New Separation Axiom on $\hat{\Omega}$-Closed Sets

M.Lellis Thivagar
School Of Mathematics,
Madurai Kamaraj University
Madurai-625021,
Tamil Nadu, INDIA.

M.Anbuchelvi
Department of Mathematics,
V.V.Vanniaperumal College For Women
Virudhunar-626001,
Tamil Nadu, INDIA.

ABSTRACT

In this paper we introduce and investigate a new separation axiom associated with $\hat{\Omega}$-closed sets and characterize it by using $\hat{\Omega}$-closure operator,$\hat{\Omega}$-kernel $\hat{\Omega}$-derived set and $\hat{\Omega}$-shell of singletons. Also we find some of their applications.

Keywords:

$\hat{\Omega}$-closed sets,$\hat{\Omega}$-closure, $Ker_{\hat{\Omega}}(\{x\})$, $\hat{\Omega}$-derived set and $\hat{\Omega}$-shell.,
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ifx

1. INTRODUCTION

[3] Generalized closed sets play a vital role in General Topology and many separation axioms were introduced and studied by using it.By using various kinds of generalized closed sets many authors introduced and investigated many types of separation axioms.[1]Thivagar.et.al introduced and investigated a new classes of sets known as $\hat{\Omega}$-closed sets which lies properly between the classes of [6]δ-closed sets and that of [5]ω-closed sets.Also it is independent of open sets.Main feature of this classes of sets is "This structure forms a topology".The aim of the paper is to introduce and investigate the new notion namely $\hat{\Omega}$-$T_{\frac{1}{2}}$-space by utilizing $\hat{\Omega}$-open set.Also we characterize $\hat{\Omega}$-$T_{\frac{1}{2}}$ and $\hat{\Omega}$-T_0 spaces in terms of $\hat{\Omega}$-$d(\{x\})$ and $\hat{\Omega}$-$shl(\{x\})$.Also we introduce and investigate some more spaces like $\hat{\Omega}$-C_0, $\hat{\Omega}$-C_1,weakly $\hat{\Omega}$-C_0 and sober $\hat{\Omega}$-R_0.

2. PRELIMINARIES

Throughout this paper (X, τ) (or briefly X) represent a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, cl(A), int(A) and A^c denote the closure of$A$, the interior of$A$ and the complement of A respectively.The family of all $\hat{\Omega}$-open (resp.$\hat{\Omega}$-closed) subsets of X is denoted by $\hat{\Omega}O(X)$. (resp.$\hat{\Omega}C(X)$) and $\hat{\Omega}O(X, x) = \{U \in X : x \in U \in \hat{\Omega}O(X)\}$,$\hat{\Omega}C(X, x) = \{F \in X : x \in F \in \hat{\Omega}C(X)\}$.
Let us recall the following definitions, which are useful in the sequel.

DEFINITION 2.1. *[6] A subset A of X is called δ-closed in a topological space (X, τ) if $A = \delta cl(A)$, where $\delta cl(A) = \{x \in X : int(cl(U)) \cap A \neq \emptyset, U \in O(X, x)\}$. The complement of δ-closed set in (X, τ) is called δ-open set in (X, τ).*

DEFINITION 2.2. *A subset A of a topological space (X, τ) is called*

(i) Semi-open [4]if $A \subseteq cl(int(A))$.

(ii) $\hat{\Omega}$-open [1] if $\delta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X, τ).

The complement of semi open and $\hat{\Omega}$-closed set in (X, τ) is called semi closed and $\hat{\Omega}$-open set in (X, τ).

DEFINITION 2.3. *A topological space (X, τ) is*

(i) $\hat{\Omega} - T_0$ [2] if for any distinct pair of points x and y of X, there exists a $\hat{\Omega}$-open set U of X containing x but not y (or) containing y but not x.

(ii) $\hat{\Omega} - T_1$ [2] if for any distinct pair of points x and y of X, there exists a $\hat{\Omega}$-open set U of X containing x but not y and a $\hat{\Omega}$-open set V of X containing y but not x.

DEFINITION 2.4. *[1] Let A be a subset of a topological space(X, τ). Then the $\hat{\Omega}$-closure of A is defined to be the intersection of all $\hat{\Omega}$-closed sets containing A and it is denoted by $\hat{\Omega}cl(A)$. That is $\hat{\Omega}cl(A) = \bigcap\{F : A \subseteq F, F \in \hat{\Omega}C(X)\}$. Always $A \subseteq \hat{\Omega}cl(A)$.*

DEFINITION 2.5. *[2] Let (X, τ) be a space and $A \subseteq X$. Then the $\hat{\Omega}$-kernel of A, denoted by $Ker_{\hat{\Omega}}(A)$ is defined as $Ker_{\hat{\Omega}}(A) = \bigcap\{G \in \hat{\Omega}O(X) : A \subseteq G\}$.*

DEFINITION 2.6. *[1] A point x of a space (X, τ) is called a $\hat{\Omega}$-limit point of a subset A of (X, τ) if for each $\hat{\Omega}$-open set U containing x intersects A other than x. That is $A \cap (U - \{x\}) \neq \emptyset$. The set of all limit points of A is denoted by $D_{\hat{\Omega}}(A)$ and is called the $\hat{\Omega}$-derived set of A.*

DEFINITION 2.7. *A subset A of a topological space (X, τ) is said to be degenerate if it is either a null set or a singleton set.*

3. $\hat{\Omega}$-$T_{\frac{1}{2}}$-SPACE.

