On Strong Form of Irresolute Functions

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ABSTRACT

A strong form of Λ_a-irresolute function called completely Λ_a-irresolute function is introduced and several characterizations of such functions are investigated. The relationships among completely Λ_a-irresolute functions, separation axioms and covering properties are also investigated.

Keywords

Λ_a-closed sets, Λ_a-open sets, completely Λ_a-irresolute functions, Λ_a-compact spaces, Λ_a-connected spaces and Λ_a-normal spaces.

1. INTRODUCTION

In 1972, Crossley and Hildebrand [2] introduced the notion of irresoluteness. Various types of irresolute functions have been introduced over the course of years. Recently Thivagar et al.[5],introduced a new class of sets called Λ_a-sets via a-closed sets and investigated several properties of such sets. The purpose of this paper is to introduce a new form of irresolute function called completely Λ_a-irresolute function which is stronger than Λ_a-irresolute functions. We also investigate the relationships among completely Λ_a-irresolute functions, separation axioms and covering properties.

2. PRELIMINARIES

Throughout the paper (X,τ) and (Y,σ) and (Z,η) (or simply X, Y and Z) represent topological spaces on which no separation axioms are assumed. For a subset A of X , cl(A), int(A) and A^c denote the closure of A, interior of A and the complement of A respectively. A subset A of a topological space X is called δ-closed if A = cl_δ(A) where cl_δ(A) = $\{x\in X$: int (cl(U)) $\cap A \neq \phi$, $U\in\tau$ and $x\in U\}$.The complement of δ- closed set is called δ-open set. A subset A of a topological space X is called regular open if A = int (cl(A)). The complement of regular open set is called regular closed set. A subset A of a topological space X is called an a-open set [3] if A $\subset$ int (cl (int$_\delta$ (A))).The complement of an a-open set is called an a-closed set. A subset A of a topological space X is called a δ-semiopen [7] if A $\subset$ cl (int$_\delta$ (A)).The complement of a δ-semiopen set is called a δ-semiclosed set.

Definition 2.1. A subset A of a topological space (X,τ) is said to be a Λ_a-set [5] if Λ_a(A) = A where Λ_a(A) = $\cap\{O \in aO(X,\tau) : A \subset O\}$.

Definition 2.2. A subset A of a topological space (X,τ) is said to be Λ_a-closed [5] if A=T$\cap$ C where T is a Λ_a-set and C is an a-closed set. A is said to be Λ_a-open if X - A is Λ_a- closed.

Definition 2.3. A function f: $(X,\tau)\rightarrow(Y,\sigma)$ is called

(i) strongly continuous [4] if f^{-1}(V) is clopen in X for every subset V in Y.

(ii) completely continuous [8] if f^{-1}(V) is regular open in X for every open set V in Y.

(iii) almost a-continuous [3] if f^{-1}(V) is a-open in X for every regular open set V in Y.

(iv) Λ_a-continuous [5] if f^{-1}(V) is Λ_a-open in X for every open set V in Y.

(v) Λ_a-irresolute [5] if f^{-1}(V) is Λ_a-open in X for every Λ_a-open set V in Y.

(vi) quasi Λ_a-irresolute [5] if f^{-1}(V) is Λ_a-open in X for every a-open set V in Y.

(vii) completely α-irresolute [10] if f^{-1}(V) is regular open in X for every α-open set V in Y.

(viii) completely δ-semi-irresolute [8] if f^{-1}(V) is regular open in X for every δ-semiopen set V in Y.

(ix)R-map [8] if f^{-1}(V) is regular open in X for every regular open set V in Y.

(x) a-irresolute [3] if f^{-1}(V) is a-open in X for every a-open set V in Y.

(xii) a*-closed [3] if f (V) is a-closed in X for every a-closed set V in Y.

3. COMPLETELY Λ_a-IRRESOLUTE FUNCTIONS

In this section we introduce completely Λ_a-irresolute functions and obtain several properties concerning such functions.

Definition 3.1. A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be completely Λ_a-irresolute function if the inverse image of every Λ_a-open subset of Y is regular open in X.

Example3.2. Let X= {a,b,c,d}=Y, τ ={ ϕ ,{a},{b},{a,b}, {c,d},{a,c,d},{b,c,d},X}and σ ={ ϕ ,{a},{c},{a,b},{a,c}, {a,b,c},{a,c,d},Y}.Define a function f : (X, τ) $\rightarrow$ (Y, σ) by f(a) = d, f(b) = c ,f(c) = a and f(d) = b. Then f is completely Λ_a-irresolute.

