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Fixed Points of Mappings in Fuzzy Normed Spaces

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ABSTRACT

Chugh and Rathi [3] introduced the concept of Fuzzy normed space. In this paper, a common fixed point theorem for a pair of operators in fuzzy normed spaces is established.

Keywords

Fixed point, Fuzzy normed spaces.

1. INTRODUCTION

The concept of Random normed space was introduced by Serstnev [12] as a generalization of ordinary normed space. A random normed space is Menger space if we set $G_{x,y} = F_{x-y}$. Fixed point theorems for contraction mappings in RN-spaces were first investigated by Boscan [2]. Thus many fixed point theorems for metric space have an immediate analogue in Random normed spaces. For topological preliminaries in RN - spaces, Schweizer and Sklar [11] and Serstnev [12] are excellent readings.

The notion of probabilistic metric space described a situation between two points, when distance is unknown. Though the probabilities of the possible value of this distance in known. Also the probabilities theory is a study about uncertainty and randomness. Further the study of mathematics explore the restricted zone fuzziness, which also a kind of uncertainty. Zadeh brings the concept of fuzzy metric spaces as an extension of probabilistic metric spaces in 1965. The concept of fuzzy metric spaces has been introduced in different ways by Erceg [4], Kaleva *et al.* [7] and Kramosil et al. [8]. In addition to this Grabic followed Kramosil and Michalek [8] obtained the fuzzy version of Banach contraction principle. George and Verramani [5] introduced the improved concept of fuzzy metric spaces. George et al. [5] established a relation between fuzzy metric spaces and metric spaces as $M(x, y, t) = t/[t+d(x,y)]$. Every metric space can be made to fuzzy metric space by the above relation. Chugh and Rathi [3] introduced the concept of Fuzzy normed space and fuzzy normed space is a fuzzy metric space if $G(x, y,t) = M(x-y, t)$. In this paper, a common fixed point theorem for a pair of operators in fuzzy normed spaces is established.

2 PRELIMINRIES

2.1 Definition

A fuzzy set A in X is a function with domain X and values in [0, 1].

2.2 Definition

A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if * satisfies the following conditions :

(i) * is associative and commutative,

(ii) * is continuous,

(iii) $a * 1 = a$ for all $a \in [0, 1]$,

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, where $a, b, c, d \in [0, 1]$.

2.3 Example

$a*b = ab$, $a*b = \min\{a, b\}$.

2.4 Definition

A triplet (X, M, *) is called a fuzzy normed space (briefly FN - space) if X is a real vector space, * is a continuous t-norm and M is a fuzzy set on $X \times [0, \infty)$ satisfying the following conditions

(FN – 1) $M(x, 0) = 0$,

(FN – 2) $M(x, t) = 1$ for all $t > 0$ if and only if $x = 0$,

(FN–3) $M(\alpha x, t) = M(x, \frac{t}{|\alpha|})$ for all $\alpha \in R$, $\alpha \ne 0$,

(FN–4) $M(x + y, t + s) \ge M(x, t) * M(y, s)$ for all $x, y \in X$ and $t, s \in R+$

(FN–5) $M(x, .) : [0, \infty) \to [0, 1]$ is left continuous for all $x \in X$

(FN–6) $\lim_{t\to\infty} M(x, t) = 1$ for all x in X and $t \in R$.

2.5 Notation

M(x, t) can be thought of as the degree of nearness of norm of x with respect to t.

2.6 Definition

The natural topology t(M) is said to be topological if for each x in X and any $\in > 0$

$U_x(\in) = \{y : M(x-y, \in) > 1-\in\}$ is a neighborhood of x in t(M).

2.7 Definition

A sequence $\{x_n\}$ in a fuzzy normed space is said to be convergent if for each r , $0 < r < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$$M(x_n-x, t) > 1-r \quad \text{for all } n \ge n_0 .$$

2.8 Definition

A sequence $\{x_n\}$ in a fuzzy normed space is said to be a Cauchy if for each r, $0 < r < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$$M(x_n- x_m, t) > 1-r \quad \text{for all } n, m \ge n_0 .$$

2.9 Definition

A fuzzy normed space is said to be complete if every Cauchy sequence is convergent.

2.10 Example

Let X = R. Define $a * b = ab$ and

$M(x, t) = \left[\exp\left(\frac{|x|}{t}\right)\right]^{-1}$ for all $x \in X$ and $t \in [0, \infty)$.

Then (X, M, *) is a fuzzy normed space.

Proof. (i) M(x, 0) = 0

(ii) M(x, t) = 1 implies $\left[\exp\left(\frac{|x|}{t}\right)\right]^{-1} = 1$

or $\exp\left(\frac{|x|}{t}\right) = 1$

or $\exp\left(\frac{|x|}{t}\right) = \exp(0)$

or $\left(\frac{|x|}{t}\right) = 0$

$\Rightarrow$ $|x| = 0 \Rightarrow x = 0$

If x = 0 , then M(x, t) = 1 .

