Embedding of C_n^2 and C_{n-1}^2+K_1 into Arbitrary Tree

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ABSTRACT

We present an approach to find the edge congestion sum and dilation sum forembedding of square of cycle on n vertices, C_n^2, and $C_n{}^2{}_{-1}$ + K_1 into arbitrary tree. The embedding algorithms use a technique based on consecutive label property. Our algorithm calculates edge congestion in linear time.

General Terms

Graph embedding

Keywords

Embedding, dilation, congestion, cycles, wheel.

1. INTRODUCTION

Definition: A **graph**$G(V,E)$ is a pair where $V$the set of all vertices in *G* and *E* is is the set of all edges in *G*. We call a graph *G(V,E)* is finite if *V* and *E* both are finite.

Definition: Let *G(V,E)* and $H(V',E')$ be two finite graphs. A 1-1 mapping $f: V \rightarrow V'$ is called an **embedding**. Graph H is called a **host graph** and graph G is called **guest or virtual graph.**

The Dilation Problem

Definition: Let *G(V,E)* and $H(V',E')$ be two finite graphs. Let f be an embedding of G into H. Then the dilation of G into H with respect to f ,denoted by $D_f(G,H)$, is defined as

$$D_f(G,H) = \max\{d_H(f(u), f(v)) | (u,v)\epsilon E\},$$

where$d_H(f(u), f(v))$ denotes the length of the shortest path between$f(u)$ and $f(v)$ in H.

Definition: The dilation of G in to H is denoted by $D(G,H)$, is defined as $D(G,H) = min\, D_f(G,H)$where the minimum is taken over all embedding f of *G* in to *H*.

Definition: The dilation problem is to find an embedding of G onto H that gives minimum dilation.

The Dilation Sum Problem

Definition: The dilation sum of an embedding f of G into H is denoted by $D'_f(G,H)$ and is defined as

$D'_f(G,H)$=$\Sigma_{(u,v)\epsilon E} d_H((f(u), f(v))$

Definition: The dilation sum of *G* into *H*, denoted by *D'(G,H),* is defined as *D'(G,H)*=min $D'_f(G,H)$

where the minimum is taken over all embeddings$f$ of *G* into *H*.

The Congestion Sum Problem

Definition: The **congestion** of an embedding is the maximum number of edges of the guest graph that are embedded to any single edge of the host graph. For an embedding f, of G in to H, let there is a unique path, for every edge (u,v)in $E(G)$, in H from $f(u)$ to $f(v)$. Let $P_f(f(u), f(v))$denotes this path and$C_f(G, H(e))$ denotes congestion on the edge e in $E(H)$.Then

$C_f(G, H(e)) = |\{(u,v)\epsilon E(G) | e\epsilon P_f(f(u), f(v))\}|$**Definition:** Let f be an arbitrary embedding of G in to H. Then the **congestion sum** of $f$is defined as $C_f(G,H) = \Sigma_{e\epsilon E(H)} C_f(G, H(e))$

The minimum congestion sum of G in to $H$is defined as

$C(G,H)$=min. $C_f(G,H)$

where the minimum is taken over all embedding f of G into H.

Definition: The **congestion sum problem** is to find an embedding of G on to H that gives minimum congestion sum.

We shall denote $C_f(G, H(e))$by $C_f(e)$.

REMARK: The congestion sum problem and dilation sum problem are same.

2. OVERVIEW OF THE ARTICLE

The dilation-sum of a graph embedding arises from VLSI designs, data structures and data representations, net-works for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architectures, structural engineering, and so on [20]. The dilation-sum problem of an arbitrary graph on a path is called the linear layout or the linear arrangement problem in the VLSI literature [20]. The concept of embedding is widely studied in the literature of fixed interconnection parallel architectures [21].The dilation problem is *NP*-complete for two classes of'almost' catepillars on a path [18] and trees of maximum degree 3 on paths [20, 27]. From the above *NP*-complete results, the dilation-sum problem is expected to be harder than the dilation problem [27]. That is why, even though there are numerous results and discussions on the dilationsum problem and the congestion-sum problem, most of them deal with only approximation results.