In a topological space X,the closure,the derived set, the kernel and the shell of a singleton set $\{x\}$ are denoted by $cl(\{x\}), d(\{x\}), ker(\{x\})$ and $shl(\{x\})$ respectively.In a similar way,we introduce the notations $\hat{\Omega}cl(\{x\}), \hat{\Omega}d(\{x\}), ker_{\hat{\Omega}}(\{x\})$ and $\hat{\Omega}shl(\{x\})$ for the $\hat{\Omega}$-closure, $\hat{\Omega}$-derived set,$\hat{\Omega}$-kernel and $\hat{\Omega}$-shell of a singleton set $\{x\}$ respectively.

DEFINITION 3.1. *In a topological space (X, τ) ,for any $x \in X$,*
$\hat{\Omega}cl(\{x\}) = \bigcap\{F : F \in \hat{\Omega}C(X, x)\}$.
—$Ker_{\hat{\Omega}}(\{x\}) = \bigcap\{G : G \in \hat{\Omega}O(X, x)\}$.
—$\hat{\Omega}cl(\{x\}) = \{y : x \in Ker_{\hat{\Omega}}(\{y\})\}$.
—$Ker_{\hat{\Omega}}(\{x\}) = \{y : x \in \hat{\Omega}cl(\{y\})\}$.
—$\hat{\Omega}d(\{x\}) = \hat{\Omega}cl(\{x\}) \setminus \{x\} = \{y : x \in Ker_{\hat{\Omega}}(\{y\}), y \neq x\}$.

—$\hat{\Omega}shl(\{x\}) = Ker_{\hat{\Omega}}(\{x\}) \setminus \{x\} = \{y : x \in \hat{\Omega}cl(\{y\}), y \neq x\}$.

LEMMA 3.2. *In a topological space (X, τ) ,the following statements are true.*

(i) $A \subseteq Ker_{\hat{\Omega}}(A)$ for any $A \subseteq X$.
(ii) $Ker_{\hat{\Omega}}(A) = Ker_{\hat{\Omega}}(Ker_{\hat{\Omega}}(A))$ for any $A \subseteq X$.

PROOF. *(i)* Suppose that A is any subset of X.If $x \notin Ker_{\hat{\Omega}}(A)$,then there exists $U \in \hat{\Omega}O(X)$ such that $A \subseteq U$ and $x \notin U$.Therefore,$x \notin A$.Thus,it is proved.
(ii) Suppose that A is any subset of $X$and $x \in Ker_{\hat{\Omega}}(Ker_{\hat{\Omega}}(A))$.Let $U \in \hat{\Omega}O(X)$ such that $A \subseteq U$.By the definition, $Ker_{\hat{\Omega}}(A) \subseteq U$. Again by definition,$x \in Ker_{\hat{\Omega}}(Ker_{\hat{\Omega}}(A))$ implies that $x \in U$. Thus,$Ker_{\hat{\Omega}}(Ker_{\hat{\Omega}}(A)) \subseteq Ker_{\hat{\Omega}}(A)$. By (i),$Ker_{\hat{\Omega}}(A) \subseteq Ker_{\hat{\Omega}}Ker_{\hat{\Omega}}(A)$. Hence it holds.

□

LEMMA 3.3. *In a topological space (X, τ) ,$Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{x\})) = \hat{\Omega}shl(\{x\})$ for any $x \in X$.*

PROOF. By the lemma 3.1 (i),$\hat{\Omega}shl(\{x\}) \subseteq Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{x\}))$.On the other hand,if $y \in Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{x\}))$ such that $y \neq x$.Then by [2] lemma 3.2,$\hat{\Omega}cl(\{y\}) \cap \hat{\Omega}shl(\{x\}) \neq \emptyset$.Therefore, there exists $z \in X$ such that $z \in \hat{\Omega}cl(\{y\}) \cap \hat{\Omega}shl(\{x\})$.By remark 3.1,$z \in \hat{\Omega}shl(\{x\})$ implies that $x \in \hat{\Omega}cl(\{z\})$ such that $z \neq x$.By [1] remark 5.2,$z \in \hat{\Omega}cl(\{y\})$ implies that $\hat{\Omega}cl(\{z\}) \subseteq \hat{\Omega}cl(\{y\})$. Therefore,$x \in \hat{\Omega}cl(\{z\}) \subseteq \hat{\Omega}cl(\{y\})$. Thus,$y \in \hat{\Omega}shl(\{x\})$.Therefore, $Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{x\})) \subseteq \hat{\Omega}shl(\{x\})$ and hence proved. □

DEFINITION 3.4. *A topological space X is said to be $\hat{\Omega}$-$T_{\frac{1}{2}}$-Space if for every pair of distinct points x and y in X, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or $\{x\}$ or $\{y\}$.*

EXAMPLE 3.5. $X = \{a, b, c\}$ $\tau = \{\emptyset, \{a, b\}, X\}$. *Then* $\hat{\Omega}O(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. *It is* $\hat{\Omega}$-$T_{\frac{1}{2}}$.

