Control Chart for Waiting Time in System of (M/M/1): (∞/FCFS) Queuing Model

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ABSTRACT
Queue is a very volatile situation which always cause unnecessary delay and reduce the service effectiveness of establishments or service industries. Long queues may create negative effects like wasting of man power, unnecessary congestion which leads to suffocation; even develop complications to customers and also to the establishments. This necessitates the study of waiting time of the customers and the facility. Control chart technique may be applied to analyze the waiting time of the customers in the system to improve the services and the effective performance of concerns. Control chart constructed for random variable W, the time spent in the system, provides the control limits for W. The prior idea about the expected waiting time, maximum waiting time and minimum waiting time from the parameters of the constructed chart makes effective use of time and guarantees customer's satisfaction. Keeping this in view, the construction of control charts for waiting time is proposed for M /M /1queueing model.

Keywords
Waiting time, Control limits, Poisson arrival and Exponential service.

1. INTRODUCTION
Every manufacturing organization is concerned with the quality of its product. Stiff competition in the national and international level and customers' awareness require production of quality goods and services for survival and growth of the company. The most essential ingredient required to meet this ever growing competition is quality. This warrants every manufacturing organization to be concerned with the quality of its product. Quality is to be planned, improved and monitored continuously.

Shewhart developed control chart techniques based on data of one or several quality related characteristics of the product or service to identify whether a production process or service having goods of set quality standards. Montgomery [2] proposed a number of applications of control charts in assuring quality in manufacturing industries.

Shore [3] developed control chart for random queue length, N of M /M /1queueing model by considering the first three moments. Khaparde and Dhabe [4] constructed N_1, Shewhart control chart and N_2, the control chart using method of weighted variance for random queue length N for M/M/1queueing model. With reference to individuals sorting for facilities the time spent is more influential than the number in the queueing system. Thus the analysis of time spent in the system by the control chart provides improvement of the performance of the system and hence customer satisfaction. In this paper, an attempt is made to construct Shewhart control chart for waiting time, W of M/M/1 queueing model. This model finds applications in a number of fields like assembly and repairing of machines, aircrafts, ATM facility of banks etc. where the system is having a single server.

2. M/M/1 MODEL DESCRIPTION
M/M/1 model has single server, Poisson input, exponential service time and infinite capacity with First Come First Serve (FCFS) queue discipline. Let λ be the mean arrival rate and μ be the average service rate.

2.1 Steady state equations
The steady state equations of this model are by [1].

Let

$P_n(t)$ = Probability that there are n customers in the system (waiting and in service) at time t.

$$P_0(t+\Delta t) = P_0(t)\,(1 - \lambda\Delta t) + P_1(t)\,\mu\,\Delta t + o(\Delta t)$$

$$P_n(t+\Delta t) = P_n(t)\,(1-(\lambda+\mu)\Delta t) + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + o(\Delta t),\quad n \geq 1 \qquad (1)$$

Equation (1) gives

$$P_0'(t) = -\lambda\, P_0(t) + \mu\, P_1(t)$$

$$P_n'(t) = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t),\quad n \geq 1 \qquad (2)$$

The steady state equations corresponding to (2) are

$$0 = -\lambda\, P_0 + \mu\, P_1$$

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1},\quad n \geq 1 \qquad (3)$$

Let $\rho = \frac{\lambda}{\mu}$ be the traffic intensity. Equation (3) yields

$$P_0 = (1-\rho)$$

$$P_n = (1-\rho)\,\rho^n \qquad (4)$$

2.2 Performance measures
(i) $E(L_s)$ = Average number of customers in the system

$$= \sum_{n=0}^{\infty} n\, P_n$$

$$= \frac{\rho}{(1-\rho)} \quad (5)$$

(ii) E (L_q) = Average number of customers in the queue

$$= \sum_{n=1}^{\infty} (n-1) P_n$$

$$= \frac{\rho^2}{(1-\rho)} \quad (6)$$

(iii) E (W_s) = Average waiting time of a customer in the system

$$= \frac{1}{\mu(1-\rho)} \quad (7)$$

(iv) E (W_q) = Average waiting time of a customer in the queue

$$= \frac{\rho}{\mu(1-\rho)} \quad (8)$$

Let W denote the waiting time of a customer in the system which includes both the waiting time and the service time. The pdf of the random variable W is given by [5]

$$f(w) = (\mu - \lambda)\, e^{-(\mu-\lambda) w}, \qquad w > 0$$

which is an exponential distribution with parameter (μ -λ) .