(iii) $M(\alpha x, t) = M(x, \frac{t}{|\alpha|})$ is obvious

(iv) To prove
M(x+y, t+s) ≥ M(x, t) * M(y, s)

Since $\frac{|x+y|}{t+s} \le \frac{|x|+|y|}{t+s}$

$= \frac{|x|}{t+s} + \frac{|y|}{t+s}$

$\le \frac{|x|}{t} + \frac{|y|}{s}$

$\exp\left(\frac{|x+y|}{t+s}\right) \le \exp\left(\frac{|x|}{t} + \frac{|y|}{s}\right)$

$= \exp\left(\frac{|x|}{t}\right) . \exp\left(\frac{|y|}{s}\right)$

Taking inverse, we have

$$\left[\exp\left(\frac{|x+y|}{t+s}\right)\right]^{-1} \ge \left[\exp\left(\frac{|x|}{t}\right)\right]^{-1} \left[\exp\left(\frac{|y|}{s}\right)\right]^{-1}$$

Hence M(x+y, t+s) ≥ M(x, t) * M(y, s)

(v) $M(x, .) : [0, \infty) \to [0, 1]$ is left continuous

(vi) $\lim_{t\to\infty} M(x, t) = 1$

Hence (X, M, *) is a fuzzy normed space.

2.11 Example

Let M be a fuzzy set on $X \times [0, \infty)$ defined by

$M(x, t) = \frac{t}{t+|x|}$ for all $x \in X$, $t > 0$ and * is a t-norm defined by a*b = ab. Then (X, M, *) is a fuzzy normed space.

3. MAIN RESULT

We need the following lemma to prove the theorem.

3.1 Lemma

Let $\{y_n\}$ be a sequence in a fuzzy normed space (X, M, *), where * is continuous and satisfies $*(x, x) \ge x$ for every $x \in \{0, 1\}$. If there exists a constant $k \in (0, 1)$ such that

(3.1) $M(y_n - y_{n+1}, kx) \ge M(y_{n-1} - y_n, x)$ for all n,

then $\{y_n\}$ is a Cauchy sequence in X.

Proof. Let $\in$, λ be positive reals. Then for $m \ge n$, we have by (FN–4).

$M(y_n - y_m, \in) \ge M(y_n - y_{n+1}, \in - k\in) * M(y_{n+1} - y_m, k\in)$

$M(y_n - y_m, \in) \ge M(y_n - y_{n+1}, \in - k\in) * M(y_{n+1} - y_m, k\in)$

$\ge M(y_0 - y_1, (\in - k\in)\ k^{-n}) * M(y_{n+1} - y_m, k\in)$ (by 2.1)

Taking $(\in - k\in)\ k^{-n} = h$, we get

$M(y_n - y_m, \in) \ge M(y_0 - y_1, h) * (M(y_{n+1} - y_{n+2}, k\in - k^2\in) *$
$M(y_{n+2} - y_m, k^2\in))$

$\ge M(y_0 - y_1, h) * (M(y_0 - y_1, h) * M(y_{n+2} - y_m, k^2 \in)$

Repeating these arguments

$M(y_n - y_m, \in) \ge M(y_0 - y_1, h) * M(y_{m-1} - y_m, k^{m-n-1} \in)$

$\ge M(y_0 - y_1, h) * M(y_0 - y_1, k^{-n} \in)$

$\ge M(y_0 - y_1, h) * M(y_0 - y_1, h)$

$\ge M(y_0 - y_1, (\in - k\in)\ k^{-n})$

Therefore, if N be so chosen that

$M(y_0 - y_1, (\in - k\in)\ k^{-N}) > 1-r$

It follows that

$M(y_n - y_m, \in) \ge 1-r$ for all $m > n \ge N$.

Hence $\{y_n\}$ is a Cauchy sequence.

3.2 Theorem

Let (X, M, *) be a complete fuzzy normed space with t norm * satisfying $*(x, x) \ge x$ for every $x \in [0, 1]$ and f, g be two mappings from X to itself such that for $k \in (0, 1)$

$\{M(fu - gv, kx)\}^2 \ge M(u - fu, x)\ M(v - gv, x) * M(u - gv, 2x)$
$M(v - fu, x) * M(u - fu, x)\ M(u - gv, 2x) *$
$M(v - fu, x)\ M(v - gv, 2x).$ (3.2.1)

holds for all $u, v \in X$ and $x \ge 0$. Then f and g have a common fixed point.