The dilation-sum problem has been studied for binary trees into paths [8, 12], hypercubes into grids [5], complete graphs into hypercubes [19]. The bounded cost of dilation and congestion has been estimated for the embedding on binary trees [27]. Most of the work on the dilation-sum problem and the dilation problem are for the particular case in which the host graph is a path, or a cycle [20].The concept of cutwidth is a special case of congestion when the host graph is a path [11, 26, 29]. There are several results on the congestion problem for various architectures such as trees into cycles [11], trees into stars [28], trees into hypercubes [4, 22], hypercubes into grids [5, 6, 25], complete binary trees into grids [23], and ladders and caterpillars into hypercubes [7, 10]. There are also other general results on embeddings [2]. There are algorithms for the embedding of Cycles and wheel into arbitrary tree [30] and k sequential m –ary into hypercube[31]. We apply Lemma 1 for estimation, and use the consecutive label property for characterization. We use the characterization results to construct optimal embeddings of these graphs into trees to solve the congestion-sum problem in polynomial time. We use the estimation results to show the proof of correctness of the algorithms. Since the well knowninorder, preorder, postorder traversals satisfy the consecutive label property, we use them to construct linear time embeddings. In this article we produce an embedding based on consecutive label property which gives us minimum congestion sum.

3. THE FUNDAMENTAL LEMMA

Definition: Let $G(V,E)$ is a finite graph with n vertices. Then we say that G is a **complete graph** if there is a direct edge between every pair of vertices in G. The complete graph on n vertices is denoted by $\boldsymbol{K_n}$. A graph $G(V,E)$ on n vertices {1,2,3,……,n} is called a **cycle,** denoted by$\boldsymbol{C_n}$, if $(i, i+1)$ is an edge for all $i = 1,2, \ , n-1$, and $(n,1)$ is also an edge.

Definition: Let $\boldsymbol{K_n}$ and $\boldsymbol{C_n}$ be complete graph and cycle on n vertices respectively. Let $V(C_{n-1}) = \{1,2,\ldots\ldots,n-1\}$and $V(K_1) = \{n\}$. Then a **wheel** on n vertices $W_n = C_{n-1} + K_1$ is a graph obtained by $C_{n-1} \cup K_1$ by joining each vertex of C_{n-1} to each vertex of K_1 with an edge.

We assume that a wheel has at least 5 vertices.

Definition: A tree is a connected acyclic graph. If a node of the tree is labeled as the root then it is called a rooted tree.

Definition: Let T be an ordered rooted tree with vertex labels$1,2,\ldots n$. A sub tree T_1 of T is **consecutively labeled** if the labels of sub tree T_1 are consecutive numbers.

Definition: Let T be an ordered rooted tree with vertex labels $1,2,\ldots n$. A labeling of T satisfies the **consecutive label property** if for every vertex $v$of T, the sub trees of T rooted at $v$are consecutively labeled.

Definition: Let $G(V,E)$ be a graph, then for each $v \in V$,define the **eccentricity** of v, $\mu(v)$ as

$\mu(v) = max\{\, d(u,v) \mid u \epsilon V\}$.

A vertex with minimum eccentricity is called a **central vertex.**

The eccentricity of G is denoted by $\pi(G) = min.\{\mu(v) \mid v \epsilon V\}$.

Definition: For each $v \in V$, we define $\delta(v) = \sum_{u \epsilon V} d(u,v)$.

A vertex v for which $\delta(v)$ is minimum is called **median** of G. The median eccentricity of G is denoted by $\delta(G)$, is defined as

$$\delta(G) = min\{\, \delta(v) | v \,\epsilon V\}.$$

Definition: A graph $H(V,E')$ is called square of a graph $G(V,E)$ if E' contains all edges (u,v) where v,u are in V such that distance between u and v is less than or equal to 2 in G. Similarly $H(V,E')$ is called the r th power of $G(V,E)$if E' contains all edges (u,v) , where u,v are in V and distance between u and v is less than or equal to r in G.

Lemma 1: Let f be an embedding of a graph G in to arbitrary tree T. Let e be an edge of T and T_1 is component of $T-e$. Then the congestion on the edge e, $C_f(e)$ is given by

$$C_f(e) = \sum_{v \epsilon G_1} d_G(v) - 2|E(G_1)|$$

whereG_1 is a sub graph of G induced by vertices $\{f^{-1}(v) \mid v \,\epsilon\, T_1\}$ and $d_G(v)$ denotes the degree of v in G.

Proof: Let $A = \{(u,v) \epsilon\, E(G) | f(u) \epsilon\, T_1 , f(v) \epsilon\, T_2\}$ where T_1 and T_2 are components of $T - e$. By definition of $C_f(e)$

$C_f(e) = |A|$……………(1)

Also by definition of A,

$|A| = \sum_{v \epsilon G_1} d_G(v) - 2|E(G_1)|$……………….(2)

By (1) and (2), we have

$$C_f(e) = \sum_{v \epsilon G_1} d_G(v) - 2|E(G_1)|$$

4. EMBEDDING OF $\boldsymbol{C_n^2}$ AND $\boldsymbol{C_{n-1}^2 + K_1}$ IN TO ARBITRARY TREE

Now we denote $\boldsymbol{C_{n-1}^2 + K_1}$ by W_n^2 and the square of $C_n$by C_n^2. Here K_1consists of a vertex n only.