Theorem 3.3 The following are equivalent for a function f : (X, τ) $\rightarrow$ (Y, σ)

(i) f is completely Λ_a-irresolute.

(ii) the inverse image of every Λ_a-closed subset of Y is regular closed in X.

Proof: (i) $\Rightarrow$ (ii) Suppose f is completely Λ_a-irresolute. Let V be a Λ_a-closed subset of Y .Then Y-V is Λ_a-open in Y. By (i), f^{-1}(Y-V) = X- f^{-1}(V) is regular open in X which implies f^{-1}(V) is regular closed in X. Thus (ii) holds.

Similarly (ii) $\Rightarrow$ (i) holds.

Remark 3.4. It is clear that every strongly continuous function is completely Λ_a-irresolute. However the converse is not true as shown by the following example.

Example 3.5. Let X and τ be same as in example 3.2.Then f is completely Λ_a-irresolute but not strongly continuous since f^{-1}{b}={d} is not clopen in X.

Theorem 3.6. Every completely Λ_a-irresolute function is

(i) Λ_a-irresolute.

(ii) a-irresolute.

(iii) quasi- Λ_a-irresolute.

(iv) a R-map.

(v) almost a-continuous.

Proof :

(i) Let f : (X, τ) $\rightarrow$ (Y, σ) be a completely Λ_a-irresolute function and V be Λ_a-open in Y. Since f is completely Λ_a-irresolute, f^{-1}(V) is regular open in X. Since every regular open set is a-open [7], f^{-1}(V) is a-open in X. By proposition 4.20[5], f^{-1}(V) is Λ_a-open in X which implies f is Λ_a-irresolute.

(ii) Let f : (X, τ) $\rightarrow$ (Y, σ) be a completely Λ_a-irresolute function and V be an a-open in Y. By proposition 4.20[5], V is Λ_a-open in Y. Since f is completely Λ_a-irresolute, f^{-1}(V) is regular open in X. Since every regular open set is a-open [7], f^{-1}(V) is a-open in X which implies f is a - irresolute.

(iii) Let f : (X, τ) $\rightarrow$ (Y, σ) be a completely Λ_a-irresolute function and V be an a-open in Y. By proposition 4.20[5], V is Λ_a-open in Y. Since f is completely Λ_a-irresolute, f^{-1}(V) is regular open in X. Since every regular open set is a-open [7], f^{-1}(V) is a-open in X. By proposition 4.20[5], f^{-1}(V) is Λ_a-open in X which implies f is quasi Λ_a-irresolute.

(iv) Let f : (X, τ) $\rightarrow$ (Y, σ) be a completely Λ_a-irresolute function and V be a regular open set in Y. Since every regular open set is a-open [7], V is a-open in Y. By proposition 4.20[5], V is Λ_a-open in Y. Since f is completely Λ_a-irresolute, f^{-1}(V) is regular open in X which implies f is a R-map.

(v) Let f : (X, τ) $\rightarrow$ (Y, σ) be a completely Λ_a-irresolute function and V be a regular open set in Y. Since every regular open set is a-open [7], V is a-open in Y. By proposition 4.20[5], V is Λ_a-open in Y. Since f is completely Λ_a-irresolute, f^{-1}(V) is regular open in X which implies f^{-1}(V) is a-open in X and hence f is almost a-continuous.

Remark 3.7. The converses of the above theorem are not true as shown by the following examples.

Example 3.8.Let X = {a,b,c,d}=Y, τ ={ ϕ ,{a},{c},{a,b}, {a,c},{a,b,c},{a,b,d},X} and σ ={ ϕ ,{a},{b,c},{a,b,c},Y}. Define a function f : (X, τ) $\rightarrow$ (Y, σ) by f(a) = b, f(b) = c, f(c) = a and f(d) = d. Then f is a-irresolute and R-map but not completely Λ_a-irresolute since f^{-1}({a,d})={c,d} is not regular open in X where {a,d} is Λ_a-open in Y.

