On Strongly g^*-continuous Maps and Pasting Lemma in Topological Spaces

V.Subramonia Pillai
Research Scholar
Karpagam University,
Coimbatore,Tamil Nadu, India

R.Parimelazhagan
Department of Science and Humanities
Karpagam College of Engineering,
Coimbatore -32. Tamil Nadu, India

ABSTRACT

The objective of the present paper is to introduce new classes of functions called Strongly g^*-continuous maps .We obtain some characterizations of these classes and several properties are studied.Also we prove Pasting lemma for Strongly g^*-continuous maps.

General Terms:

AMS Classification(2000): 54A05.

Keywords:

strongly g^* -continuous mapsifx

1. INTRODUCTION

Strong forms of continuous maps have been introduced and investigated by several mathematicians.Strongly continuous maps,perfectly continuous maps,completely continuous maps,super continuous maps were introduced by Levine[6],Noiri[9],Munshi[8] and Tong[15]respectively. Noiri[10] introduced a new concept called strongly θ continuity which is stronger than continuity.Lang[5] studied strongly θ continuous functions.Balachandran et al[2] have introduced and studied generalized semi-continuous maps,semi-locally continuous maps ,semi -generalized locally continuous maps and generalized locally continuous maps.Sundaram[14] introduced and studied g-continuous functions.Maki[7]studied the Pasting Lemma for α- continuous maps.Parimelazhagan[12]introduced and studied strongly g^*-closed sets.

In this paper we introduce and study the concepts of a new class of maps,namely Strongly g^*-continuous maps which includes the class of continuous maps .Also we prove a pasting lemma for strongly g^*-continuous maps.

2. PRELIMINARIES

Before entering into our work, we recall the following definitions which are due to Levine.

Definition 2.1[6]: A function f: X $\rightarrow$Y is said to be strongly continuous if f^{-1}(V) is both open and closed in X for each subset V of Y.

Definition 2.2[1]:A function f: X $\rightarrow$ Y is said to be completely continuous if f^{-1}(V)is regular open in X for each open set V of Y

Definition 2.3[11]:A function f: X $\rightarrow$Y is said to be perfectly continuous if f^{-1}(V) is both open and closed in X for each open set V of Y.

Definition 2.4[9]: A function f: X $\rightarrow$Y is said to be α continuous or strongly semi continuous if f^{-1}(V)is α open in X for each open set V of Y.

Definition 2.5[3]: A function f: X $\rightarrow$Y is said to be generalized continuous(g-continuous) if f^{-1}(V) is g-open in X for each open set V of Y

Definition 2.6[12]: Let (X, τ) be a topological space and A be its subset , then A is strongly g^*-closed set if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is g-open.

Definition 2.7[13]: A map f: X $\rightarrow$ Y from a topological space X into a topological space Y is called strongly g^* irresolute(sg^* -irresolute) if the inverse image of every sg^*- closed set in Y is sg^* -closed in X.

3. STRONGLY G^*-CONTINUOUS FUNCTIONS

In this section we have introduce the concept of strongly g^*-continuous functions in topological space.

Definition 3.1: Let X and Y be topological spaces. A map f: X $\rightarrow$ Y is said to be strongly g^* continuous (sg^* - continuous) if the inverse image of every open set Y is sg^*-open in X.

Theorem 3.2: If a map f: X $\rightarrow$Y from a topological space X in to a topological space Y is continuous then it is sg^*- continuous .

Proof: Let V be an open set in Y.Since f is continuous f^{-1}(v) is open is X .As open set is sg^*- open ,f^{-1}(v) is sg^*-open in X. Therefore f is sg^* -continuous.

Remark 3.3: The converse of the above theorem neednot be true as seen from the following example.

Example 3.4: Let X=Y= {a,b,c} with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{b\}, \{b, c\}\}$ Let f :X$\rightarrow$ Y be defined by f(a)=b,f(b)=c,f(c)=a then f is sg^*-continuous but not continuous as the inverse image of the open set {b,c} in Y is {a,b} is not open in X.

Theorem 3.5: A map f:X$\rightarrow$Y is sg^* -continuous if and only if the inverse image of every closed set in Y is sg^*- closed in X.

Proof: Let F be closed in Y.Then F^c is open in Y. Since f is sg*-continuous,f^{-1}(F)is sg^*-open in X. But $f^{-1}(F^c)$ =X - f^{-1}(F) and so f^{-1}(F) is sg^*-closed in X.

