Time Dependent Solution of M $^{[X]}$/G/1 Queuing Model with Bernoulli Vacation and Balking

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ABSTRACT
In this paper we consider a queueing model, wherein the customers are arriving as batches following compound Poisson process. With one of the customer behavior, Balking such that the batch upon arrival may refuses to enter in to the system due to some reasons. Also after completing a service the server may opt for a vacation with probability p, or remain stay back in the system to serve the next customer if any, with probability 1-p. In this model, the customer behavior balking is considered in both the busy time and server vacation time of the system. For this mode. We obtain the time dependent solution and the corresponding steady state solutions. Also, we derive the performance measures, the mean queue size and the average waiting time explicitly.

AMS Subject Classification : 60K25,60K30

Keywords
Batch Arrival, Single server, Balking, Bernoulli vacation, Transient state solution, Steady state Analysis.

1. INTRODUCTION
The research study on queueing theory has been increased tremendously during the last few decades. Most Recently many researchers have been concentrating on the study of queueing models with vacations. In this area the contributions of eminent authors, like Baba (1986), Lee et al. (1995), Doshi (1986) and Takagi (1990) are worthy to be addressed. Keilson and Servi (1987) have studied about the Dynamics of the M/G/1 vacation model. Choudhury and Borthakur (1997) discussed vacation queues with batch arrivals.

The concept of customer impatience has been studied in 1950's. Haight (1957) has first studied about the concept of customer behavior called balking, which deals the reluctance of a customer to join a queue upon arrival, since then a remarkable attention has been given on many queueing models with customer impatience. Madan (2012) analyzed the steady state batch arrival queueing system with balking and Re-service in a Vacation queue, having two types of heterogeneous services. Kumar (2012) have studied Markovian queueing model with balking and reneging.

The study of transient behavior of queueing models have been increased recently. Takagi (1990) studied the time-dependent analysis of an M/G/1 vacation models with exhaustive service. Madan (1992) has obtained time dependent solution of M/G/1 model with compulsory vacation. Krishna kumar et al. (2002) have obtained time dependent solution of many of queueing models.

In our model we have obtained the transient solution by assum ing that the batch arrival units may decide not to join the system (balks) by estimating the duration of waiting time for a service to get completed or by witnessing the long length of the queue and considering the Bernoulli vacation such that, the server may opt for vacation with probability p or with probability 1-p, remain stay back in the system to provide the service for the next customer.

The rest of the paper has been organized as follows: in section 2, the mathematical description of our model has been found, in section 3, the transient solution of the system has been derived, in section 4, the steady state analysis has been discussed in section 5, the conclusion has been given.

2. MATHEMATICAL DESCRIPTION OF THE QUEUEING MODEL
To describe the required queueing model, we assume the following.

◦ Let $\lambda c_i \Delta t$; i = 1, 2, 3, ... be the first order probability of arrival of 'i' customers in batches in the system during a short period of time(t, t+dt) where $0 \le ci \le 1$; $\sum_{i=1}^{n} c_i = 1$ $\lambda > 0$ is the mean arrival rate of batches.

◦ There is a single server which provides service following a general (arbitrary) distribution with distribution function. B(v) and density function b(v). Let $\mu(x)dx$ be the conditional probability density function of service completion during the interval (x, x+dx] given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1-b(x)} \tag{1}$$

and therefore

$$b(v) = \mu(v)e^{-\int_0^{\infty} \mu(x)dx} \tag{2}$$

° As soon as, a service is completed, the server may take a vacation of random length with probability p or he may stay in the system providing service with probability (1-p).

° The vacation time of the server follows general (arbitrary) distribution with distribution function V(s) and the density function v(s). Let $\nu(x)dx$ be the conditional probability of a completion of a vacation during the interval (x, x+dx] given that the elapsed vacation time is x so that

$$\nu(x) = \frac{v(x)}{1 - V(x)} \tag{3}$$

$$v(s) = \nu(s) e^{-\int_0^{vs} \nu(x)dx} \tag{4}$$

° We assume that $(1 - a_1)$ $(0 \le a_1 \le 1)$ is the probability is the probability that an arriving customer balks during the period

° when the server is busy and $(1 - a_2)$ $(0 \le a_2 \le 1)$ is the probability that an arriving customer balks during the period when the server is on vacation.

° The customers are served according to the first come -first served queue discipline.

2.1 Definitions And Equations Governing The System

We let,

(i) P_n (x, t) = probability that at time 't' the server is active providing service and there are 'n' $n \ge 1$ customers in the queue including the one being served and the elapsed service time for this customer is x. Consequently P_n (t) denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in service irrespective of the value of x.