Example 3.9. Le X= {a,b,c,d}=Y, τ ={ ϕ ,{a},{c},{d}, {a,c},{a,d},{c,d},{a,c,d},X} and σ ={ ϕ ,{c},{a,b},{a,b,c}, {a,b,d},Y}.Define a function f : (X, τ) $\rightarrow$ (Y, σ) by f(a) = c, f(b) = d, f(c) = b and f(d) = a. Then f is Λ_a-irresolute and almost a- continuous but not completely Λ_a-irresolute since f^{-1}({a,b,d})={b,c,d} is not regular open in X where {a,b,d} is Λ_a-open in Y.

Example3.10 Let X={a,b,c,d}=Y, τ ={ ϕ ,{a},{b},{a,b}, X} and σ ={ ϕ ,{a},{b,c},{a,b,c},Y}.Define a function f : (X, τ) $\rightarrow$ (Y, σ) by f(a) = c, f(b) = d, f(c) = a and f(d) = b. Then f is quasi- Λ_a -irresolute but not completely Λ_a -irresolute since f^{-1}({a,d})={b,c} is not regular open in X where {a,d} is Λ_a -open in Y.

Definition 3.11 A space (X, τ) is said to be Λ_a -space [5] if every Λ_a -closed subset of X is a-closed in X.

Theorem 3.12 Let f : (X, τ) $\rightarrow$ (Y, σ) be a completely δ -semi-irresolute function where Y is a Λ_a -space ,then f is completely Λ_a -irresolute.

Proof : Let V be a Λ_a -closed subset of Y. Since Y is a Λ_a -space, V is a-closed in Y. Since every a-closed set is δ -semiclosed [7], V is δ -semiclosed in Y. Now f is completely δ -semi-irresolute implies f^{-1}(V) is regular closed in X and so f is completely Λ_a -irresolute.

Theorem 3.13 Let f : (X, τ) $\rightarrow$ (Y, σ) be a completely α -irresolute function where Y is a Λ_a -space ,then f is completely Λ_a -irresolute.

Proof : Let V be a Λ_a -closed subset of Y. Since Y is a Λ_a -space, V is a-closed in Y. Since every a-closed set is α -closed [7], V is α -closed in Y. Now f is completely α -irresolute implies f^{-1}(V) is regular closed in X and so f is completely Λ_a -irresolute.

Theorem 3.14 Let f : (X, τ) $\rightarrow$ (Y, σ) and g : (Y, σ) $\rightarrow$ (Z, η) be functions. Then the following properties hold:

(i) If f is completely Λ_a -irresolute and g is Λ_a -continuous, then g $\circ$ f is completely continuous.

(ii) If f is completely Λ_a -irresolute and g is Λ_a -irresolute, then g $\circ$ f is completely Λ_a -irresolute.

(iii) If f is almost a-continuous and g is completely Λ_a -irresolute, then g $\circ$ f is Λ_a -irresolute.

(iv) If f is completely continuous and g is completely Λ_a -irresolute, then g $\circ$ f is completely Λ_a -irresolute.

(v) If f is a R-map and g is completely Λ_a -irresolute , then g $\circ$ f is completely Λ_a -irresolute.

(vi) If f is completely Λ_a -irresolute and g is a R-map, then g $\circ$ f is almost a-continuous.

(vii) If f is almost a-continuous and g is completely Λ_a -irresolute, then g $\circ$ f is a-irresolute.

Proof. (i) Let V be an open set in Z. Since g is Λ_a -continuous, g^{-1} (V) is Λ_a -open in Y. Since f is completely Λ_a -irresolute, f^{-1}(g^{-1}(V))= (g $\circ$ f) $^{-1}$(V) is regular open in X and hence g $\circ$ f is completely continuous.

Proofs of (ii) – (vii) can be obtained similarly.

Theorem 3.15 If f : (X, τ) $\rightarrow$ (Y, σ) is a surjective, a*-closed function and g : (Y, σ) $\rightarrow$ (Z, η) is a function such that g $\circ$ f : (X, τ) $\rightarrow$ (Z, η) is completely Λ_a -irresolute, then g is Λ_a -irresolute.

Proof. Let V be a Λ_a -closed set in Z. Since g $\circ$ f is completely Λ_a -irresolute, (g $\circ$ f) $^{-1}$(V) = f^{-1}(g^{-1}(V)) is regular closed in X. Since every regular closed set is a-closed [7], f^{-1}(g^{-1}(V)) is a-closed in X. Now f is a*-closed and surjective implies f(f^{-1}(g^{-1}(V)))= g^{-1}(V) is Λ_a -closed in Y. Thus g is Λ_a -irresolute.

