Some Identities of Multiplicative Coupled Fibonacci Sequences

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ABSTRACT

In the recent years, there has been much interest in development of coupled Fibonacci sequences. The concept of coupled Fibonacci sequences was first introduced by Atanassov, K. T. in 1985. He deliberated multiplicative coupled Fibonacci sequences of second order in 1995. Multiplicative coupled Fibonacci sequences are less known. In this paper we present some identities of multiplicative coupled Fibonacci sequences of second order under three specific schemes.

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1. INTRODUCTION

The concept of additive coupled Fibonacci sequence was first introduced by Atanassov, K. T. [1] in 1985. He defined four different schemes of additive coupled Fibonacci sequences [1] and called them 2-Fibonacci sequence (or 2-F sequences).

In 1995, Atanassov, K. T. [2] deliberated multiplicative coupled Fibonacci sequences of second order. He notified four different schemes of multiplicative coupled Fibonacci sequences.

Let $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ be two infinite sequences with initial values *a*, *b*, *c* and *d*. Then four different schemes of multiplicative coupled Fibonacci sequences of second order [2] are defined as follows:

First Scheme
$$\alpha_{n+2} = \beta_{n+1}.\,\beta_n, \quad n \ge 0 \qquad\qquad (1.1)$$
$$\beta_{n+2} = \alpha_{n+1}.\,\alpha_n, \quad n \ge 0.$$

Second Scheme
$$\alpha_{n+2} = \alpha_{n+1}.\,\beta_n, \quad n \ge 0 \qquad\qquad (1.2)$$
$$\beta_{n+2} = \beta_{n+1}.\,\alpha_n, \quad n \ge 0.$$

Third Scheme
$$\alpha_{n+2} = \beta_{n+1}.\,\alpha_n, \quad n \ge 0 \qquad\qquad (1.3)$$
$$\beta_{n+2} = \alpha_{n+1}.\,\beta_n, \quad n \ge 0.$$

Forth Scheme
$$\alpha_{n+2} = \alpha_{n+1}.\,\alpha_n, \quad n \ge 0 \qquad\qquad (1.4)$$
$$\beta_{n+2} = \beta_{n+1}.\,\beta_n, \quad n \ge 0.$$

Singh B. and Sikhwal [5] have studied some fundamental properties for scheme (1.1). Rathore, G. P. S., Jain, S. and Sikhwal, O. [4], presents multiplicative coupled Fibonacci sequences of third order under two specific schemes. Sikhwal, O. [6], presented some fundamental properties of coupled Fibonacci sequences of higher order and multiplicative coupled Fibonacci sequences of second order.

In this paper, some identities of multiplicative coupled Fibonacci sequences of second order are presented under various schemes.

2. MAIN RESULTS

In this section, identities under scheme (1.2), (1.3) and (1.4) will be described.

Some identities of scheme (1.2) are discussed below:

Theorem (2.1). If $n \ge 0$ is any integer, then

(*a*). $\beta_0.\alpha_{6n} = \alpha_0.\beta_{6n}$, (*b*). $\beta_1.\alpha_{6n+1} = \alpha_1.\beta_{6n+1}$,

(*c*). $\beta_2.\alpha_{6n+2} = \alpha_2.\beta_{6n+2}$, (*d*). $\alpha_0.\alpha_{6n+3} = \beta_0.\beta_{6n+3}$,

(*e*). $\alpha_1.\alpha_{6n+4} = \beta_1.\beta_{6n+4}$, (*f*). $\alpha_2.\alpha_{6n+5} = \beta_2.\beta_{6n+5}$.

Proof. Induction method will be used to derive the identities.

(a) If $n = 0$, then $\beta_0.\alpha_0 = \alpha_0.\beta_0$.
Thus the result is true for $n = 0$.