Lemma2 : Let f be an arbitrary embedding of W_n^2 into an arbitrary tree T such that $f(n) = s$. Let e be an edge of T and let $T_1$and T_2 be two components of $T - e$ such that s is in T_2. Then the minimum congestion of edge e ,$C_f(e)$ is given by

$C_f(e) \geq \begin{cases} |V(T_1)|+4 \text{ if e is a leaf edge} \\ |V(T_1)|+6 \text{ otherwise} \end{cases}$

Proof:$\sum_{v \,\epsilon G_1} d_{W_n^2}(v) = 5|V(T_1)|$ and

$$|E(G_1)| \leq 2(|V(T_1)| - 1) \; if \; |V(T_1)| = 1,$$

$$|E(G_1)| \leq 2|V(T_1)| - 3 \; if \; |V(T_1)| > 1.$$

whereG_1 is sub graph of W_n^2 induced by $\{v | f(v) \in T_1\}$.

Now using lemma 1,$C_f(e) = \sum_{v \epsilon G_1} d_G(v) - 2|E(G_1)|$,we have

$$C_f(e) \geq |V(T_1)| + 4 \; if \; e \; is \; a \; leaf \; edge$$

$$C_f(e) \geq |V(T_1)| + 6 \; otherwise.$$

This completes the proof.

Let us denotes the number of leaf nodes in the tree $T$by l_n.

Lemma 3: Let f be an arbitrary embedding of W_n^2 into an arbitrary tree T such that $f(n) = s$. Let $e = (a,b)$ be some edge of T and let $T_1$and T_2 be two components of $T - e$ such that s is in T_2 and T_1 has $m - 1$ nodes. Also say $T_1' = T_1 \; U \; \{e\}$ is rooted at a. Then

$\sum_{e \in T_1'} C_f(e) \geq \sum_{v \,\epsilon T_1} d(a,v) + 4l_{n_1} + 6(m - l_{n_1} - 1)$where$l_{n_1}$ is the number of leaf nodes in T_1.

Proof: Let T_1^vdenotes the sub tree of T_1 rooted at v then it is true that $\sum_{v \,\epsilon T_1} d(a,v) = \sum_{v \,\epsilon T_1} |V(T_1^v)|$.Now using lemma 2,we have

$\sum_{e \in T_1'} C_f(e) \geq \sum_{v \,\epsilon T_1} d(a,v) + 4l_{n_1} + 6(m - l_{n_1} - 1)$.

Lemma 4: Let f be an arbitrary embedding of W_n^2 into an arbitrary tree T such that $f(n) = s$ .Then

$C_f(W_n^2, T) \geq \sum_{v \,\epsilon T} d(s,v) + 4\, l_n + 6(n - l_n - 1)$.

Proof: Let the degree of s in T be k and $(s,u_1),(s,u_2),\dots\dots,(s,u_k)$ be edges incident to s. Let $T_1,T_2,\dots\dots,T_k$ be components of $T-s$. $T_1,T_2,\dots\dots,T_k$are sub tree rooted at $u_1,u_2,\dots\dots,u_k$ respectively. Let $T_i' = T_i \cup \{(s,u_i)\}$for $i=1,2,\dots,k$,and each T_i has m_i-1 nodes and l_{n_i} leaf nodes. $T_i'$is rooted at *s*.

$\sum_{e\,\epsilon\,T} C_f(e) =$
$\sum_{e\in T_1'} C_f(e) + \sum_{e\in T_2'} C_f(e) + \dots\dots + \sum_{e\in T_k'} C_f(e)$
$\geq(\sum_{v\,\epsilon T_1} d(a,v) + 4l_{n_1} + 6(m_1 - l_{n_1} - 1))$+($\sum_{v\,\epsilon T_2} d(a,v) + 4l_{n_2} + 6(m_2 - l_{n_2} - 1)$)+…………….+($\sum_{v\,\epsilon T_k} d(a,v) + 4l_{n_k} + 6(m_k - l_{n_k} - 1)$)=$\sum_{v\,\epsilon\,T} d(s,v) + 4\,l_n + 6(n - l_n - 1)$.

Theorem 1: The congestion sum of W_n^2 into an arbitrary tree T is at least $\delta(T) + 4l_n + 6(n - l_n - 1)$.