(ii) V_n (x, t) = probability that at time 't', the server is on vacation with elapsed vacation time x, and there are 'n' $n \ge 1$ customers waiting in the queue for service. Consequently V_n(t) denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

(iii) Q(t) = probability that at time 't' there are no customers in the system and the server is idle but available in the .

$$\frac{\partial}{\partial t} P_n(x,t) + \frac{\partial}{\partial x} P_n(x,t) + (\lambda + \mu(x)) P_n(x,t) =$$

$$\lambda(1-a_1) P_n(x,t) + a_1 \lambda \sum_{i=1}^{n} c_i P_{n-i}(x,t); n \ge 1 \text{\_\_\_\_\_\_} (5)$$

$$\frac{\partial}{\partial t} V_n(x,t) + \frac{\partial}{\partial x} V_n(x,t) + (\lambda + \nu(x)) V_n(x,t) =$$

$$\lambda(1-a_2) V_n(x,t) + a_2 \lambda \sum_{i=1}^{n} c_i V_{n-i}(x,t); n \ge 1 \text{\_\_\_\_\_\_} (6)$$

$$\frac{\partial}{\partial t} V_0(x,t) + \frac{\partial}{\partial x} V_0(x,t) + (\lambda + \nu(x)) V_0(x,t) =$$

$$\lambda(1-a_2) V_0(x,t) \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (7)$$

$$\frac{d}{dt} Q(t) = -\lambda Q(t) + \lambda(1-a_1) Q(t) +$$

$$(1-p)\int_0^\infty P_1(x,t)\mu(x)dx + \int_0^\infty V_0(x,t)\nu(x)dx \text{\_\_\_\_\_} (8)$$

The above equations are to be solved subject to the boundary condition given below at x=0

$$P_n(0,t) = (1-p)\int_0^\infty P_{n+1}(x,t)\mu(x)dx +$$

$$\int_0^\infty V_n(x,t)\nu(x)dx + \lambda a_1 c_n Q(t); n \ge 1 \text{\_\_\_\_\_\_\_\_} (9)$$

$$V_n(0,t) = p\int_0^\infty P_{n+1}(x,t)\mu(x)dx \text{\_\_\_\_\_\_\_\_\_\_\_\_} (10)$$

Assuming there are no customers in the system initially so that the server is idle.

V0 (0) = 0; V_n (0) = 0; Q(0) = 1; P_n (0) = 0, n = 1, 2, 3, ...(11)

3. TIME DEPENDENT SOLUTION

Generating functions of the queue length

Define Laplace transform

$$\bar{f}(s) = \int_0^\infty f(t) e^{-st} dt \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (12)$$

Taking Laplace transforms of equations (5) to (10)

$$\frac{\partial}{\partial x} \bar{P}_n(x,s) + (s + \lambda a_1 + \mu(x)) \bar{P}_n(x,s) =$$

$$a_1 \lambda \sum_{i=1}^{n} c_i \bar{P}_{n-i}(x,s); n \ge 1 \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (13)$$

$$\frac{\partial}{\partial x} \bar{V}_n(x,s) + (s + \lambda a_2 + \nu(x)) \bar{V}_n(x,s) =$$

$$a_2 \lambda \sum_{i=1}^{n} c_i \bar{V}_{n-i}(x,s); n \ge 1 \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (14)$$

$$\frac{\partial}{\partial x} \bar{V}_0(x,s) + (s + \lambda a_2 + \nu(x)) \bar{V}_0(x,s) = 0 \tag{15}$$

$$(s + \lambda a_1) \bar{Q}(s) = 1 + (1-p)\int_0^\infty \bar{P}_1(x,s)\mu(x)dx +$$

$$\int_0^\infty \bar{V}_0(x,s)\nu(x)dx \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (16)$$

$$\bar{P}_n(0,s) = (1-p)\int_0^\infty \bar{P}_{n+1}(x,s)\mu(x)dx +$$

$$\int_0^\infty \bar{V}_n(x,s)\nu(x)dx + \lambda a_1 c_n \bar{Q}(s) \text{\_\_\_\_\_\_\_\_\_} (17)$$

$$\bar{V}_n(0,s) = p\int_0^\infty \bar{P}_{n+1}(x,s)\mu(x)dx \text{\_\_\_\_\_\_\_} (18)$$

We define the probability generating functions

$$P_q(x,z,t)=\sum_{n=0}^{\infty}P_n(x,t)z^n$$
$$P_q(z,t)=\sum_{n=0}^{\infty}P_n(t)z^n$$
$$V_q(x,z,t)=\sum_{n=0}^{\infty}V_n(x,t)z^n$$
$$V_q(z,t)=\sum_{n=0}^{\infty}V_n(t)z^n$$
$$C(z)=\sum_{n=1}^{\infty}c_n z^n \quad (19)$$

Which are convergent inside the circle given by $|z|\le 1$.