Now assume that the result is true for some integer $n \ge 1$. Then,

$$\beta_0.\,\alpha_{6n+6} = \beta_0(\alpha_{6n+5}.\,\beta_{6n+4}) \quad \text{(By scheme 1.2)}$$
$$= \beta_0(\alpha_{6n+4}.\,\beta_{6n+3})\beta_{6n+4} \quad \text{(By scheme 1.2)}$$
$$= \beta_0.\,\alpha_{6n+4}(\beta_{6n+2}.\,\alpha_{6n+1})\,\beta_{6n+4} \quad \text{(By scheme 1.2)}$$
$$= \beta_0.\,\alpha_{6n+4}(\beta_{6n+1}.\,\alpha_{6n})\alpha_{6n+1}.\beta_{6n+4} \quad \text{(By scheme 1.2)}$$
$$= \alpha_{6n+4}.\beta_{6n+1}(\beta_0.\alpha_{6n})\alpha_{6n+1}.\beta_{6n+4} \quad \text{(By scheme 1.2)}$$
$$= \alpha_{6n+4}.\beta_{6n+1}(\alpha_0.\beta_{6n})\alpha_{6n+1}.\beta_{6n+4} \quad \text{(By hypothesis)}$$
$$= \alpha_0.\alpha_{6n+4}.\beta_{6n+1}.\alpha_{6n+2}.\beta_{6n+4} \quad \text{(By scheme 1.2)}$$
$$= \alpha_0.\beta_{6n+6} \quad \text{(By scheme 2.2)}$$

Hence the result is true for all integers $n \ge 0$.

Similar proofs can be given for remaining parts (b) to (f).

Theorem (2.2). If $n \geq 0$ is an integer, then

(a). $\alpha_{n+4} = \alpha_{n+2}\beta_{n+1}^2\alpha_n$,

(b). $\beta_{n+4} = \beta_{n+2}\alpha_{n+1}^2\beta_n$.

Theorem (2.3). If $n \geq 0$ is an integer, then

$\alpha_{n+2}\beta_{n+2} = (\alpha_0\beta_0)^{F_{n+1}}.(\alpha_1\beta_1)^{F_{n+2}}$.

Theorem (2.4). If $n \geq 0$ is an integer, then

(a). $\alpha_n\beta_{n+1}\alpha_{n+2} = (\alpha_0\beta_0)^{F_{n+1}}.(\alpha_1\beta_1)^{F_{n+2}}$,

(b). $\beta_n\alpha_{n+1}\beta_{n+2} = (\alpha_0\beta_0)^{F_{n+1}}.(\alpha_1\beta_1)^{F_{n+2}}$.

Theorem (2.5). If $n \geq 0$ is an integer, then

(a) $\dfrac{\alpha_{n+3}}{\beta_n} = (\alpha_0\beta_0)^{F_n}.(\alpha_1\beta_1)^{F_{n+1}}$,

(b) $\dfrac{\beta_{n+3}}{\alpha_n} = (\alpha_0\beta_0)^{F_n}.(\alpha_1\beta_1)^{F_{n+1}}$.

Theorem (2.6). If $n \geq 0$ is an integer, then

$\prod_{k=0}^{n}\alpha_k\beta_k = (\alpha_0\beta_0)^{F_{n+1}}.(\alpha_1\beta_1)^{F_{n+2}-1}$.

Theorem (2.7). If $n \geq 0$ is an integer, then

(a). $\alpha_{n+2} = \alpha_1\prod_{k=0}^{n}\beta_k$,

(b). $\beta_{n+2} = \beta_1\prod_{k=0}^{n}\alpha_k$.

Theorem (2.8). If $n \geq 0$ is an integer, then

(a). $\dfrac{\prod_{k=0}^{6n}\alpha_k}{\prod_{k=0}^{6n}\beta_k} = \dfrac{\alpha_0}{\beta_0}$, (b). $\dfrac{\prod_{k=0}^{6n+1}\alpha_k}{\prod_{k=0}^{6n+1}\beta_k} = \dfrac{\alpha_0\alpha_1}{\beta_0\beta_1}$,

(c). $\dfrac{\prod_{k=0}^{6n+2}\alpha_k}{\prod_{k=0}^{6n+2}\beta_k} = \dfrac{\alpha_1^2}{\beta_1^2}$, (d). $\dfrac{\prod_{k=0}^{6n+3}\alpha_k}{\prod_{k=0}^{6n+3}\beta_k} = \dfrac{\alpha_1^2}{\beta_1^2}\dfrac{\beta_0}{\alpha_0}$,

(e). $\dfrac{\prod_{k=0}^{6n+4}\alpha_k}{\prod_{k=0}^{6n+4}\beta_k} = \dfrac{\alpha_1\beta_0}{\beta_1\alpha_0}$, (f). $\dfrac{\prod_{k=0}^{n}\alpha_k}{\prod_{k=0}^{n}\beta_k} = 1$.