Remark 3.16 From the above results we have the following diagram where A $\rightarrow$ B represents A implies B but not conversely.

1.completely Λ_a -irresolute 2.almost a-continuous 3.a-irresolute 4.quasi Λ_a -irresolute 5. Λ_a -irresolute 6.strongly continuous

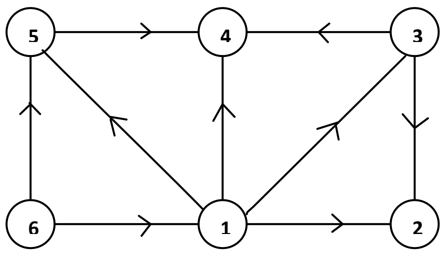

FIGURE : 1

4. CHARACTERIZATIONS

Lemma 4.1.[9] Let S be an open subset of a topological space (X, τ).Then the following hold:

(i) If U is regular open in X, then so is U $\cap$ S in the subspace (S, τ_S).

(ii) If B $\subset$ S is regular open in (S, τ_S) there exists a regular open set U in (X, τ) such that B = U $\cap$ S.

Theorem 4.2. If f : (X, τ) $\rightarrow$ (Y, σ) is completely Λ_a -irresolute and A is any open subset in X, then the restriction $f|_A$: A $\rightarrow$ Y is completely Λ_a -irresolute.

Proof. Let V be any Λ_a-open subset of Y .Since f is completely Λ_a-irresolute, $f^{-1}(V)$ is regular open in X. Since A is open in X, by lemma 4.1, $(f|_A)^{-1}(V) = A \cap f^{-1}(V)$ is regular open in A and so $f|_A$ is completely Λ_a-irresolute.

Lemma 4.3.[1] Let Y be a preopen subset of a topological space (X, τ).Then $Y \cap U$ is regular open in Y for every regular open subset U of X.

Theorem 4.4. If $f : (X,\tau) \to (Y,\sigma)$ is completely Λ_a-irresolute and A is any preopen subset of X, then the restriction $f|_A : A \to Y$ is completely Λ_a-irresolute.

Proof. Let V be any Λ_a-open subset of Y .Since f is completely Λ_a-irresolute, $f^{-1}(V)$ is regular open in X. Since A is preopen in X, by lemma 4.3, $(f|_A)^{-1}(V) = A \cap f^{-1}(V)$ is regular open in A and so $f|_A$ is completely Λ_a-irresolute.

Theorem 4.5. A topological space (X, τ) is connected if every completely Λ_a-irresolute function from a space X into any T_0-space Y is constant

Proof. Suppose X is not connected and every completely Λ_a-irresolute function from a space X into Y is constant. Since X is not connected, there exists a proper nonempty clopen subset A of X. Let Y={a,b} and $\tau = \{\phi,\{a\},\{b\},Y\}$ be a topology for Y. Let $f : X \to Y$ be a function such that f(A)={a} and f(X - A) ={b}. Then f is a non-constant completely Λ_a-irresolute function such that Y is T_0, a contradiction. Hence X must be connected.

Definition 4.6. A topological space (X, τ) is said to be

(i) Λ_a-connected [5] if X cannot be written as a disjoint union of two nonempty Λ_a-open subsets in X.

(ii) r-connected [10] if X cannot be written as a disjoint union of two nonempty regular open subsets in X.

(iii) hyperconnected [8] if every open subset of X is dense.

Theorem 4.7. If $f : (X,\tau) \to (Y,\sigma)$ is completely Λ_a-irresolute surjection and X is r-connected, then Y is Λ_a-connected.

Proof. Suppose Y is not Λ_a-connected. Then $Y = A \cup B$ where A and B are disjoint nonempty Λ_a-open subsets in Y. Since f is completely Λ_a-irresolute surjection, $f^{-1}(A)$ and $f^{-1}(B)$ are regular open sets in X such that $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$ which shows that X is not r-connected, a contradiction. Hence Y is Λ_a-connected.

Theorem 4.8. Completely Λ_a-connected images of hyperconnected spaces are Λ_a-connected.