Now multiplying equation (13) by z^n and take summation over all possible ' n'

$$\frac{\partial}{\partial x}\bar{P}_q(x,z,s)+(s+\lambda a_1(1-C(z))+\mu(x))\bar{P}_q(x,z,s)=0 \quad (20)$$

multiplying equation (14) by z^n and adding (15), we have

$$\frac{\partial}{\partial x}\bar{V}_q(x,z,s)+(s+\lambda a_2(1-C(z))+\nu(x))\bar{V}_q(x,z,s)=0 \quad (21)$$

for the boundary equations , multiplying (17) and (18) by appropriate powers of z and taking summation over all possible values of' n' respectively

$$z\bar{P}_q(0,z,s)=(1-p)\int_0^{\infty}\bar{P}_q(x,z,s)\mu(x)dx+z\int_0^{\infty}\bar{V}_q(x,z,s)\nu(x)dx+\lambda a_1 z(C(z)-1)\bar{Q}(s)+(1-s\bar{Q}(s))z \quad (22)$$

$$z\bar{V}_q(0,z,s)=p\int_0^{\infty}\bar{P}q(x,z,s)\mu(x)dx \quad (23)$$

integrate (20) from 0 to x yields

$$\bar{P}_q(x,z,s)=\bar{P}_q(0,z,s)e^{-(s+a_1\lambda(1-C(z)))x-\int_0^x\mu(t)dt} \quad (24)$$

where $\bar{P}_q(x,z,s)$ is given by equation (22).

Again integrate equation (24) by parts with respect to x yields

$$\bar{P}_q(z,s)=\bar{P}_q(0,z,s)\left[\frac{1-\bar{B}(s+\lambda a_1(1-C(z)))}{(s+\lambda a_1(1-C(z))}\right] \quad (25)$$

Where

$$\bar{B}(s+\lambda a_1(1-C(z)))=\int_0^{\infty}e^{-(s+\lambda a_1(1-C(z)))x}dB(x) \quad (26)$$

is Laplace - Stieltjes transform of the service time B(x).

Now multiplying both sides of equation (24) by μ(x) and integrating over x, we get

$$\int_0^{\infty}\bar{P}_q(x,z,s)\mu(x)dx=\bar{P}_q(0,z,s)\bar{B}(s+\lambda a_1(1-C(z))) \quad (27)$$

Similarly integrating equation (21) from 0 to x yields

$$\bar{V}_q(x,z,s)=\bar{V}_q(0,z,s)e^{-(s+a_2\lambda(1-C(z)))x-\int_0^x\nu(t)dt} \quad (28)$$

$\bar{V}_q(x,z,s)$ is given by equation (23),

where

Again integrating equation (28) by parts with respect to x yields

$$\bar{V}_q(z,s)=\bar{V}_q(0,z,s)\left[\frac{1-\bar{V}(s+\lambda a_2(1-C(z)))}{(s+\lambda a_2(1-C(z))}\right] \quad (29)$$

Where

$$\bar{V}(s+\lambda a_2(1-C(z)))=\int_0^{\infty}e^{-(s+\lambda a_2(1-C(z)))x}dV(x) \quad (30)$$

is Laplace - Stieltjes transform of the service time V(x).

Now multiplying both sides of equation (24) by μ(x) and integrating over x, we get

$$\int_0^{\infty}\bar{V}_q(x,z,s)\nu(x)dx=\bar{V}_q(0,z,s)\bar{V}(s+\lambda a_2(1-C(z))) \quad (31)$$

using (27) and (31) in (22)

$$z\bar{P}_q(0,z,s)=(1-p)\bar{P}_q(0,z,s)\bar{B}(s+\lambda a_1(1-C(z)))+z\bar{V}_q(0,z,s)\bar{V}(s+\lambda a_2(1-C(z)))+\lambda a_1 z(C(z)-1)\bar{Q}(s)+(1-s\bar{Q}(s))z \quad (32)$$

using (23) and (27) in (32)

$$z\bar{P}_q(0,z,s)=(1-p)\bar{P}_q(0,z,s)\bar{B}(s+\lambda a_1(1-C(z)))+p\bar{P}_q(0,z,s)\bar{B}(s+\lambda a_1(1-C(z)))\bar{V}(s+\lambda a_2(1-C(z)))+\lambda a_1 z(C(z)-1)\bar{Q}(s)+(1-s\bar{Q}(s))z \quad (33)$$

$$z\bar{V}_q(0,z,s)=p\bar{P}_q(0,z,s)\bar{B}(s+\lambda a_1(1-C(z))) \quad (34)$$

solving (33) we get

$$\bar{P}_q(0,z,s) = \frac{\lambda a_1 z(C(z)-1)\bar{Q}(s) + (1 - s\bar{Q}(s))z}{Dr} \quad (35)$$

$$Dr = z - [(1-p) +$$

$$p\bar{V}(s + \lambda a_2(1 - C(z)))]\bar{B}(s + \lambda a_1(1 - C(z))) \_\_\_\_\_\_ (36)$$

substituting (35) and (36) in (25) and (29) respectively.

