2k$

$\phi(x_i) = (1/2)\, q – 4i + 2 \quad , i = 1, 2, 3\ldots 2k$

$\phi(h_i) = 2q – 2i – 4k + 1 \quad , i = 1, 2, 3\ldots 2k$

$\phi(a_i) = q – 4i + 2 \quad , i = 1, 2, 3\ldots 2k$

$\phi(b_i) = (1/4)\, q – 2i + 1 \quad , i = 1, 2, 3\ldots 2k$

$\phi(c_i) = (3/2)\, q – 4i + 2 \quad , i = 1, 2, 3\ldots 2k$

$\phi(d_i) = q – 2i + 1 \quad , i = 1, 2, 3\ldots 2k$

$\phi(v_i) = (5/4)\, q – 8i + 4 \quad , i = 1, 2, 3\ldots k$

a)

$$\max_{v\in V(G)} \phi(v) = \max\{0, \max_{1\le i\le 2k}(2q - 2i + 1), \max_{1\le i\le 2k}((1/2)\, q - 4i + 2), \max_{1\le i\le 2k}(2q - 2i - 4k + 1), \max_{1\le i\le 2k}(q - 4i + 2), \max_{1\le i\le 2k}((1/4)\, q - 2i + 1), \max_{1\le i\le 2k}((3/2)\, q - 4i + 2), \max_{1\le i\le 2k}(q - 2i + 1), \max_{1\le i\le k}((5/4)\, q - 8i + 4)\}$$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi(v) \in \{0, 1, 2 \ldots 2q – 1\}$

b) Clearly, The function ϕ is one-to-one mapping from the vertex set of G to the set $\{0,1,2\ldots 2q - 1\}$

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1,2q\text{-}1]$ and that's as following:

The range of $| \phi(u_i) - \phi(u) | = \{2q – 2i + 1 , i = 1, 2 \ldots 2k\}$

$= \{2q - 1, 2q – 3 \ldots 2q – 4k + 1\}$

The range of $| \phi(u_i) - \phi(x_i) | = \{(3/2)\, q + 2i – 1, i = 1, 2 \ldots 2k\}$

$= \{(3/2)\, q + 1, (3/2)\, q + 3 \ldots (3/2)\, q + 4k - 1\}$

The range of $| \phi(h_i) - \phi(x_i) | = \{(3/2)\, q + 2i – 4k - 1, i = 1,2\ldots 2k\}$

$= \{(3/2)\, q – 4k + 1, (3/2)\, q – 4k + 3 \ldots (3/2)\, q – 1\}$

The range of $| \phi(h_i) - \phi(a_i) | = \{q + 2i – 4k - 1 , i = 1, 2 \ldots 2k\}$

$= \{q – 4k + 1, q – 4k + 3 \ldots q – 1\}$

The range of $| \phi(a_i) - \phi(b_i) | = \{(3/4)\, q - 2i + 1, i = 1, 2 \ldots 2k\}$

$= \{(3/4)\, q – 1, (3/4)\, q – 3 \ldots (3/4)\, q – 4k + 1\}$

The range of $| \phi(c_i) - \phi(b_i) | = \{(5/4)\, q - 2i + 1, i = 1, 2 \ldots 2k\}$

$= \{(5/4)\, q - 1, (5/4)\, q - 3 \ldots (5/4)\, q – 4k + 1\}$

The range of $| \phi(c_i) - \phi(d_i) | = \{(1/2)\, q - 2i + 1, i = 1, 2 \ldots 2k\}$

$= \{(1/2)\, q - 1, (1/2)\, q - 3 \ldots (1/2)\, q - 4k + 1\}$

The range of $| \phi(v_i) - \phi(d_{2i-1}) | = \{(1/4)\, q - 4i + 1, i = 1,2,3\ldots k\}$

$= \{(1/4)\, q - 3, (1/4)\, q - 7 \ldots (1/4)\, q – 4k + 1\}$

The range of $| \phi(v_i) - \phi(d_{2i}) | = \{(1/4)\, q - 4i + 3, i = 1,2,3 \dots k\}$

$= \{(1/4)\, q - 1, (1/4)\, q - 5 \dots (1/4)\, q - 4k + 3\}$

Hence, $\{| \phi(u) - \phi(v) | : u\, v \in E(G)\} = \{1, 3, 5 \dots 2q - 1\}$.