Mean E (W) = $\frac{1}{(\mu-\lambda)}$ and variance V (W) = $\frac{1}{(\mu-\lambda)^2}$

3. CONTROL CHART FOR WAITING TIME, W

Shewhart type control charts are constructed by approximating the statistic under consideration by a normal distribution. The parameters of the control chart are given by

$$\left.\begin{aligned} UCL &= E(W) + 3\sqrt{V(W)} \\ CL &= E(W) \\ LCL &= E(W) - 3\sqrt{V(W)} \end{aligned}\right\} \quad (9)$$

For M /M /1queueing model the parameters of the control chart for waiting time of the customer in the system are given by

$$UCL = \frac{4}{(\mu-\lambda)}$$

$$CL = \frac{1}{(\mu-\lambda)}$$

$$LCL = \frac{-2}{(\mu-\lambda)}$$

4. NUMERICAL ANALYSIS

Assessment of waiting time in the system by means of control chart is carried out with numerical illustrations for certain selected values of λ and μ. Table.1 gives the parameters of the control chart for various values of the arrival rate λ and a constant service rate μ = 6.

Table.1 Parameters of control chart with constant service rate

λ	μ	ρ	CL	UCL	LCL
1.00	6	0.1667	0.2000	0.8000	-0.4000
1.50	6	0.2500	0.2222	0.8889	-0.4444
2.00	6	0.3333	0.2500	1.0000	-0.5000
2.50	6	0.4167	0.2857	1.1429	-0.5714
3.00	6	0.5000	0.3333	1.3333	-0.6667
3.50	6	0.5833	0.4000	1.6000	-0.8000
4.00	6	0.6667	0.5000	2.0000	-1.0000
4.50	6	0.7500	0.6667	2.6667	-1.3333
5.00	6	0.8333	1.0000	4.0000	-2.0000
5.50	6	0.9167	2.0000	8.0000	-4.0000

From the calculated values of parameters, it is clear that the increase in arrival rate with constant service rate increases the average waiting time and the expected upper limit of waiting time.Table.2 gives the parameters of the control chart for a constant arrival rate λ = 2 and various values of service rate μ.

Table.2 Parameters of control chart with constant arrival rate

λ	μ	ρ	CL	UCL	LCL
2	2.50	0.8000	2.0000	8.0000	-4.0000
2	3.00	0.6667	1.0000	4.0000	-2.0000
2	3.50	0.5714	0.6667	2.6667	-1.3333
2	4.00	0.5000	0.5000	2.0000	-1.0000
2	4.50	0.4444	0.4000	1.6000	-0.8000
2	5.00	0.4000	0.3333	1.3333	-0.6667
2	5.50	0.3636	0.2857	1.1429	-0.5714
2	6.00	0.3333	0.2500	1.0000	-0.5000

From the above table it is seen that if the service rate increases, the average waiting time and the expected upper control limit of waiting time decrease for a constant arrival rate. LCL values in both the cases are considered as 0, since the values are negative.

5. APPLICATION

As an application of the above theoretical calculations, a real situation, relating to a grocery shop is considered.

A grocery shop has a single server for billing which starts at 8.00 a.m. An arrival moves immediately into the service facility if it is empty. On the other hand, if the server is busy, the arrival will wait in the queue. Customers are served on first come first served basis. Observed inter arrival time and service time of 200 customers in the system is given in Table.3. In this table customers, arrival time (min.), inter-arrival time, starting time of service, service time (min.), ending time of service and customer's waiting time in system (min.) are given in columns I,II,III,IV,V,VI and VII respectively. Control charts are constructed using the theoretical formula and also estimated values of observed data.