THEOREM 3.6. *Every $\hat{\Omega}$-T_1-space is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space.*

PROOF. Let x and y be any pair of distinct points in X.Since $X$is $\hat{\Omega}$-T_1-space,[2] theorem 3.19 (iii),$Ker_{\hat{\Omega}}(\{x\}) = \{x\}, Ker_{\hat{\Omega}}(\{y\}) = \{y\}$.Therefore,$Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \emptyset$.Thus,$X$ is $\hat{\Omega}$-$T_{\frac{1}{2}}$. □

REMARK 3.7. *Converse of the above is not always possible from the example 3.2.*

THEOREM 3.8. *Every $\hat{\Omega}$-$T_{\frac{1}{2}}$-space is $\hat{\Omega}$-T_0-space.*

PROOF. If x and y are any pair of distinct points in X.Since $X$is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or $\{x\}$ or $\{y\}$.Therefore,we have the following three cases.

Case(i) If $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{x\}$,then $\{x\} \subseteq Ker_{\hat{\Omega}}(\{y\})$.By lemma 3.2 (ii) and by [2] lemma 3.5,$Ker_{\hat{\Omega}}(\{x\}) \subseteq Ker_{\hat{\Omega}}(\{y\})$.Therefore, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = Ker_{\hat{\Omega}}(\{x\}) = \{x\}$.That is,$y \in Ker_{\hat{\Omega}}(\{y\})$ but $y \notin Ker_{\hat{\Omega}}(\{x\})$. Therefore,$Ker_{\hat{\Omega}}(\{x\}) \neq Ker_{\hat{\Omega}}(\{y\})$.
Case(ii) In a similar way,if $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{y\}$, then $Ker_{\hat{\Omega}}(\{y\}) = \{y\}$and hence $Ker_{\hat{\Omega}}(\{x\}) \neq Ker_{\hat{\Omega}}(\{y\})$.
Case(iii) If $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = emptyset$,then $Ker_{\hat{\Omega}}(\{x\}) \neq Ker_{\hat{\Omega}}(\{y\})$.
In all the three cases by[2] theorem 3.18,X is $\hat{\Omega}$-T_0-space.

□

REMARK 3.9. *Converse of the above is not always possible from the following example.*

EXAMPLE 3.10. $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$.*Then* (X, τ) *is a $\hat{\Omega}$-T_0-space which is not a $\hat{\Omega}$-$T_{\frac{1}{2}}$-space.*

The following theorem states that the necessary condition under which the reversible of the theorem 3.8 holds.

THEOREM 3.11. *If $Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{x\})) \cap Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{y\})) = \emptyset$ for any pair of distinct points $x, y \in X$ in a $\hat{\Omega}$-T_0 space (X, τ) ,then X is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space.*

PROOF. Let x and y be any pair of distinct points in a $\hat{\Omega}$-T_0 space X.By hypothesis, $Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{x\})) \cap Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{y\})) = \emptyset$. By lemma 3.3,$\hat{\Omega}shl(\{x\}) \cap \hat{\Omega}shl(\{y\}) = \emptyset$.Then, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or $\{x\}$ or $\{y\}$ or $\{x, y\}$.If $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{x, y\}$,then $Ker_{\hat{\Omega}}(\{x\}) = Ker_{\hat{\Omega}}(\{y\})$, a contradiction to X is $\hat{\Omega}$-T_0 space.Therefore, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or $\{x\}$ or $\{y\}$ and hence X is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space. □

Let us prove a characterizations of $\hat{\Omega}$-$T_{\frac{1}{2}}$-space.

THEOREM 3.12. *The following statements are equivalent in a topological space.*

Case(i) (X, τ) is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space.
Case(ii) For every pair of distinct points $x, y \in X$, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or one of the points has an empty $\hat{\Omega}$-shell.
(iii) (X, τ) is $\hat{\Omega}$-T_0 and $\hat{\Omega}shl(\{x\}) \cap \hat{\Omega}shl(\{y\}) = \emptyset$ for every pair of distinct points $x, y \in X$.
(iv) (X, τ) is $\hat{\Omega}$-T_0 and $Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{x\})) \cap Ker_{\hat{\Omega}}(\hat{\Omega}shl(\{y\})) = \emptyset$ for every pair of distinct points $x, y \in X$.

PROOF. $(i) \Rightarrow (ii)$ Suppose that x, y are any two distinct points in X.Since X is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or $\{x\}$ or $\{y\}$.If $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{x\}$,then $\{x\} \subseteq Ker_{\hat{\Omega}}(\{y\})$.By lemma 3.2 (ii) and by [2] lemma 3.5,$Ker_{\hat{\Omega}}(\{x\}) \subseteq Ker_{\hat{\Omega}}(\{y\})$. Therefore, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = Ker_{\hat{\Omega}}(\{x\}) = \{x\}$.Thus,$\hat{\Omega}shl(\{x\}) = \emptyset$.
$(ii) \Rightarrow (i)$ If $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \emptyset$ for every pair of distinct points $x, y \in X$,then there is nothing to prove.If not,by hypothesis,we assume that $\hat{\Omega}shl(\{x\}) = \emptyset, \hat{\Omega}shl(\{y\}) \neq \emptyset$, and $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) \neq \emptyset$.Then, $Ker_{\hat{\Omega}}(\{x\}) = \{x\}$ and $Ker_{\hat{\Omega}}(\{y\})$ contains points more than y.Therefore,$x \in Ker_{\hat{\Omega}}(\{y\})$ and hence $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{x\}$.Similarly if $\hat{\Omega}shl(\{y\}) = \emptyset, \hat{\Omega}shl(\{x\}) \neq \emptyset$, and $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) \neq \emptyset$.Then, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{y\}$.Therefore, for every pair of distinct points $x, y \in X$,$Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or $\{x\}$ or $\{y\}$ and hence X is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space.