So the revised friendship graph $F(kC_{16})$ is odd graceful.

Theorem 2.7: The revised friendship graph $F(kC_{20})$ is odd graceful, where k is any positive integer.

Proof:

Let $G = F(kC_{20})$ has q edges and p vertices. The graph G consists of the vertices u, $u_1u_2u_3 \dots u_{2k}$, $x_1x_2x_3 \dots x_{2k}$, $h_1h_2h_3 \dots h_{2k}$, $a_1a_2a_3 \dots a_{2k}$, $b_1b_2b_3 \dots b_{2k}$, $c_1c_2c_3 \dots c_{2k}$, $d_1d_2d_3 \dots d_{2k}$, $z_1z_2z_3 \dots z_{2k}$, $g_1g_2g_3 \dots g_{2k}$ and $v_1v_2v_3 \dots v_k$, where the graph G consisting k copies of C_{20} with a vertex u in common, such that the vertex u is the common vertex, g_i is put between u and v_j, z_i is put between u and g_i, d_i is put between u and z_i, c_i is put between u and d_i, b_i is put between u and c_i, a_i is put between u and b_i, h_i is put between u and a_i, x_i is put between u and h_i, u_i is put between u and x_i where $i = 1,2,3..2k$ and $j = 1, 2, 3 \dots k$. The graph G has $q = 20k$ and $p = 19k + 1$, as shown in the next figure.

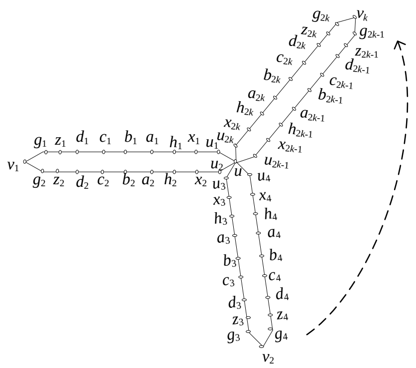

Figure 8: the revised friendship graph $F(kC_{20})$

Define $\phi : V(G) \to \{0, 1, 2 \dots 2q\text{-}1\}$ as following:

$\phi(u) = 0$

$\phi(u_i) = 2q - 2i + 1$, $i = 1, 2, 3 \dots 2k$

$\phi(x_i) = (2/5)\, q - 4i + 2$, $i = 1, 2, 3 \dots 2k$

$\phi(h_i) = 2q - 2i - 4k + 1$, $i = 1, 2, 3 \dots 2k$

$\phi(a_i) = (4/5)\, q - 4i + 2$, $i = 1, 2, 3 \dots 2k$

$\phi(b_i) = 2q - 2i - 8k + 1$, $i = 1, 2, 3 \dots 2k$

$\phi(c_i) = (6/5)\, q - 4i + 2$, $i = 1, 2, 3 \dots 2k$

$\phi(d_i) = (7/5)\, q - 2i + 1$, $i = 1, 2, 3 \dots 2k$

$\phi(z_i) = (8/5)\, q - 4i + 2$, $i = 1, 2, 3 \dots 2k$

$\phi(g_i) = (1/5)\, q - 2i + 1$, $i = 1, 2, 3 \dots 2k$

$\phi(v_i) = q - 8i + 4$, $i = 1, 2, 3 \dots k$

a)

$$\max_{v \in V(G)} \phi(v) = \max\{0, \max_{1\le i\le 2k}(2q - 2i + 1), \max_{1\le i\le 2k}((2/5)\, q - 4i + 2), \max_{1\le i\le 2k}(2q - 2i - 4k + 1), \max_{1\le i\le 2k}((4/5)\, q - 4i + 2), \max_{1\le i\le 2k}(2q - 2i - 8k + 1), \max_{1\le i\le 2k}((6/5)\, q - 4i + 2), \max_{1\le i\le 2k}((7/5)\, q - 2i + 1), \max_{1\le i\le 2k}((8/5)\, q - 4i + 2), \max_{1\le i\le 2k}((1/5)\, q - 2i + 1), \max_{1\le i\le k}(q - 8i + 4)\}$$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi(v) \in \{0, 1, 2 \dots 2q - 1\}$

b) Clearly, The function ϕ is one-to-one mapping from the vertex set of G to the set $\{0,1,2 \dots 2q - 1\}$