From Table.3 the average inter arrival time is 2.755 min. and the average service time is 2.59 min. Arrival rate $\lambda = 21.78$ customers/hr. and service rate $\mu = 22.90$ customers/hr. Using (9) the parameters of the control chart are calculated as CL = 43.3 min., UCL = 173.2 min. and LCL = 0. The estimated parameters of the control chart for waiting time are calculated as

$$CL = E(\hat{W}) = 6.16 \text{ min}$$

$$UCL = E(\hat{W}) + 3\sqrt{V(\hat{W})} = 19.84 \text{ min}$$

$$LCL = E(\hat{W}) - 3\sqrt{V(\hat{W})} = 0,$$

based on sample observations.

Table.3 Observed waiting time of customers

I	II	III	IV	V	VI	VII
1	8.01	1	8.01	4	8.05	4
2	8.03	2	8.05	3	8.08	5
3	8.04	1	8.08	5	8.13	9
4	8.08	4	8.13	4	8.17	9
5	8.09	1	8.17	2	8.19	10
6	8.1	1	8.19	8	8.27	17
7	8.12	2	8.27	7	8.34	22
8	8.16	4	8.34	10	8.44	28
9	8.19	3	8.44	1	8.45	26
10	8.23	4	8.45	3	8.48	25
11	8.25	2	8.48	1	8.49	24
12	8.27	2	8.49	1	8.5	23
13	8.3	3	8.5	1	8.51	21
14	8.33	3	8.51	1	8.52	19
15	8.37	4	8.52	1	8.53	16
16	8.41	4	8.53	2	8.55	14
17	8.44	3	8.55	2	8.57	13
18	8.47	3	8.57	1	8.58	11
19	8.48	1	8.58	2	9	12
20	8.53	5	9	1	9.01	8
21	8.57	4	9.01	2	9.03	6
22	8.58	1	9.03	2	9.05	7
23	8.59	1	9.05	3	9.08	9
24	9.03	4	9.08	2	9.1	7
25	9.04	1	9.1	3	9.13	9
26	9.08	4	9.13	2	9.15	7
27	9.1	2	9.15	4	9.19	9
28	9.12	2	9.19	2	9.21	9
29	9.15	3	9.21	3	9.24	9
30	9.18	3	9.24	2	9.26	8
31	9.2	2	9.26	2	9.28	8
32	9.22	2	9.28	3	9.31	9
33	9.26	4	9.31	2	9.33	7
34	9.29	3	9.33	2	9.35	6
35	9.3	1	9.35	1	9.36	6
36	9.34	4	9.36	2	9.38	4
37	9.37	3	9.38	3	9.41	4
38	9.4	3	9.41	2	9.43	3
39	9.46	6	9.46	3	9.49	3
40	9.51	5	9.51	3	9.54	3
41	9.55	4	9.55	6	10.01	6
42	9.56	1	10.01	4	10.05	9
43	9.57	1	10.05	2	10.07	10
44	10.01	4	10.07	2	10.09	8
45	10.02	1	10.09	3	10.12	10
46	10.06	4	10.12	2	10.14	8
47	10.08	2	10.14	2	10.16	8
48	10.1	2	10.16	3	10.19	9
49	10.13	3	10.19	2	10.21	8
50	10.16	3	10.21	3	10.24	8
51	10.2	4	10.24	4	10.28	8
52	10.24	4	10.28	3	10.31	7
53	10.25	1	10.31	5	10.36	11
54	10.29	4	10.36	4	10.4	11
55	10.3	1	10.4	2	10.42	12
56	10.34	4	10.42	1	10.43	9
57	10.36	2	10.43	1	10.44	8
58	10.38	2	10.44	1	10.45	7
59	10.41	3	10.45	2	10.47	6