$(i) \Rightarrow (iii)$ By the theorem 3.8,X is $\hat{\Omega} - T_0$.It is enough to claim that $\hat{\Omega}shl(\{x\}) \cap \hat{\Omega}shl(\{y\}) = \emptyset$ for every pair of distinct points $x, y \in X$.Suppose that x, y are any two distinct points in X.Since X is $\hat{\Omega}$-$T_{\frac{1}{2}}$-space,$Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\})$ is either $\emptyset$ or $\{x\}$ or $\{y\}$.
case(i).If $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \emptyset$,then $[Ker_{\hat{\Omega}}(\{x\}) \setminus \{x\}] \cap [Ker_{\hat{\Omega}}(\{y\}) \setminus \{y\}] = \emptyset$ and hence

$\hat{\Omega}shl(\{x\}) \cap \hat{\Omega}shl(\{y\}) = \emptyset$.
case(ii).If $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{x\}$,then $\{x\} \subseteq Ker_{\hat{\Omega}}(\{y\})$.By lemma 3.2 (ii) and by [2] lemma 3.5,$Ker_{\hat{\Omega}}(\{x\}) \subseteq Ker_{\hat{\Omega}}(\{y\})$.Therefore, $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = Ker_{\hat{\Omega}}(\{x\}) = \{x\}$.Therefore, $\hat{\Omega}shl(\{x\}) = \emptyset$ and hence $\hat{\Omega}shl(\{x\}) \cap \hat{\Omega}shl(\{y\}) = \emptyset$.
case(iii).In a similar way,if $Ker_{\hat{\Omega}}(\{x\}) \cap Ker_{\hat{\Omega}}(\{y\}) = \{y\}$,then $\hat{\Omega}shl(\{x\}) \cap \hat{\Omega}shl(\{y\}) = \emptyset$.
$(iii) \Rightarrow (iv)$ By lemma 3.3,it follows.

$(iv) \Rightarrow (i)$ By the theorem 3.11,it holds. □

DEFINITION 3.13. *A subset A of a topological space (X, τ) is said to be weakly $\hat{\Omega}$ separated from a set B of X if there exists a $\hat{\Omega}$-open set U such that $A \subseteq U$ and $U \cap B = \emptyset$ or $A \cap \hat{\Omega}cl(B) = \emptyset$.*

Characterize $\hat{\Omega}cl(\{x\}), Ker_{\hat{\Omega}}(\{x\}), \hat{\Omega}d(\{x\})$ and $\hat{\Omega}shl(\{x\})$ in terms of weakly $\hat{\Omega}$ separated sets.

LEMMA 3.14. *In a topological space (X, τ) for any $x, y \in X$,we have*

(i) $\hat{\Omega}cl(\{x\}) = \{y : \{y\}$ *is not weakly* $\hat{\Omega}$ *separated from* $\{x\}\}$.

(ii) $Ker_{\hat{\Omega}}(\{x\}) = \{y : \{x\}$ *is not weakly* $\hat{\Omega}$ *separated from* $\{y\}\}$.

(iii) $\hat{\Omega}d(\{x\}) = \{y : y \neq x, \{y\}$ *is not weakly* $\hat{\Omega}$ *separated from* $\{x\}\}$.

(iv) $\hat{\Omega}shl(\{x\}) = \{y : y \neq x, \{x\}$ *is not weakly* $\hat{\Omega}$ *separated from* $\{y\}\}$.

(v) $y \in \hat{\Omega}cl(\{x\})$ *if and only if* $x \in Ker_{\hat{\Omega}}(\{y\})$.

PROOF. *(i)* By [1] theorem 5.11,$y \in \hat{\Omega}cl(\{x\})$ if and only if every $U \in \hat{\Omega}C(X, y)$ contains $\{x\}$. Therefore,$\hat{\Omega}cl(\{x\}) = \{y : \{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}\}$.
(ii) By [2] lemma 3.2,$y \in ker_{\hat{\Omega}}(\{x\})$ if and only if $\{x\} \cap \hat{\Omega}cl(\{y\}) \neq \emptyset$. Therefore, $Ker_{\hat{\Omega}}(\{x\}) = \{y : \{x\}$ is not weakly $\hat{\Omega}$ separated from $\{y\}\}$.
(iii) Since $\hat{\Omega}d(\{x\}) = \hat{\Omega}cl(\{x\}) \setminus \{x\}$ and by (i),$\hat{\Omega}d(\{x\}) = \{y : y \neq x, \{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}\}$.
(iv) Since $\hat{\Omega}shl(\{x\}) = Ker_{\hat{\Omega}}(\{x\}) \setminus \{x\}$ and by (ii), $\hat{\Omega}shl(\{x\}) = \{y : y \neq x, \{x\}$ is not weakly $\hat{\Omega}$ separated from $\{y\}\}$.
(v) If $x \in ker_{\hat{\Omega}}(\{y\})$,then by (ii),$\{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}$.By (i), $y \in \hat{\Omega}cl(\{x\})$. In a similar way, if $y \in \hat{\Omega}cl(\{x\})$ then by (i), $\{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}$.By (ii), $x \in ker_{\hat{\Omega}}(\{y\})$.Hence it is proved.