Conversely assume that the inverse image of every closed set in Y is sg^*-closed in X.Let V be an open set in Y.Then V^c is closed in Y. By hypothesis $f^{-1}(V^c)$= X - f^{-1}(V) is sg^*- closed in X and so f^{-1}(V) is sg^*- open in X. Thus f is sg^*- continuous.

Theorem 3.6: Let X and Y be topological spaces.If a map f: X $\rightarrow$ Y is sg^*-continuous then it is g continuous .

Proof: Assume that a map $f : X \rightarrow$ Y is sg*-continuous. Let V be an open set in Y.Since f is continuous f^{-1}(V) is sg* -open and hence g- open in X.Therefore f is g-continuous.

Remark 3.7: The converse of the above theorem need not be true as seen from the following example.

Example 3.8: Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\phi, Y, \{a, c\}\}$ and f be identity map. Then f is g-continuous but not sg^*-continuous as the inverse image of the openset$\{a, c\}$ in Y is $\{a, c\}$ in X is not sg^*-open.

Theorem 3.9: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map from a topological space (X, τ) into a topological space (Y, σ)

(i)The following statements are equivalent. a)f is sg^*- continuous. b).The inverse image of each open set in Y is sg^*-open in X.
(ii).If $f : (X, \tau) \to (Y, \sigma)$ is sg^*-continuous then f($cl^*(A)$) $\subset \overline{f(A)}$ for every subset A of X.(Here cl^*(A) is the closure of A as defined by Dunham[4]).
iii).(a)For each point $x \in X$ and each open set V containing f(x),there exist a sg^*-open set U containing x suchthat $f(V) \subset V$ (b)For every subset A of X, $f(cl^*(A)) \subset \overline{f(A)}$ holds (c)The map $f : (X, \tau^*) \to (Y, \sigma)$ from a topological space(X, τ^*)defined by Dunham[6] into topological space (Y, σ) is continuous.
Proof: (i) Assume that f : X $\to$Y is sg^* continuous. Let G be open in Y. Then G^c is closed in Y. Since f is sg^* continuous $f^{-1}(G^c)$ is sg^*-closed in X. But $f^{-1}(G^c)$=X - f^{-1}(G).Thus X - f^{-1}(G) is sg^*-closed in X and so f^{-1}(G) is sg^*-open in X. Therefore (a) implies (b). Conversely assume that the inverse image of each open set in Y is sg^*-open in X. Let F be any closed set in Y. Then F^c is open in Y. By assumption, $f^{-1}(F^c)$ is sg^* open in X. But $f^{-1}(F^c)$ = X - f^{-1}(F)). Thus X - f^{-1}(F) is sg^*-open in X and so f^{-1}(F) is sg^* -closed in X. Therefore f is sg^*-continuous. Hence(b) implies (a). Thus (a) and (b) are equivalent.
(ii). Assume that f is sg^* continuous. Let A be any subset of X. Then $\overline{f(A)}$ is closed set in Y. Since f is sg^* continuous,$f^{-1}\overline{f(A)}$ is sg^* closed in X and it contains A. But cl^*(A) is the intersection of all sg^* -closed sets containing A. Therefore $cl^*(A) \subset f^{-1}(\overline{f(A)}$ and so f(cl^*(A))$\subset$ $\overline{f(A)}$ (iii). $(a) \Rightarrow (b)$ Let $Y \in f(cl^*(A))$ and let V be any open neighbourhood of Y. Then there exist a point $x \in$ and a sg^* -openset V suchthat f(x)=y, $x \in V, x \in cl^*$(A) and $f(v) \subset V$. Since $x \in cl^*$(A), $V \cap A \neq \phi$ holds and hence $f(A) \cap V \neq \phi$. Therefore we have $y = f(x) \in \overline{f(A)}$
$(b) \Rightarrow (a)$. Let $x \in X$ and V be any openset containing f(x). Let A=$f^{-1}(V^c)$,then $x \notin A$. Now $cl^*(A) \subset f^{-1}(f(cl^*(A))) \subset f^{-1}(V^c)$ =A. i.e.$cl^*(A) \subset A$. But $A \subset cl^*$(A) Therefore A=cl^*(A),then since $x \notin cl^*$(A) there exist a sg^*-openset U containing x suchthat $U \cap A = \phi$ and hence $f(U) \subset f(A^c) \subset V$. (b) $\Rightarrow$ (c). By assumption $f(cl^*(A)) \subset \overline{f(A)}$. Therefore f is sg^* continuous. (c)$\Rightarrow$(b)Let A be any subset of X. Then $\overline{f(A)}$ is a closed set in Y. Since f is continuous $f^{-1}(\overline{f(A)}$is closed in X. Now $A \subset f^{-1}(\overline{f(A)} \Rightarrow cl^*(A) \subset cl^*f^{-1}(\overline{f(A)} = f^{-1}(\overline{f(A)}$.Since $f^{-1}(\overline{f(A)}$ is closed in X.$\Rightarrow f(cl^*(A)) \subset \overline{f(A)}$