Some identities of scheme (1.3) are discussed below:

Theorem (2.9). If $n \geq 0$ is an integer, then

(a). $\alpha_{3n+3} = \alpha_{3n+1}\beta_{3n+2}$,

(b). $\beta_{3n+3} = \beta_{3n+1}\alpha_{3n+2}$.

Theorem (2.10). If $n \geq 0$ is an integer, then

(a). $\alpha_{n+4} = \alpha_{n+2}\beta_{n+1}^2\alpha_n$,

(b). $\beta_{n+4} = \beta_{n+2}\alpha_{n+1}^2\beta_n$.

Theorem (2.11). If $n \geq 0$ is an integer, then

$\alpha_{n+2}\beta_{n+2} = (\alpha_0\beta_0)^{F_{n+1}}.(\alpha_1\beta_1)^{F_{n+2}}$.

Theorem (2.12). If $n \geq 0$ is an integer, then

(a). $\alpha_n\beta_{n+1}\alpha_{n+2} = \begin{cases} \alpha_0^{2F_{n+1}}.\beta_1^{2F_{n+2}}, & n \text{ is even} \\ \beta_0^{2F_{n+1}}.\alpha_1^{2F_{n+2}}, & n \text{ is odd} \end{cases}$.

(b). $\beta_n\alpha_{n+1}\beta_{n+2} = \begin{cases} \beta_0^{2F_{n+1}}.\alpha_1^{2F_{n+2}}, & n \text{ is even} \\ \alpha_0^{2F_{n+1}}.\beta_1^{2F_{n+2}}, & n \text{ is odd} \end{cases}$.

Theorem (2.13). If $n \geq 0$ is an integer, then

(a) $\dfrac{\alpha_{n+3}}{\beta_n} = \begin{cases} \beta_0^{2F_n}.\alpha_1^{2F_{n+1}}, & n \text{ is even} \\ \alpha_0^{2F_n}.\beta_1^{2F_{n+1}}, & n \text{ is odd} \end{cases}$.

(b) $\dfrac{\beta_{n+3}}{\alpha_n} = \begin{cases} \alpha_0^{2F}.\beta_1^{2F_{n+1}}, & n \text{ is even} \\ \beta_0^{2F_n}.\alpha_1^{2F_{n+1}}, & n \text{ is odd} \end{cases}$.

Theorem (2.14). If $n \geq 0$ is an integer, then

$\prod_{k=0}^{n}\alpha_k\beta_k = (\alpha_0\beta_0)^{F_{n+1}}.(\alpha_1\beta_1)^{F_{n+2}-1}$.

Theorem (2.15). If $n \geq 0$ is an integer, then

(a) $\dfrac{\alpha_{3n+5}}{\beta_{3n+2}} = \begin{cases} \beta_0^{2F_{3n+2}}.\alpha_1^{2F_{3n+3}}, & n \text{ is even} \\ \alpha_0^{2F_{3n+2}}.\beta_1^{2F_{3n+3}}, & n \text{ is odd} \end{cases}$,

(b) $\dfrac{\beta_{3n+5}}{\alpha_{3n+2}} = \begin{cases} \alpha_0^{2F_{3n+2}}.\beta_1^{2F_{3n+3}}, & n \text{ is even} \\ \beta_0^{2F_{3n+2}}.\alpha_1^{2F_{3n+3}}, & n \text{ is odd} \end{cases}$,

(c) $\dfrac{\alpha_{3n+6}}{\beta_{3n+3}} = \begin{cases} \alpha_0^{2F_{3n+3}}.\beta_1^{2F_{3n+4}}, & n \text{ is even} \\ \beta_0^{2F_{3n+3}}.\alpha_1^{2F_{3n+4}}, & n \text{ is odd} \end{cases}$,

(d) $\dfrac{\beta_{3n+6}}{\alpha_{3n+3}} = \begin{cases} \beta_0^{2F_{3n+3}}.\alpha_1^{2F_{3n+4}}, & n \text{ is even} \\ \alpha_0^{2F_{3n+3}}.\beta_1^{2F_{3n+4}}, & n \text{ is odd} \end{cases}$,

(e) $\dfrac{\alpha_{3n+7}}{\beta_{3n+4}} = \begin{cases} \beta_0^{2F_{3n+4}}.\alpha_1^{2F_{3n+5}}, & n \text{ is even} \\ \alpha_0^{2F_{3n+4}}.\beta_1^{2F_{3n+5}}, & n \text{ is odd} \end{cases}$,

(f) $\dfrac{\beta_{3n+7}}{\alpha_{3n+4}} = \begin{cases} \alpha_0^{2F_{3n+4}}.\beta_1^{2F_{3n+5}}, & n \text{ is even} \\ \beta_0^{2F_{3n+4}}.\alpha_1^{2F_{3n+5}}, & n \text{ is odd} \end{cases}$.