□

DEFINITION 3.15. *In a topological space (X, τ) for any $x \in X$,we define*

(i) $\hat{\Omega}$-*N-D* $= \{x \in X : \hat{\Omega}d(\{x\}) = \emptyset\}$.
(ii) $\hat{\Omega}$-*N-shl* $= \{x \in X : \hat{\Omega}shl(\{x\}) = \emptyset\}$.
(iii) $\hat{\Omega}\text{-}\langle x\rangle = \hat{\Omega}cl(\{x\}) \cap Ker_{\hat{\Omega}}(\{x\})$.

THEOREM 3.16. *For any $x, y \in X$,the following statements hold.*

(i) $y \in \hat{\Omega}shl(\{x\})$ *if and only if* $x \in \hat{\Omega}d(\{y\})$.
(ii) $y \in \hat{\Omega}cl(\{x\}) \Rightarrow \hat{\Omega}cl(\{y\}) \subseteq \hat{\Omega}cl(\{x\})$.
(iii) $y \in ker_{\hat{\Omega}}(\{x\}) \Rightarrow ker_{\hat{\Omega}}(\{y\}) \subseteq ker_{\hat{\Omega}}(\{x\})$.

PROOF. *(i)* By (iii) and (iv) of lemma 3.14,it holds.
(ii) Assume that $y \in \hat{\Omega}cl(\{x\})$.Let $z \in \hat{\Omega}cl(\{y\})$ be arbitrary and U be any $\hat{\Omega}$-open set in X containing $\{z\}$.Since $z \in \hat{\Omega}cl(\{y\})$,by (i) of lemma 3.4,$\{z\}$ is not weakly $\hat{\Omega}$ separated from $\{y\}$. By the definition of weakly $\hat{\Omega}$ separated,$y \in U$.By hypothesis, $\{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}$.Again by the definition of weakly $\hat{\Omega}$ separated,$x \in U$. Therefore, $\{z\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}$.By (i) of lemma 3.4,$z \in \hat{\Omega}cl(\{x\})$.Therefore, $\hat{\Omega}cl(\{y\}) \subseteq \hat{\Omega}cl(\{x\})$.
(iii) Suppose that $y \in ker_{\hat{\Omega}}(\{x\})$.If $z \in ker_{\hat{\Omega}}(\{y\})$,then by (v) of lemma 3.14,$y \in \hat{\Omega}cl(\{z\})$. By (ii),$\hat{\Omega}cl(\{y\}) \subseteq \hat{\Omega}cl(\{z\})$. By hypothesis,$y \in ker_{\hat{\Omega}}(\{x\})$ and by (v)of lemma 3,14, $x \in \hat{\Omega}cl(\{y\})$.By (ii),$\hat{\Omega}cl(\{x\}) \subseteq \hat{\Omega}cl(\{y\})$. Therefore,$\hat{\Omega}cl(\{x\}) \subseteq \hat{\Omega}cl(\{y\}) \subseteq \hat{\Omega}cl(\{z\})$and hence $x \in \hat{\Omega}cl(\{z\})$.Again by (v) of lemma 3.14,$z \in ker_{\hat{\Omega}}(\{x\})$. Therefore,$ker_{\hat{\Omega}}(\{y\}) \subseteq ker_{\hat{\Omega}}(\{x\})$.

□

THEOREM 3.17. *In a topological space (X, τ) the following statements are true.*

(i) For every $x \in X, \hat{\Omega}shl(\{x\})$ is degenerate if and only if $\hat{\Omega}d(\{x\}) \cap \hat{\Omega}d(\{y\}) = \emptyset$ for any y different from x.

(ii) For every $x \in X, \hat{\Omega}d(\{x\})$ is degenerate if and only if $\hat{\Omega}shl(\{x\}) \cap \hat{\Omega}shl(\{y\}) = \emptyset$ for any y different from x.

PROOF. *(i)* **Necessity-** Assume that $\hat{\Omega}shl(\{x\})$ is degenerate for every $x \in X$ and suppose that there exist some $y \in Y$ such that $\hat{\Omega}d(\{x\}) \cap \hat{\Omega}d(\{y\}) \neq \emptyset$. Choose $z \in X$ such that $z \in \hat{\Omega}d(\{x\})$ and $z \in \hat{\Omega}d(\{y\})$.Therefore,by the definition 3.1,$x \neq y \neq z$ and $x, y \in Ker_{\hat{\Omega}}(\{z\})$.Then, $x, y \in Ker_{\hat{\Omega}}(\{z\}) \setminus \{z\} = \hat{\Omega}shl(\{z\})$ such that $x \neq y$,a contradiction to $\hat{\Omega}shl(\{z\})$ is degenerate.
Sufficiency- Assume the contrary that there exist some $x \in X$ such that $\hat{\Omega}shl(\{x\})$ contains two distinct points p, q of X.Then by the definition 3.1,$x \neq p \neq q$ such that $x \in Ker_{\hat{\Omega}}(\{p\}) \cap Ker_{\hat{\Omega}}(\{q\})$ which contradicts the hypothesis.Thus,$\hat{\Omega}shl(\{x\})$ is degenerate for every $x \in X$.
(ii) In a similar way by the lemma 3.16 (i),it can be proved.