Proof: Let f be an arbitrary embedding of W_n^2into an arbitrary tree T such that $f(n) = s$. Then by lemma 4

$C_f(W_n^2,T) \geq \sum_{v\,\epsilon\,T} d(s,v) + 4\,l_n + 6(n - l_n - 1), \sum_{v\,\epsilon\,T} d(s,v)$is minimum when s is median of T and its value is *δ(T)*.Thus

$$C(W_n^2,T) = min\,C_f(W_n^2,T) \geq \sum_{v\,\epsilon\,T} d(s,v) + 4\,l_n + 6(n - l_n - 1), \sum_{v\,\epsilon\,T} d(s,v) \geq \delta(T) + 4\,l_n + 6(n - l_n - 1).$$

The minimum is taken over all embeddings$f$ of W_n^2 into *T*.

Theorem2: The congestion sum of W_n^2 into an arbitrary tree T is$\delta(T) + 4l_n + 6(n - l_n - 1)$.

Proof: Let f be an embedding of W_n^2 on to arbitrary tree *T*. Let f satisfies consecutive label property and $f(n) = s$, where s is the median of T . Let e be an edge of T and let $T_1$and T_2 be two components of $T - e$ such that s is in T_2.Then by using lemma 1,

$$C_f(e) = \sum_{v\epsilon G_1} d_G(v) - 2|E(G_1)|$$

whereG_1 is a sub graph of G induced by vertices $\{f^{-1}(v) \mid v\,\epsilon\,T\}$.

Taking G= W_n^2 and G_1 is sub graph of W_n^2 induced by $\{f^{-1}(v) \mid v\,\epsilon\,T_1\}$. Then consecutive label property implies that the vertices of $T_1$are consecutive numbers. If e contain a leaf node and s is not in T_1 then T_1 contains only single vertex hence number of edges in G_1 is zero i.e. $2(|V(T_1)| - 1)$.If e does not contain a leaf node and s is not in T_1 then number of edges in $G_1$is $2|V(T_1)| - 3$.

Using these in lemma 2 we get

$$C_f(e) = \begin{cases} |V(T_1)|+4 & \text{if e is a leaf edge} \\ |V(T_1)|+6 & \text{otherwise} \end{cases}$$

Using these in lemma 3 and lemma 4 we have

$\sum_{e\,\epsilon\,T} C_f(e)$=$\sum_{v\,\epsilon\,T} d(s,v) + 4\,l_n + 6(n - l_n - 1)$.

Since $f(n) = s$, is the medianof T, then $\sum_{v\,\epsilon\,T} d(s,v)$ $=\delta(T)$.

$\sum C_f(W_n^2,T) = \sum_{v\,\epsilon\,T} d(s,v) + 4\,l_n + 6(n - l_n - 1)$= $\delta(T) + 4\,l_n + 6(n - l_n - 1)$.

Now using theorem 1, we have

$C(W_n^2,T)$= $\delta(T) + 4\,l_n + 6(n - l_n - 1)$.

Corollary1: The congestion sum of C_n^2 on an arbitrary tree is$l_n + 6(n - l_n - 1)$.

Proof: there is no contribution of the term $\delta(T)$ when the graph is of C_n^2 .So using theorem 2 we have

$C(C_n^2,T) = 4\,l_n + 6(n - l_n - 1)$.

5. INORDER EMBEDDING ALGORITHM

Now, we give an algorithm named inorder algorithm, to find an embedding that gives minimum congestion sum.

Input: The input of the algorithm is a graph G and an ordered rooted tree T.Gcan be C_n^2 or $W_n^2$where n is the centre of the graph when it is W_n^2.

Algorithm: First find the median of given tree. The centre is mapped to the median of the tree if the guest graph is W_n^2. The remaining vertices $1,2,\dots\dots,n-1$ of *G* are mapped to the vertices of tree in inorder traversal.

Output: The output is an embedding with minimum congestion sum.

As the inorder of a tree holds the consecutive labeling property, so this mapping gives us minimum congestion sum, using theorem 2.

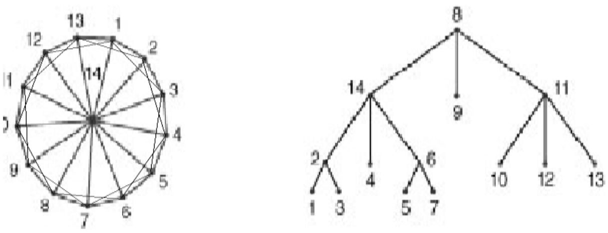

Fig.1Optimal embedding of W_n^2 into arbitrary tree.

6. FUTURE SCOPE

We have solved the congestion-sum problem for square of cycles into arbitrary trees. The question that arisesnow is how far we can carry out similar extensions to larger powers r.The embeddings we have constructed in this article aresimple and elegant. They produce the optimal congestionsumin linear time. In this article, the host graph is anarbitrary tree. The domain of host graphs may be extendedto a few architectures such as *X* trees and pyramids.

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