Remark 3.10: The converse of the theorem 3.9(ii) need not be true as seen from the following example
Example 3.11: Let $X = Y = \{a, b, c\}, \tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, Y, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined by f(a)=b,f(b)=a and f(c)=c.Then for every subset A,f(cl^*(A))
Theorem 3.12: If a map f:X $\to$Y from a topological space X into a topological space Y is continuous then it is sg^*- continuous.
Proof: Let f: X $\to$Y be continuous . Let F be any closed set in Y. Then the inverse image f^{-1}(F) is closed in Y. Since every closed set is sg^* -closed f^{-1}(F) is sg^*-closed in X. Therefore f is sg^* -continuous.
Remark 3.13: The converse need not be true as seen from the following example.
Example 3.14: Let$X = \{a, b, c\}\tau = \{\phi, X, \{a\}\}, Y = \{p, q\}$ and $\sigma = \{\phi, \{p\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by f(a)=f(c)=q,f(b)=p. Then f is sg^*- continuous. But f is not continuous since for the openset $G = \{p\}$ in Y,f^{-1}(G)= $\{b\}$ is not open in X.
Theorem 3.15: Let X and Y be any topological spaces and Y be a $T_{1/2}$ spaces.Then the composition $gof : X \to Z$ of the sg^* -continuous maps $f : X \to Y$ and $g : Y \to Z$ is also sg^*-continuous.
proof: Let F be a closed in Z. Since g is sg^* -continuous. g^{-1}(F) is sg^*- closed in Y. But Y is $T_{1/2}$ space and so g^{-1}(F) is closed. Since f is sg^* -continuous. $f^{-1}(g^{-1}$(F)) is sg^*- closed in X. But $f^{-1}(g^{-1}(F)) = (gof)^{-1}$(F). Therefore gof is sg^* -continuous.
Remark 3.16: The following example shows that the above theorem neednot be true if Y is not $T_{1/2}$
Example 3.17: Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$, $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$, $\eta = \{\phi, \{a, c\}, Z\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by f(a)=c, f(b)=b ,f(c)=c .Let $g : (Y, \sigma) \to (Z, \eta)$ be the identity map.Then f and g are sg^*- continuous . But gof is not a g continuous. Since $F = \{b\}$ is closed in Z. g^{-1}(F)=F and $f^{-1}(g^{-1}$(F))=F is not g closed in X. Therefore gof is non sg^*- continuous.
Theorem 3.18: Let f: X $\to$Y be a sg^*- continuous maps from a topological space X into a topological space Y and let H be a closed subset of X. Then the restriction $f/_H : H \to$Y is sg^* continuous where H is endowed with the relative topology
Proof: Let F be any closed subset in Y. Since f is sg^*- continuous. f^{-1}(F) is sg^* -closed in X.Levine[13]) has proved that intersection of a closed set is closed.Pari[25] has proved that intersection of two sg^* -closed set is sg^*- closed set. Thus if $f^{-1}(F) \cap H = H_1$ then H_1 is a sg^* -closed set in X.Since $(f/_H)^{-1}(F) = H_1$,it is sufficient to show that H_1 is sg^*-closed in H. Let G_1 be any open set of H suchthat $G_1 \supset H_1$. Let $G_1 = G \cap H$ where G is open in X.Now $H_1 \subset G \cap H \subset$ G.SinceH_1 is sg^*- closed in X. $\overline{H_1} \subset G$. Now $cl_H(H_1) = \overline{H_1} \cap H \subset G \cap H = G1$ where $cl_H(A)$ is the closure of a subset $A \subset H$ in a subspace H of X. Therefore f/H is sg^*- continuous.
Remark 3.19: In the above theorem the assumption of closedness of H cannot be removed as seen from the following example.
Example 3.20: Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$, $Y = \{p, q\}$ and $\sigma = \{\phi, \{p\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by f(a)=f(c)=q, f(b)=p. Then f is sg^*- continuous. Now $H = \{a, b\}$ is not closed in X. Then f is sg^* -continuous but the restriction $f/_H$ is not sg^*- continuous.Since for the closed set $F = \{q\}$ in Y . $f^{-1}(F) = \{a, c\}$ and $f^{-1}(F) \cap H = \{a\}$ is not sg^*- closed in H.