□

THEOREM 3.18. *In a topological space (X, τ) ,if $y \in \hat{\Omega}\text{-}\langle x\rangle$,then $\hat{\Omega}\text{-}\langle x\rangle = \hat{\Omega}\text{-}\langle y\rangle$.*

PROOF. If $y \in \hat{\Omega}\text{-}\langle x\rangle$,then by the definition 3.15 (iii),$y \in Ker_{\hat{\Omega}}(\{x\})$ and $y \in \hat{\Omega}cl(\{x\})$.By theorem 3.16 (ii) and(iii),$\hat{\Omega}cl(\{y\}) \subseteq \hat{\Omega}cl(\{x\})$ and $Ker_{\hat{\Omega}}(\{y\}) \subseteq Ker_{\hat{\Omega}}(\{x\})$.Therefore, $\hat{\Omega}cl(\{y\}) \cap Ker_{\hat{\Omega}}(\{y\}) \subseteq \hat{\Omega}cl(\{x\}) \cap Ker_{\hat{\Omega}}(\{x\})$. Thus,$\hat{\Omega}\text{-}\langle y\rangle \subseteq \hat{\Omega}\text{-}\langle x\rangle$.To prove the reversible inclusion, by lemma 3.14 (v), $y \in Ker_{\hat{\Omega}}(\{x\})$ and $y \in \hat{\Omega}cl(\{x\})$ implies that $x \in \hat{\Omega}cl(\{y\})$ and $x \in Ker_{\hat{\Omega}}(\{y\})$.Hence By theorem 3.16 (ii) and(iii),$\hat{\Omega}cl(\{x\}) \subseteq \hat{\Omega}cl(\{y\})$ and $Ker_{\hat{\Omega}}(\{x\}) \subseteq Ker_{\hat{\Omega}}(\{y\})$.Therefore, $\hat{\Omega}cl(\{x\}) \cap Ker_{\hat{\Omega}}(\{x\}) \subseteq \hat{\Omega}cl(\{y\}) \cap Ker_{\hat{\Omega}}(\{y\})$. Therefore,$\hat{\Omega}\text{-}\langle x\rangle \subseteq \hat{\Omega}\text{-}\langle y\rangle$.Thus,$\hat{\Omega}\text{-}\langle x\rangle = \hat{\Omega}\text{-}\langle y\rangle$. □

THEOREM 3.19. *In a topological space (X, τ) either $\hat{\Omega}\text{-}\langle x\rangle \cap \hat{\Omega}\text{-}\langle y\rangle = \emptyset$ or $\hat{\Omega}\text{-}\langle x\rangle = \hat{\Omega}\text{-}\langle y\rangle$ for any $x, y \in X$.*

PROOF. If $\hat{\Omega}\text{-}\langle x\rangle \cap \hat{\Omega}\text{-}\langle y\rangle \neq \emptyset$,then there exist $z \in X$ such that $z \in \hat{\Omega}\text{-}\langle x\rangle$ and $z \in \hat{\Omega}\text{-}\langle y\rangle$.By theorem 3.18,$\hat{\Omega}\text{-}\langle x\rangle = \hat{\Omega}\text{-}\langle y\rangle = \hat{\Omega}\text{-}\langle z\rangle$. Thus, either $\hat{\Omega}\text{-}\langle x\rangle \cap \hat{\Omega}\text{-}\langle y\rangle = \emptyset$ or $\hat{\Omega}\text{-}\langle x\rangle = \hat{\Omega}\text{-}\langle y\rangle$ for any $x, y \in X$. □

4. $\hat{\Omega}$-T_0 VIA WEAKLY $\hat{\Omega}$ SEPARATED SETS,SOBER $\hat{\Omega}$-R_0

Let us give some characterizations of $\hat{\Omega}$-T_0 space.

THEOREM 4.1. *X is $\hat{\Omega}$-T_0 if and only if either $\{x\}$ is weakly $\hat{\Omega}$ separated from $\{y\}$ or $\{y\}$ is weakly $\hat{\Omega}$ separated from $\{x\}$.*

PROOF. **Necessity-** Let $x, y \in X$ be any two points such that $x \neq y$.By the definition of $\hat{\Omega}$-T_0 space,there exists $U \in \hat{\Omega}O(X)$ such that either U contains x but not y or contains y but not x.By the definition 3.13,either $\{x\}$ is weakly $\hat{\Omega}$ separated from $\{y\}$ or $\{y\}$ is weakly $\hat{\Omega}$ separated from $\{x\}$.Hence it holds.
Sufficiency-Suppose that x and y are any two points in X such that $x \neq y$.By hypothesis,either $\{x\}$ is weakly $\hat{\Omega}$ separated from $\{y\}$ or $\{y\}$ is weakly $\hat{\Omega}$ separated from $\{x\}$.If $\{x\}$ is weakly $\hat{\Omega}$ separated from $\{y\}$,then there exists $\hat{\Omega}$-open set U in X such that $x \in U$ and $U \cap \{y\} = \emptyset$.Therefore,U satisfies $\hat{\Omega}$-T_0 axiom.Similarly if $\{y\}$ is weakly $\hat{\Omega}$ separated from $\{x\}$,then there exists $\hat{\Omega}$-open set U in X such that $y \in U$ and $x \notin U$.Thus X is $\hat{\Omega}$-T_0 space. □

THEOREM 4.2. *A topological space (X, τ) is $\hat{\Omega}$-T_0 if and only if $y \in \hat{\Omega}cl(\{x\}) \Rightarrow x \notin \hat{\Omega}cl(\{y\})$ for any distinct points $x, y \in X$.*