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1,2q\text{-}1]$ and that's as following:

The range of $| \phi(u_i) - \phi(u) | = \{2q - 2i + 1, i = 1, 2 \dots 2k\}$

$= \{2q - 1, 2q - 3 \dots 2q - 4k + 1\}$

The range of $| \phi(u_i) - \phi(x_i) | = \{(8/5)\, q + 2i - 1, i = 1, 2 \dots 2k\}$

$= \{(8/5)\, q + 1, (8/5)\, q + 3 \dots (8/5)\, q + 4k - 1\}$

The range of $|\phi(h_i) - \phi(x_i)| = \{(8/5)\, q + 2i - 4k - 1, i = 1,2 \dots 2k\}$

$= \{(8/5)\, q - 4k + 1, (8/5)\, q - 4k + 3 \dots (8/5)\, q - 1\}$

The range of $| \phi(h_i) - \phi(a_i) | = \{(6/5)\, q + 2i - 4k - 1, i = 1,2 \dots 2k\}$

$= \{(6/5)\, q - 4k + 1, (6/5)\, q - 4k + 3 \dots (6/5)\, q - 1\}$

The range of $| \phi(a_i) - \phi(b_i) | = \{(6/5)\, q + 2i - 8k - 1, i = 1,2 \dots 2k\}$

$= \{(6/5)\, q - 8k + 1, (6/5)\, q - 8k + 3 \dots (6/5)\, q - 4k - 1\}$

The range of $| \phi(c_i) - \phi(b_i) | = \{(4/5)\, q + 2i - 8k - 1, i = 1,2 \dots 2k\}$

$= \{(4/5)\, q - 8k + 1, (4/5)\, q - 8k + 3 \dots (4/5)\, q - 4k - 1\}$

The range of $| \phi(c_i) - \phi(d_i) | = \{(1/5)\, q + 2i - 1, i = 1, 2 \dots 2k\}$

$= \{(1/5)\, q + 1, (1/5)\, q + 3 \dots (1/5)\, q + 4k - 1\}$

The range of $| \phi(d_i) - \phi(z_i) | = \{(1/5)\, q - 2i + 1, i = 1, 2 \dots 2k\}$

$= \{(1/5)\, q - 1, (1/5)\, q - 3 \dots 1\}$

The range of $| \phi(z_i) - \phi(g_i) | = \{(7/5)\, q - 2i + 1, i = 1, 2 \dots 2k\}$

$= \{(7/5)\, q - 1, (7/5)\, q - 3 \dots (7/5)\, q - 4k + 1\}$

The range of $| \phi(v_i) - \phi(g_{2i-1}) | = \{(4/5)\, q - 4i + 1, i = 1,2,3 \dots k\}$

$= \{(4/5)\, q - 3, (4/5)\, q - 7 \dots (4/5)\, q - 4k + 1\}$

The range of $| \phi(v_i) - \phi(g_{2i}) | = \{(4/5)\, q - 4i + 3, i = 1,2, 3 \dots k\}$

$= \{(4/5)\, q - 1, (4/5)\, q - 5 \dots (4/5)\, q - 4k + 3\}$

Hence, $\{| \phi(u) - \phi(v) | : u\, v \in E(G)\} = \{1, 3, 5 \dots 2q - 1\}$.

So the revised friendship graph $F(kC_{20})$ is odd graceful.

Now, we introduce a new conjecture and that's as shown.

Conjecture 2.8: The revised friendship graph $F(kC_n)$ is odd graceful where k is any positive integer and n = 0 (mod 4)".

3. Conclusion

Graceful and odd gracefulness of a graph are two entirely different concepts. A graph may posses one or both of these or neither. In this paper we introduced the odd graceful labeling of the revised friendship graphs $F(kC_4)$, $F(kC_8)$, $F(kC_{12})$, $F(kC_{16})$ and $F(kC_{20})$ where k is any positive integer. Finally, we introduced a new conjecture " The revised friendship graph $F(kC_n)$ is odd graceful where k is any positive integer and $n = 0$ (*mod* 4).

4. References

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