Proof Let $u_0 \in X$. Construct the sequence $\{y_n\}$ by taking

$u_{2n+1} = fu_{2n}$, $u_{2n+2} = gu_{2n+1}$ (3.2.2)

We shall prove

$M(u_{2n+1} - u_{2n+2}, kx) \ge M(u_{2n} - u_{2n+1}, x)$ for which we suppose

$M(u_{2n+1} - u_{2n+2}, kx) < M(u_{2n} - u_{2n+1}, x)$ (3.2.3)

Now by (3.2.1) and (3.2.2),

$\{M(u_{2n+1} - u_{2n+2}, kx)\}^2 = \{M(fu_{2n} - gu_{2n+1}, kx)\}^2$

$\ge M(u_{2n} - u_{2n+1}, x)\ M(u_{2n+1} - u_{2n+2}, x) * M(u_{2n} - u_{2n+2}, 2x)$

$M(u_{2n+1} - u_{2n+1}, x) * M(u_{2n}-u_{2n+1}, x)\ M(u_{2n}-u_{2n+2}, 2x) *$

$M(u_{2n+1} - u_{2n+1}, x)\ M(u_{2n+1} - u_{2n+2}, 2x)$

$\geq M(u_{2n}-u_{2n+1}, x)\ M(u_{2n+1} - u_{2n+2}, x) * M(u_{2n}-u_{2n+1}, x) *$

$M(u_{2n+1} - u_{2n+2}, x) * M(u_{2n}-u_{2n+1}, x)^2 * M(u_{2n}-u_{2n+1}, x)$

$M(u_{2n+1} - u_{2n+1} - u_{2n+2}, x) * M(u_{2n+1} - u_{2n+2}, 2x)$

$> (M(u_{2n+1} - u_{2n+2}, kx))^2 * M(u_{2n+1} - u_{2n+2}, kx)*$

$M(u_{2n+1}-u_{2n+2}, kx)* (M(u_{2n+1}- u_{2n+2}, kx))^2*$

$(M(u_{2n+1} - u_{2n+2}, kx))^2 * M(u_{2n+1} - u_{2n+2}, kx)$

$\left\{\begin{array}{l}\text{Using (3.2.3) and observing that}\\ M(u_{2n+1} - u_{2n+2}, kx) < M(u_{2n+1} - u_{2n+2}, x)\end{array}\right.$

$> (M(u_{2n+1} - u_{2n+2}, kx))^2$

which is a contradiction so

$$M(u_{2n+1} - u_{2n+2}, kx) \geq M(u_{2n} - u_{2n+1}, x)$$

Similarly, $M(u_{2n+2} - u_{2n+3}, kx) \geq M(u_{2n+1} - u_{2n+2}, x)$.

In general, $M(u_{n+1} - u_{n+2}, kx) \geq M(u_n - u_{n+1}, kx)$

So by the above Lemma $\{u_n\}$ is a Cauchy sequence and converges to a point z in X.

We now prove that gz = z. Let U_{gz} ($\in$, r) be an ($\in$, r) neighborhood of gz. Since $u_n \to z$, for $\in$, r > 0 there is an integer N such that

$$M(u_{2n} - u_{2n+1}, \in/k) > 1-r \qquad (3.2.4)$$

and $M(z - u_{2n+1}, \in/k) > 1-r$, for all $n \geq N(\in, r)$.

Now by (3.2.1) and (3.2.2)

$\{M(u_{2n+1} - gz, \in)\}^2 = \{M(fu_{2n} - gz, \in)\}^2$

$\geq M(u_{2n} - u_{2n+1}, \in/k)\ M(z-gz, \in/k) * M(u_{2n} - gz, 2\in/k)$

$M(z- u_{2n+1}, \in/k)* M(u_{2n}-u_{2n+1}, \in/k)\ M(u_{2n}-gz, 2\in/k) *$

$M(z-u_{2n+1}, \in/k)\ M(z-gz, 2\in/k)\}$

$> M(u_{2n} - u_{2n+1}, \in/k)\ M(z-u_{2n+1},(1+k) \in/2k)*$

$M(u_{2n}-u_{2n+1}, \in/k)\ M(u_{2n+1} - gz, (1+k)\in/2k) *$

$M(u_{2n} - u_{2n+1}, \in/k)\ M(z-u_{2n+1}, \in/k) * M(u_{2n+1} - gz, \in/k)$

$M(z-u_{2n+1}, \in/k)* (M(u_{2n}-u_{2n+1}, \in/k))^2* M(u_{2n}-u_{2n+1}, \in/k)$

$M(u_{2n+1} - gz, \in/k) * (M(z-u_{2n+1}, \in/kx))^2* M(z-u_{2n+1}, \in/k)$

$M(u_{2n+1}-gz, \in/k)\}$

$> (M(u_{2n} - u_{2n+1}, \in/k))^2* (M(z-u_{2n+1}, \in/k))^2$

$> (1-\lambda)^2$, by (3.2.4)

This implies that $M(u_{2n+1} - gz, \in) > 1-\lambda$

Since $u_{2n+1} \to z$, $gz \to z$. Similarly fz = z. To prove the uniqueness of z as a common fixed point of f and g, let y be another common fixed point. Then by (3.2.1), for some x > 0 , we have

$\{M(z-y, kx)\}^2 \geq M(z-z, x)\ M(y-y, x) *M(z-y, 2x)\ M(y-z, x)$

$M(z-z, x)M(z-y,2x)*M(y-z, x)\ M(y-y, 2x)$

$= M(z-y, x) \geq \ldots. \geq M(z-y, x/k^n) \to 1 \text{ as } n\to\infty$

This proves y = z .

This completes the proof of the theorem.

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