PROOF. **Necessity-** Assume that X is $\hat{\Omega}$-T_0 and $y \in \hat{\Omega}cl(\{x\})$.By lemma 3.14 (i),$\{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}$. Since X is $\hat{\Omega}$-T_0 space and by theorem 4.1,$\{x\}$ is weakly $\hat{\Omega}$ separated from $\{y\}$.Again by lemma 3.14 (i),$x \notin \hat{\Omega}cl(\{y\})$.
Sufficiency-If $x, y \in X$ are any two distinct points such that $\{x\}$ is not weakly $\hat{\Omega}$ separated from $\{y\}$.By lemma 3.14 (i),$x \in \hat{\Omega}cl(\{y\})$.By hypothesis,$y \notin \hat{\Omega}cl(\{x\})$.Again by lemma 3.14 (i),$\{y\}$ is weakly $\hat{\Omega}$ separated from $\{x\}$.By theorem 4.1,X is $\hat{\Omega}$-T_0 space. □

THEOREM 4.3. *A topological space (X, τ) is $\hat{\Omega}$-T_0 if and only if $\hat{\Omega}d(\{x\}) \cap \hat{\Omega}shl(\{x\}) = \emptyset$ for every $x \in X$.*

PROOF. **Necessity-**Suppose that X is $\hat{\Omega}$-T_0 space and there exists $x \in X$ such that $\hat{\Omega}d(\{x\}) \cap \hat{\Omega}shl(\{x\}) \neq \emptyset$.Choose $y \in X$ such that $y \in \hat{\Omega}d(\{x\})$ and $y \in \hat{\Omega}shl(\{x\})$.By lemma 3.14 (iii) and (iv),$y \neq x$,$\{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}$ and $\{x\}$ is not weakly $\hat{\Omega}$ separated from $\{y\}$ a contradiction to X is $\hat{\Omega}$-T_0 space.Therefore,$\hat{\Omega}d(\{x\}) \cap \hat{\Omega}shl(\{x\}) = \emptyset$ for every $x \in X$.
Sufficiency-On contrary,if there exists two distinct points $x, y \in X$ such that both $\{x\}$ is not weakly $\hat{\Omega}$ separated from $\{y\}$ and $\{y\}$ is not weakly $\hat{\Omega}$ separated from $\{x\}$ a contradiction to hypothesis.Therefore,either $\{x\}$ is weakly $\hat{\Omega}$ separated from $\{y\}$ or $\{y\}$ is weakly $\hat{\Omega}$ separated from $\{x\}$.By theorem 4.1,X is $\hat{\Omega}$-T_0 space. □

COROLLARY 4.4. *If a topological space (X, τ) is $\hat{\Omega}$-T_0,then $\hat{\Omega}$-$\langle x \rangle = \{x\}$ for any $x \in X$.*

THEOREM 4.5. *A topological space (X, τ) is $\hat{\Omega}$-T_1 if and only if $\hat{\Omega}d(\{x\}) = \emptyset$ (resp.$\hat{\Omega}shl(\{x\}) = \emptyset$) for any $x \in X$ or $\hat{\Omega}$-N-$D = X$ (resp.$\hat{\Omega}$-N-shl$= X$).*

PROOF. By [2] theorem 3.19,X is $\hat{\Omega}$-T_1 if and only if $\hat{\Omega}cl(\{x\}) = \{x\}(resp.ker_{\hat{\Omega}}(\{x\}) = \{x\})$ for every $x \in X$.Therefore,X is $\hat{\Omega}$-T_1 if and only if $\hat{\Omega}cl(\{x\}) \setminus \{x\} = \emptyset(resp.ker_{\hat{\Omega}}(\{x\}) \setminus \{x\} = \emptyset)$ for every $x \in X$.That is,X is $\hat{\Omega}$-T_1 if and only if $\hat{\Omega}d(\{x\}) = \emptyset(resp.\hat{\Omega}shl(\{x\}) = \emptyset)$ for every $x \in X$ or $\hat{\Omega}$-N-$D = X$ (resp.$\hat{\Omega}$-N-shl $= X$). □

DEFINITION 4.6. *A topological space (X, τ) is said to be $\hat{\Omega}$-C_0, if for every pair of distinct points $x, y \in X$,there exists $\hat{\Omega}$-open set U in X such that $\hat{\Omega}cl(U)$ contains any one of x and y but not other.*

DEFINITION 4.7. *A topological space (X, τ) is said to be $\hat{\Omega}$-C_1, if for every pair of distinct points $x, y \in X$,there exists $\hat{\Omega}$-open sets U and V in X such that $x \in \hat{\Omega}cl(U), y \notin \hat{\Omega}cl(U), y \in \hat{\Omega}cl(V), x \notin \hat{\Omega}cl(V)$.*

DEFINITION 4.8. *A topological space (X, τ) is said to be weakly $\hat{\Omega}$-C_0, if $\bigcap_{x \in X} ker_{\hat{\Omega}}(\{x\}) = \emptyset$.*

DEFINITION 4.9. *A topological space (X, τ) is said to be sober $\hat{\Omega}$-R_0, if $\bigcap_{x \in X} \hat{\Omega}cl(\{x\}) = \emptyset$.*

EXAMPLE 4.10. *$X = \{a, b, c\}$ $\tau = \{\emptyset, \{a, b\}, X\}$ then $\hat{\Omega}O(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. It is $\hat{\Omega}$-C_0 and weakly $\hat{\Omega}$-C_0,but not $\hat{\Omega}$-C_1.*

EXAMPLE 4.11. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\hat{\Omega}O(X) = P(X)$. It is $\hat{\Omega}$-C_1.*

EXAMPLE 4.12. *$X = \{a, b, c\}$ $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then $\hat{\Omega}O(X) = \tau$. It is sober $\hat{\Omega}$-R_0.*

THEOREM 4.13. *Every $\hat{\Omega}$-C_1 space is $\hat{\Omega}$-C_0.*

PROOF. It follows from their definitions. □

REMARK 4.14. *But $\hat{\Omega}$-C_0 space is not always $\hat{\Omega}$-C_1 from the example 4.9.*

Let us prove a characterization of Sober $\hat{\Omega}$-R_0 as follows.