4. PASTING LEMMA FOR SG* - CLOSED SETS

In this section we have introduce the concept of Pasting Lemma for sg^*- continuous maps in topological space.
Theorem 4.1: Let $X = A \cup B$ be a topological space with topology τ and Y be a topological space with topology σ. Let $f : (A, \tau/_A \to (Y, \sigma)$ and $g : (B, \tau/_B \to (Y, \sigma)$ be sg^* continuous maps suchthat f(x)=g(x) for every $x \in A \cap B$. Suppose that A and B are sg^* closed in X. Then the combination $\alpha : (X, \tau) \to (Y, \sigma)$ is sg^* continuous.
Proof: Let F be any closed set in Y. Clearly $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$ where $C = f^{-1}(F)$ and D=g^{-1}(F).But C is sg^* closed in A and A is sg^* closed in X and so C is sg^* closed in X.Since previous paper proved that if $B \subset A \subset X$.B is sg^* closed in A and A is sg^* closed in X then B is sg^* closed in X. Simillary D is sg^* closed in X. Also $C \cup D$ is sg^* closed in X. Therefore α^{-1}(F) is sg^* closed in X. Hence α is sg^*- continuous.

5. FURTHER STUDY ON STRONGLY G^*IRRESOLUTE MAPS

In this section we have introduce the continuation study on strongly g^*-irresolute maps in topological space.
Theorem 5.1:A map f:X $\to$Y is sg^*- irresolute if and only if the inverse image of every sg^*- open set in sg^* is open in X.
Proof: Assume that f is sg^* -irresolute. Let A be any sg^* -open set in Y. Then A^c is sg^*- closed in Y. Since f is sg^* -irresolute ,f^{-1} (A^c) is sg^* -closed in X.But f^{-1} (A^c) = X - $f^{-1}(A)$ and so $f^{-1}(A)$ is sg^* -open in X. Hence the inverse image of every sg^* -open set in Y is sg^*- open in X. Conversely assume that the inverse image of every sg^*- open in Y is sg^*- open X. Let A be any sg^* -closed set in Y. Then A^c is sg^*- open in Y. By assumption f^{-1} (A^c) is sg^*- open in X.But f^{-1} (A^c) = X - f^{-1}(A) and so f^{-1}(A) is sg^* -closed in X. Therefore f is sg^*-irresolute.

Theorem 5.2: A map f: X $\rightarrow$Y is sg^* -irresolute then it is sg^* -continuous .

Proof: Assume that f is sg^*- irresolute . Let F be any closed set in Y. Since every closed set is sg^*- closed . F is sg^*- closed in Y. Since f is sg^* -irresolute ,f^{-1}(F) is sg^*- closed in X. Therefore f is sg^*- continuous.

Remark 5.3: The converse neednot be true as seen from the following example

Example 5.4: Let $X = Y = \{a, b, c\}$ $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by f(a)=f(c)=a ,and f(b)=b. Then f is sg^*- continuous. However $\{a, c\}$ is sg^*- closed in Y but f^{-1} $\{a, c\} = \{a, c\}$ is not sg^* closed in X. Therefore f is not sg^* -irresolute.

Theorem 5.5: Let X,Y and Z be any topological spaces. For any sg^*- irresolute map f:X $\rightarrow$Y and any sg^*- continuous map g:Y $\rightarrow$Z the composition gof:X $\rightarrow$Z is sg^*- continuous

proof: Let F be any closed set in Z.Since g is sg^*- continuous g^{-1}(F) is sg^* -closed in Y. Since f is sg^* -irresolute . $f^{-1}(g^{-1}$(F)) =$(gof)^{-1}$(F).Therefore gof is sg^*-continuous

Theorem 5.6: If f:X $\rightarrow$Y from a topological space X into a topological space Y is bijective,open and sg^*-continuous then f is sg^*-irresolute

Proof: Let A be a sg^* -closed set in Y. Let $f^{-1}(A) \subset O$ where O is open in X. Therefore A $\subset f(0)$ holds. Since f(0) is open and A is sg^*- closed in Y. $\overline{A} \subset f(0)$ holds and hence $\overline{f^{-1}(\overline{A})} \subset O$. Since f is s$g^*$ -continuous and $\overline{A}$ is closed in Y. $\overline{f^{-1}(\overline{A})} \subset O$ and so $\overline{f^{-1}(A)} \subset O$.Therefore f^{-1}(A) is sg^* -closed in X .Hence f is sg^*-irresolute.