60	10.44	3	10.47	3	10.5	6
61	10.46	2	10.5	4	10.54	8
62	10.48	2	10.54	2	10.56	8
63	10.51	3	10.56	3	10.59	8
64	10.54	3	10.59	2	11.01	7
65	10.55	1	11.01	2	11.03	8
66	10.59	4	11.03	3	11.06	7
67	11.02	3	11.06	2	11.08	6
68	11.05	3	11.08	1	11.09	4
69	11.11	6	11.11	1	11.12	1
70	11.16	5	11.16	4	11.2	4
71	11.2	4	11.2	2	11.22	2
72	11.21	1	11.22	5	11.27	6
73	11.22	1	11.27	3	11.3	8
74	11.26	4	11.3	2	11.32	6
75	11.27	1	11.32	3	11.35	8
76	11.31	4	11.35	2	11.37	6
77	11.33	2	11.37	4	11.41	8
78	11.35	2	11.41	2	11.43	8
79	11.38	3	11.43	3	11.46	8
80	11.41	3	11.46	2	11.48	7
81	11.43	2	11.48	2	11.5	7
82	11.45	2	11.5	3	11.53	8
83	11.49	4	11.53	2	11.55	6
84	11.52	3	11.55	2	11.57	5
85	11.53	1	11.57	1	11.58	5
86	11.57	4	11.58	2	12	3
87	12	3	12	3	12.03	3
88	12.03	3	12.03	2	12.05	2
89	12.09	6	12.09	6	12.15	6
90	12.14	5	12.15	5	12.2	6
91	12.18	4	12.2	7	12.27	9
92	12.19	1	12.27	1	12.28	9
93	12.23	4	12.28	2	12.3	7
94	12.27	4	12.3	2	12.32	5
95	12.28	1	12.32	3	12.35	7
96	12.32	4	12.35	2	12.37	5
97	12.34	2	12.37	2	12.39	5
98	12.36	2	12.39	1	12.4	4
99	12.39	3	12.4	2	12.42	3
100	12.42	3	12.42	2	12.44	2

101	12.46	4	12.46	1	12.47	1
102	12.47	1	12.47	2	12.49	2
103	12.48	1	12.49	2	12.51	3
104	12.52	4	12.52	2	12.54	2
105	12.53	1	12.54	6	1	7
106	12.57	4	1	1	1.01	4
107	12.59	2	1.01	2	1.03	4
108	1.01	2	1.03	2	1.05	4
109	1.04	3	1.05	7	1.12	8
110	1.07	3	1.12	1	1.13	6
111	1.09	2	1.13	2	1.15	6
112	1.11	2	1.15	2	1.17	6
113	1.14	3	1.17	1	1.18	4
114	1.17	3	1.18	2	1.2	3
115	1.18	1	1.2	5	1.25	7
116	1.22	4	1.25	1	1.26	4
117	1.25	3	1.26	2	1.28	3
118	1.28	3	1.28	1	1.29	1
119	1.29	1	1.29	4	1.33	4
120	1.34	5	1.34	4	1.38	4
121	1.38	4	1.38	2	1.4	2
122	1.39	1	1.4	2	1.42	3
123	1.4	1	1.42	3	1.45	5
124	1.44	4	1.45	2	1.47	3
125	1.45	1	1.47	3	1.5	5
126	1.49	4	1.5	2	1.52	3
127	1.51	2	1.52	2	1.54	3
128	1.53	2	1.54	4	1.58	5
129	1.56	3	1.58	3	2.01	5
130	1.59	3	2.01	2	2.03	4
131	2.01	2	2.03	2	2.05	4
132	2.03	2	2.05	3	2.08	5
133	2.07	4	2.08	3	2.11	4
134	2.1	3	2.11	2	2.13	3
135	2.11	1	2.13	3	2.16	5
136	2.15	4	2.16	3	2.19	4
137	2.18	3	2.19	3	2.22	4
138	2.21	3	2.22	6	2.28	7
139	2.22	1	2.28	1	2.29	7
140	2.27	5	2.29	1	2.3	3
141	2.31	4	2.31	2	2.33	2