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be a completely Λ_a-irresolute function such that X is hyperconnected. Assume that B is a proper Λ_a-clopen subset of Y. Then $A = f^{-1}(B)$ is both regular open and regular closed set in X as f is completely Λ_a-irresolute. This clearly contradicts the fact that X is hyperconnected. Thus Y is Λ_a-connected.

Definition 4.9. A topological space (X, τ) is said to be

(i) Λ_a-T_1 [6] if for every pair of distinct points x and y ,there exist Λ_a-open sets G and H containing x and y respectively such that $y \notin U$ and $x \notin V$.

(ii) Λ_a-T_2 [6] if for every pair of distinct points x and y ,there exist disjoint Λ_a-open sets G and H containing x and y respectively .

(iii) r -T_1 [10] if for every pair of distinct points x and y ,there exist r-open sets G and H containing x and y respectively such that $x \notin H$ and $y \notin G$.

Theorem 4.10. If $f : (X,\tau) \to (Y,\sigma)$ is completely Λ_a-irresolute injective function and Y is Λ_a-T_1, then X is r -T_1.

Proof. Since Y is Λ_a-T_1, for $x \neq y$ in X, there exist Λ_a-open sets V and W such that $f(x) \in f(y) \in W$, $f(y) \notin V$, $f(x) \notin W$. Since f is completely Λ_a-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are regular open sets in X such that $x \in f^{-1}(V)$, $y \in f^{-1}(W)$, $x \notin f^{-1}(W)$, $y \notin f^{-1}(V)$. This shows that X is r -T_1.

Theorem 4.11. If $f : (X,\tau) \to (Y,\sigma)$ is completely Λ_a-irresolute injective function and Y is Λ_a-T_2.then X is T_2.

Proof. Similar to the proof of theorem 4.10

Definition 4.12. A topological space (X, τ) is said to be

(i) Λ_a-compact [5], if every Λ_a-open cover of X has a finite subcover.

(ii) nearly compact[11], if every regular open cover of X has a finite subcover.

Theorem 4.13. If $f : (X,\tau) \to (Y,\sigma)$ is completely Λ_a-irresolute surjective function and X is nearly compact, then Y is Λ_a-compact.

Proof. Let $\{V_\alpha : \alpha \in I\}$ be a cover of Y by Λ_a-open subsets of X. Since f is completely Λ_a-irresolute,

{ $f^{-1}(V_\alpha) : \alpha \in I\}$ is a regular open cover of X. Since X is nearly compact, there exists a finite subset I_0 of I such that X = $\cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ .Since f is surjective, Y = $\cup\{V_\alpha : \alpha \in I_0\}$ and hence Y is Λ_a -compact.

Definition 4.14. A topological space (X, τ) is said to be Λ_a -normal [5], if each pair of disjoint closed sets can be separated by disjoint Λ_a -open sets.

Theorem 4.15. If f : (X, τ) $\to$ (Y, σ) is completely Λ_a -irresolute, closed injection and Y is Λ_a -normal ,then X is normal.

Proof. Let E and F be disjoint closed subsets of X. Since f is closed, f(E) and f(F) are disjoint closed subsets of Y. Since f is Λ_a -normal, there exist disjoint Λ_a -open sets U and V such that f(E) $\subset$ U and f(F) $\subset$ V. Since f is completely Λ_a -irresolute, f^{-1}(U) and f^{-1}(V) are disjoint regular open subsets in X and hence open subsets in X such that E $\subset$ f^{-1}(U), F $\subset$ f^{-1}(V) which shows that X is normal.

Theorem 4.16. Let f, g be functions. If f and g are completely Λ_a -irresolute functions and Y is a Λ_a -T_2 space, then P = { x $\in$ X : f(x) = g(x)} is δ -closed.

Proof. Let x $\notin$ P. We have f(x) $\neq$ g(x). Since Y is Λ_a -T_2, there exist disjoint Λ_a -open sets A and B in Y such that f(x) $\in$ A and g(x) $\in$ B. Since f and g are completely Λ_a -irresolute, f^{-1}(A) and f^{-1}(B) are disjoint regular open subsets in X .Put U = f^{-1}(A) $\cap$ f^{-1}(B).Then U is a regular open subset of X containing x and U $\cap$ P = ϕ and hence x $\notin$ $cl_\delta(A)$ .Hence P is δ -closed in X.

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