Finally, identities of scheme (1.4) are stated:

Theorem (2.16). If $n \geq 0$ is an integer, then

(a). $\alpha_{n+4} = \alpha_{n+2}\alpha_{n+1}^2\alpha_n$,

(b). $\beta_{n+4} = \beta_{n+2}\beta_{n+1}^2\beta_n$.

Theorem (2.17). If $n \geq 0$ is an integer, then

$\alpha_{n+2}\beta_{n+2} = (\alpha_0\beta_0)^{F_{n+1}}.(\alpha_1\beta_1)^{F_{n+2}}$.

Theorem (2.18). If $n \geq 0$ is an integer, then

(a). $\alpha_n\alpha_{n+1}\alpha_{n+2} = \alpha_0^{2F_{n+1}}.\alpha_1^{2F_{n+2}}$,

(b). $\beta_n\beta_{n+1}\beta_{n+2} = \beta_0^{2F_{n+1}}.\beta_1^{2F_{n+2}}$.

Theorem (2.19). If $n \geq 0$ is an integer, then

(a) $\dfrac{\alpha_{n+3}}{\alpha_n} = \alpha_0^{2F_n}.\alpha_1^{2F_{n+1}}$,

$$(b)\ \frac{\beta_{n+3}}{\beta_n} = \beta_0^{2F_n}.\beta_1^{2F_{n+1}}.$$

Theorem (2.20). If $n \geq 0$ is an integer, then

$$(a)\ \prod_{k=0}^{n} \alpha_k = \alpha_0^{F_{n+1}}.\alpha_1^{F_{n+2}-1},$$

$$(b)\ \prod_{k=0}^{n} \beta_k = \beta_0^{F_{n+1}}.\beta_1^{F_{n+2}-1}.$$

Theorem (2.21). If $n \geq 0$ is an integer, then

$$(a)\ \alpha_{3n+3} = \alpha_1 \prod_{k=0}^{3n+1} \alpha_k, \qquad (b)\ \beta_{3n+3} = \beta_1 \prod_{k=0}^{3n+1} \beta_k,$$

$$(c)\ \alpha_{3n+4} = \alpha_1 \prod_{k=0}^{3n+2} \alpha_k, \qquad (d)\ \beta_{3n+4} = \beta_1 \prod_{k=0}^{3n+2} \beta_k,$$

$$(e)\ \alpha_{3n+5} = \alpha_1 \prod_{k=0}^{3n+3} \alpha_k, \qquad (f)\ \beta_{3n+5} = \beta_1 \prod_{k=0}^{3n+3} \beta_k.$$

Theorem (2.22). If $n \geq 0$ is an integer, then

$$(a)\ \frac{\alpha_{3n+5}}{\alpha_{3n+2}} = \alpha_0^{2F_{3n+2}}.\alpha_1^{2F_{3n+3}},$$

$$(b)\ \frac{\beta_{3n+5}}{\beta_{3n+2}} = \beta_0^{2F_{3n+2}}.\beta_1^{2F_{3n+3}},$$

$$(c)\ \frac{\alpha_{3n+6}}{\alpha_{3n+6}} = \alpha_0^{2F_{3n+3}}.\alpha_1^{2F_{3n+4}},$$

$$(d)\ \frac{\beta_{3n+6}}{\beta_{3n+3}} = \beta_0^{2F_{3n+3}}.\beta_1^{2F_{3n+4}},$$

$$(e)\ \frac{\alpha_{3n+7}}{\alpha_{3n+4}} = \alpha_0^{2F_{3n+4}}.\alpha_1^{2F_{3n+5}},$$

$$(f)\ \frac{\beta_{3n+7}}{\beta_{3n+4}} = \beta_0^{2F_{3n+4}}.\beta_1^{2F_{3n+5}}.$$

The proof of all above identities under schemes 1.2 to 1.4 can be given by induction method.

3. CONCLUSION

This paper describes identities of multiplicative coupled Fibonacci sequences of second order under various schemes. Many similar identities can be developed for higher order multiplicative coupled Fibonacci sequences.

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