142	2.32	1	2.33	2	2.35	3
143	2.33	1	2.35	2	2.37	4
144	2.37	4	2.37	2	2.39	2
145	2.38	1	2.39	3	2.42	4
146	2.42	4	2.42	2	2.44	2
147	2.44	2	2.44	2	2.46	2
148	2.46	2	2.46	3	2.49	3
149	2.49	3	2.49	3	2.52	3
150	2.52	3	2.52	4	2.56	4
151	2.56	4	2.56	2	2.58	2
152	2.57	1	2.58	1	2.59	2
153	2.58	1	2.59	3	3.02	4
154	3.02	4	3.02	2	3.04	2
155	3.03	1	3.04	3	3.07	4
156	3.07	4	3.07	2	3.09	2
157	3.09	2	3.09	2	3.11	2
158	3.11	2	3.11	3	3.14	3
159	3.14	3	3.14	3	3.17	3
160	3.17	3	3.17	2	3.19	2
161	3.19	2	3.19	2	3.21	2
162	3.21	2	3.21	3	3.24	3
163	3.24	3	3.24	3	3.27	3
164	3.27	3	3.27	2	3.29	2
165	3.28	1	3.29	3	3.32	4
166	3.32	4	3.32	3	3.35	3
167	3.35	3	3.35	3	3.38	3
168	3.38	3	3.38	6	3.44	6
169	3.39	1	3.44	2	3.46	7
170	3.44	5	3.46	4	3.5	6
171	3.48	4	3.5	2	3.52	4
172	3.49	1	3.52	2	3.54	5
173	3.5	1	3.54	4	3.58	8
174	3.54	4	3.58	2	4	6
175	3.55	1	4	1	4.01	6
176	3.59	4	4.01	2	4.03	4
177	4.01	2	4.03	2	4.05	4
178	4.03	2	4.05	3	4.08	5
179	4.06	3	4.08	3	4.11	5
180	4.09	3	4.11	2	4.13	4
181	4.11	2	4.13	2	4.15	4
182	4.13	2	4.15	4	4.19	6
183	4.17	4	4.19	3	4.22	5
184	4.2	3	4.22	2	4.24	4
185	4.21	1	4.24	3	4.27	6
186	4.25	4	4.27	3	4.3	5
187	4.28	3	4.3	3	4.33	5
188	4.31	3	4.33	1	4.34	3
189	4.37	6	4.37	5	4.42	5
190	4.42	5	4.42	4	4.46	4
191	4.46	4	4.46	2	4.48	2
192	4.47	1	4.48	1	4.49	2
193	4.48	1	4.49	3	4.52	4
194	4.52	4	4.52	1	4.53	1
195	4.53	1	4.53	1	4.54	1
196	4.57	4	4.57	2	4.59	2
197	5.01	4	5.01	2	5.03	2
198	5.03	2	5.03	3	5.06	3
199	5.07	4	5.07	1	5.08	1
200	5.11	4	5.11	1	5.12	1

Fig.1 presents the control chart based on theoretical formula using observed data in which all observed values lie below the expected waiting time. This shows that the performance of the system is in excess than the requirement. This occurs due to the assumptions of Poisson arrival and exponential service time which require large voluminous data observed over a longer period for assessment.

Fig.2 gives the control chart based on sample observations. In this chart the observed values lie above as well as below the central limit. The sample observations corresponding to 7, 8, 9,10,11,12 and 13 lie outside the upper control limit. This reveals that the system is not in statistical control. The reason for this out of control is that, the printer (device) is out of order at that instant.

The service providing facilities need to be maintained properly. In other situations the system is in statistical control.