THEOREM 4.15. *A topological space (X, τ) is Sober $\hat{\Omega}$-R_0 if and only if $ker_{\hat{\Omega}}(\{x\}) \neq X$ for every $x \in X$.*

PROOF. **Necessity**-Suppose that (X, τ) is Sober $\hat{\Omega}$-R_0 and assume that there exists $y \in X$ such that $ker_{\hat{\Omega}}(\{y\}) = X$. Then every $\hat{\Omega}$-open set U containing y contains every $x \in X$.By[1] theorem 5.11, $y \in \hat{\Omega}cl(\{x\})$ for every $x \in X$.Therefore, $y \in \bigcap_{x \in X} \hat{\Omega}cl(\{x\})$, a contradiction.
Sufficiency- Suppose that $ker_{\hat{\Omega}}(\{x\}) \neq X$ for every $x \in X$ and suppose $\bigcap_{x \in X} \hat{\Omega}cl(\{x\}) \neq \emptyset$. Then, we can choose $y \in \bigcap_{x \in X} \hat{\Omega}cl(\{x\})$ and hence $y \in \hat{\Omega}cl(\{x\})$ for every $x \in X$.By [2] lemma 3.3,$x \in ker_{\hat{\Omega}}(\{y\})$ for every $x \in X$.Therefore, $X = ker_{\hat{\Omega}}(\{y\})$, a contradiction. Hence (X, τ) is Sober $\hat{\Omega} - R_0$. □

The following theorem proves a characterization of weakly $\hat{\Omega}$-C_0.

THEOREM 4.16. *A topological space (X, τ) is weakly $\hat{\Omega}$-C_0 if and only if there exists a proper $\hat{\Omega}$-closed set containing x for every $x \in X$.*

PROOF. **Necessity**-On contrary,if there exists some $p \in X$ is such that only a $\hat{\Omega}$-closed set containing p is X.For every $x \in X$ and for every proper $\hat{\Omega}$-open set U containing x, $X \setminus U$ is a proper $\hat{\Omega}$-closed set does not contain p.Hence, $p \in ker_{\hat{\Omega}}(\{x\})$ for every $x \in X$.Therefore,$P \in \bigcap_{x \in X} ker_{\hat{\Omega}}(\{x\})$,a contradiction. Hence the result.
Sufficiency- If X is not weakly $\hat{\Omega}$-C_0, then choose $p \in X$ such that $p \in ker_{\hat{\Omega}}(\{x\})$ for any $x \in X$. This implies that X is the only $\hat{\Omega}$-open set,contains the point p, a contradiction.

□

THEOREM 4.17. *If a topological space (X, τ) is $\hat{\Omega}$-C_0 then it is weakly $\hat{\Omega}$-C_0 space.*

PROOF. Suppose that x, y are any distinct pair of points in a $\hat{\Omega}$-C_0 space.Then,there exists $\hat{\Omega}$-open set U in X such that $\hat{\Omega}cl(U)$ containing x but not y.Since $U \neq \emptyset$,we can choose $z \in X$ such that $z \in U$.By the definition of $\hat{\Omega}$-kernel,$ker_{\hat{\Omega}}(\{z\}) \subseteq U \subseteq \hat{\Omega}cl(U)$.Since $y \notin \hat{\Omega}cl(U), y \in X \setminus \hat{\Omega}cl(U)$.Again by the definition of $\hat{\Omega}$-kernel,$ker_{\hat{\Omega}}(\{y\}) \subseteq X \setminus \hat{\Omega}cl(U)$.Therefore, $ker_{\hat{\Omega}}(\{z\}) \cap ker_{\hat{\Omega}}(\{y\}) \subseteq \hat{\Omega}cl(U) \cap [X \setminus \hat{\Omega}cl(U)] = \emptyset$. Therefore, $\bigcap_{x \in X} ker_{\hat{\Omega}}(\{x\}) = \emptyset$ and hence X is weakly $\hat{\Omega}$-C_0 space. □

REMARK 4.18. *The reversible implication is not always possible from the following example.*

EXAMPLE 4.19. $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.*Then it is weakly* $\hat{\Omega}$-C_0 *but not* $\hat{\Omega}$-C_0 *space.*

5. CONCLUSION

In this paper,weak separation axioms are studied through kernel and shell of singletons via $\hat{\Omega}$-closed sets.Also some characterizations of $\hat{\Omega}$-T_0 and $\hat{\Omega}$-T_1 spaces by using $\hat{\Omega}cl(\{x\})$and $ker_{\hat{\Omega}}(\{x\})$ are investigated.It can be extended to bitopological spaces also.

6. REFERENCES

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