Remark 5.7: The following examples show that no assumption of above theorem can be removed

Example 5.8: Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$.Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by f(a)=f(c)=a and f(b)=b. Then f is sg^*- continuous and open but it is not bijective and f is not sg^* -irresolute since for the sg^* -closed set $G = \{a, c\}$ in Y.$f^{-1}(F) = \{a, c\}$ is not sg^*- closed in X.

Example 5.9: Let (X, τ) and (Y, σ) be the topological spaces in Example 2.2 the identity map $f : (X, \tau) \rightarrow (Y, \sigma)$ is sg^*-continuous ,bijective and not open. And f is not sg^* -irresolute. Since for the sg^*- closed set $G = \{a, c\}$ in Y,f^{-1}(G)=G is not sg^* -closed in X.

Example 5.10: Let$X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and σ be the discrete topology of Y.Then the identity map $f : (X, \tau) \rightarrow (Y, \sigma)$ is bijective open and not sg^*- continuous and f is not sg^* -irresolute since for the sg^*- closed set $G = \{a\}$ in Y,f^{-1}(G)=G is not sg^* -closed in X.The following two examples shows that the concept of irresolute maps and sg^*- irresolute maps are independent of each other.

Example 5.11: Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$ Then the identityu map $f : (X, \tau) \rightarrow (Y, \sigma)$is irresolute since for the sg^*-closed set G={a} in Y f^{-1}(G) is not sg^* -closed in X.

Example 5.12: Let (X, τ) and (Y, σ) be the spaces defined in above. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by f(a)=c, f(b)=b and f(c)=a.Then f is sg^*- irresolute, but it is not irresolute. Since for the sg^* -closed set. G={b} in Y, f^{-1}(G)=G is not sg^* -closed in X.

6. REFERENCES

[1] Arya S.P and Gupta R On Strongly Continuous mappings Kyungpook Math.J.14(1974),131-143.

[2] Balachandran . K, Sundaram .P and Maki. H Generalized locally closed sets and GLC -continuous functions,Indian J.Pure.Appl. Math.27(1996),235-244.

[3] Balachandran .K,Sundaram P and Maki.H On Generalized continuous maps in Topological Spaces,Mem.Fac.Sci.Kochi.Univ.Math.12(1991),5-13.

[4] Dunham.W,New closure operator for Non T_1Topologies,Kyungpook Math.J.22(1982),55-60.

[5] Lang P.E and Herington L.L.Strongly θ continuous functions ,J.Korean.Math.Soc.,18(1981)21-28.

[6] Levine N. Strong continuity in topological spaces,Amer.Math.Monthly,67(1960), 269

[7] Maki H Munakata and T.Noiri ,The Pasting Lemma for α continuous maps,GLasnik Mathematical Vol 23(43)(1988),157-163.

[8] Munshi,B.M. and Bassan,D.S.Super continuous mappings,Indian J.Pure Appl.Math.13(1982),229-236.

[9] Noiri,T.,Between continuity and Weak Continuity Bull.Un.Mat.Ital.9(1974)647-654.

[10] Noiri,T.Super continuity and some strong forms of continuity.Indian J.Pure Appl.Mat.15(1984)240-250.

[11] Noiri ,T .Strong form of continuity in topological spaces Rend.Circ.Math.Palermo,(1986)107-113.

[12] Parimelazhagan.R and Subramonia Pillai.V,Strongly g^*- closed sets in topological spaces Int.Journal of Math.Analysis,Vol.6,2012,No.30,1481-1489.

[13] Subramonia Pillai.V and Parimelazhagan.R,'Strongly g*-irresolute and homeomorphism in Topological spaces'- International Journal of Recent Scientific Research Vol.4, Issue 1,pp.005-007,January,2013

[14] Sundaram P Studies on Generalizaations of Continuous maps in topological spaces,Ph.D Thesis,Bharathiar University ,CBE(1991).

[15] Tong ,J.A Decomposition of continuity in topological spaces.Acta. Math.Hungar,54(1989)51-55.