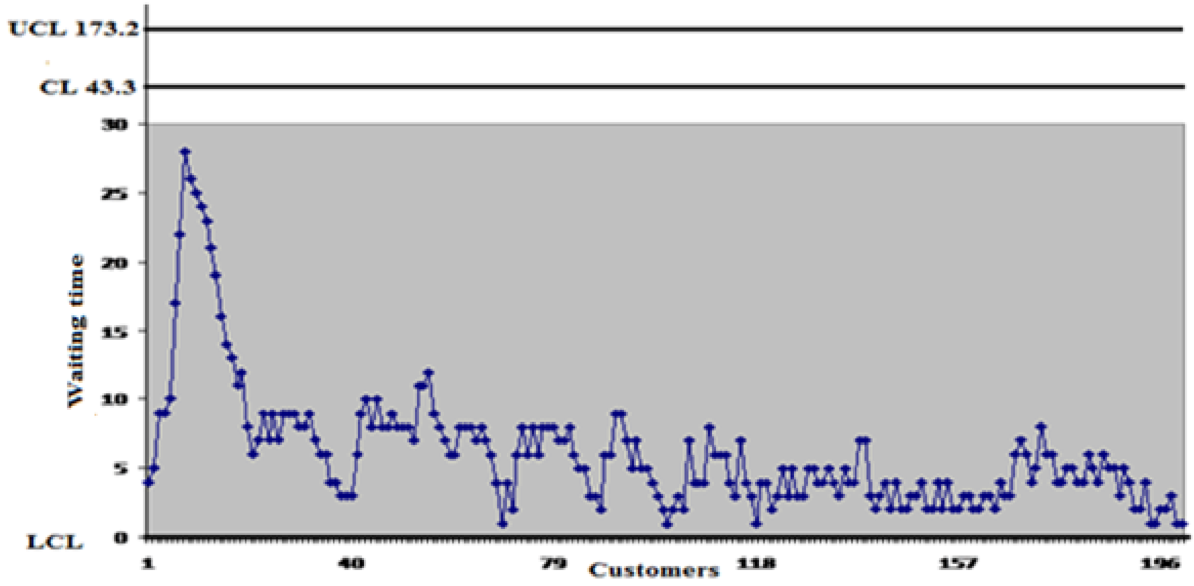

Fig.1 Control chart of waiting time of customers in the system using theoretical formula

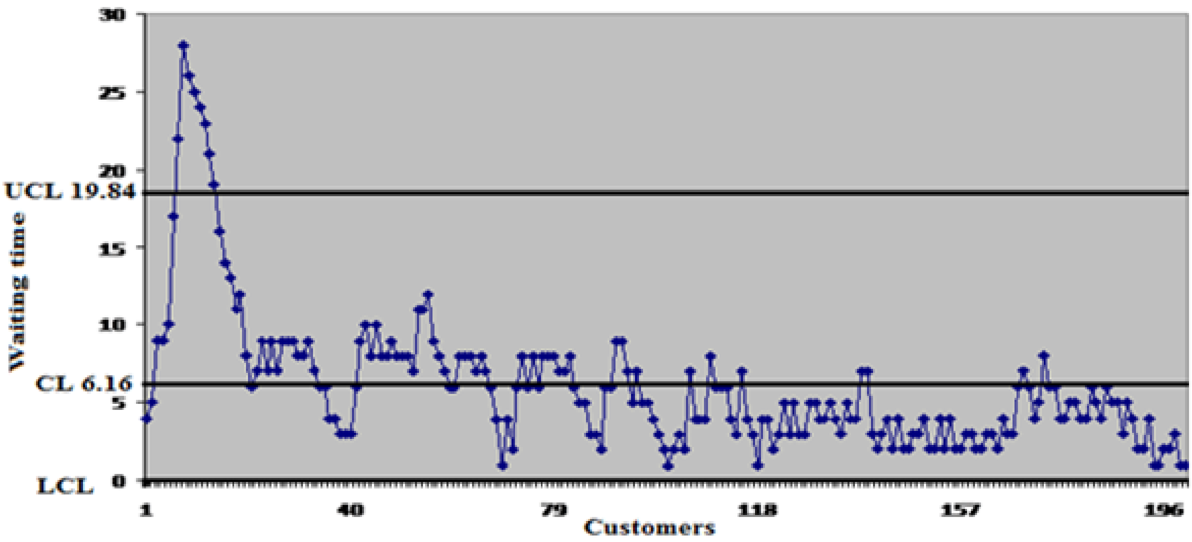

Fig.2 Control chart for waiting time of customers in the system using estimated values

6. CONCLUSION

In order to improve the performance of the queueing system, control chart for waiting time is obtained. The comparison between waiting time in the system using both theoretical and actual model shows greater variation because the system is assessed only for short period. If it is extended for a longer duration, more accurate results may be obtained which may confirm theory.

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