$$\bar{P}_q(z,s) = \frac{[\lambda a_1 z(C(z)-1)\bar{Q}(s) + (1 - s\bar{Q}(s))z]\dfrac{1-\bar{B}(s+\lambda a_1(1-C(z)))}{(s+\lambda a_1(1-C(z))}}{Dr} \_\_(37)$$

$$\bar{V}_q(z,s) = p\frac{[\lambda a_1(1-C(z))\bar{Q}(s) + (s\bar{Q}(s)-1)]\bar{B}(s+\lambda a_1(1-C(z)))\left[\dfrac{1-\bar{V}(s+\lambda a_2(1-C(z)))}{(s+\lambda a_2(1-C(z))}\right]}{Dr} \_\_(38)$$

where Dr is given by in the above.

4. THE STEADY STATE ANALYSIS

In this section derive the steady state probability distribution for our queueing model. To define the steady state probabilities, suppress. The argument 't' wherever it appears in the time dependent analysis.

By using well known Tauberian property

$$Lt_{s\to 0}\, s\,\bar{f}(s) = Lt_{t\to\infty} f(t) \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (39)$$

$$P_q(z) = z\frac{[1-\bar{B}(\lambda a_1(1-C(z)))]Q}{D(z)} \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (40)$$

$$V_q(z) = p\frac{\frac{a_1}{a_2}\bar{B}(\lambda a_1(1-C(z)))Q[1-\bar{V}(\lambda a_2(1-C(z)))]}{D(z)} \_\_\_\_\_\_\_\_ (41)$$

where

$$D(z) = [(1-p) + p\bar{V}(\lambda a_2(1-C(z)))]$$

$$\bar{B}(\lambda a_1(1-C(z))) - z \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (42)$$

Let Wq(z) denotes the probability generating function of queue size irrespective of the state of the system.

Wq(z)= Pq(z) +Vq (z)____________________________(43)

In order to find the unknown Q, using the normalizing condition

Wq (1) + Q=1___________________________________(44)

$$W_q(1) = \frac{\lambda a_1 E(I)Q[E(S) + pE(V)]}{1 - \lambda a_1 E(I)E(S) - p\lambda a_2 E(I)E(V)} \_\_\_\_ (45)$$

$$Q = \frac{1 - \lambda a_1 E(I)E(S) - p\lambda a_2 E(I)E(V)]}{1 + p\lambda(a_1 - a_2)E(I)E(V)} \_\_\_\_ (46)$$

$$\rho = 1 - Q \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (47)$$

where ρ is the stability condition under which the steady state exists, equation (46) gives the probability that the server is idle. Substitute Q from equation (46) in equation (43), Wq(z) have been completely and explicitly determined which is the the probability generating function of the queue size

The Average Queue Size

Let Lq denote the mean number of customers in the queue under the steady state, then

$$Lq = \frac{d}{dz}W_q(z)\Big|_{z=1} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (48)$$

since this formula gives 0/0 form, then we write

$$W_q(z) = \frac{N(z)}{D(z)}$$

where N{(z)} and D{(z)} are the numerator and denominator of the right hand side of equation (43) respectively, then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \_\_\_\_\_\_\_\_\_\_\_\_ (49)$$

where

$$N'(1) = Q\lambda a_1 E(I)(E(S) + pE(V)) \_\_\_\_\_\_\_\_\_\_ (50)$$

$$N''(1) = (\lambda E(I))^2\{a_1^{\,2}E(S^2) + 2pa_1^{\,2}E(S)E(V) + Pa_1a_2E(V^2)\} +$$
$$Q\lambda a_1 E(I(I-1)[E(S) + pE(V) \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (51)$$

$$D'(1) = 1 - \lambda E(I)[a_1E(S)E(V) + pa_2E(V)] \_\_\_\_\_ (52)$$

$$D''(1) = -(\lambda E(I))^2\{a_1^{\,2}E(S^2) + 2pa_1a_2E(S)E(V) +$$
$$Pa_2^{\,2}E(V^2)\} - \lambda E(I(I-1)[a_1E(S) + pa_2E(V)] \_\_\_\_\_\_\_\_\_\_ (53)$$

5. CONCLUSION

The server provides essential service to all arriving customers and the customer behavior is balking. The server vacation is based on Bernoulli schedule. This paper clearly analyzes the transient solution and the steady state solution with various performance measures of the queueing system. As a future work busy period analysis and reliability analysis will be discussed.

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