Image Denoising using K-SVD Algorithm based on Gabor Wavelet Dictionary

Wael Khedr
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt

Rehab Ali
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt

Fawzan Ismail
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt

ABSTRACT
Image denoising problem can be addressed as an inverse problem. One of the most recent approaches to solve an inverse problem is a sparse decomposition over overcomplete dictionaries. In sparse representation, images are represented as a linear combination of dictionary atoms. In this paper, we propose an algorithm for image denoising based on Orthogonal Matching Pursuit (OMP) for determining sparse representation over Gabor Wavelet adaptive dictionary by K-SVD algorithm. The results of this algorithm have more efficiency of image recovery than using DCT dictionary.

General Terms
Your general terms must be any term which can be used for general classification of the submitted material such as Pattern Recognition, Security, Algorithms et. al.

Keywords
Sparse representation, K-SVD, Gabor wavelet dictionary and OMP.

1. INTRODUCTION

Sparse representations [4,10] for signals became one of the hot topics in signal and image processing in recent years. Using an overcomplete dictionary matrix $D \in R^{n\times K}$ that contains K signal-atoms for columns $\{d_j\}_{j=1}^{K}$, a signal $y \in R^n$ can be represented as a sparse linear combination of these atoms. The sparse representation $y \cong DX$ satisfy the inequality (1), the vector $X \in R^K$ contains the representation coefficients of the signal y.

$$\|y - DX\| \le \varepsilon \tag{1}$$

In sparse approximation methods, typical norms used for measuring the deviation, are the ℓ^p -norms for $p = 1,2,\ldots$ and ∞ . In this paper, we concentrate on the case of $p = 2$. If $n < K$ and D is a full-rank matrix, an infinite number of solutions are available for the representation problem, hence constraints on the solution must be set. The solution with the fewest number of nonzero coefficients is certainly an appealing representation. This sparsest representation is the solution of either

$$\min_x \ ||X||_0 \quad \text{subject to} \quad y = DX \tag{2}$$

Or

$$\min_x \ ||X||_0 \quad \text{subject to} \quad ||y - DX|| \le \varepsilon \tag{3}$$

Where $||.||_0$ is the ℓ^0 norm, counting the nonzero entries of a vector. This problem is adequately addressed by the pursuit algorithms where the simplest ones are the matching pursuit (MP) [6] and the OMP algorithms [2,8]. These are greedy algorithms that select the dictionary atoms sequentially. These methods are very simple, involving the computation of inner products between the signal and dictionary columns, and possibly deploying some least squares solvers. Then after finding a sparse representation, the initial dictionary by DCT or Gabor Wavelet has been set then the dictionary by K-SVD algorithm has been updated for determining the best dictionary and sparse representation as in the following sections. In section 2, the pursuit algorithm OMP has is devoted to find the sparse representation. In section 3, the K-SVD algorithm is introduced to update the dictionary. In section 4, the Gabor Wavelet dictionary equations and its advantages and applications, in final, application with results is illustrated by using our algorithm in image denoising.

2. ORTHOGONAL MATCHING PURSUIT
The Orthogonal Matching Pursuit (OMP) algorithm [2, 7] is a greedy algorithm with attempts to find a sparse representation of a signal given a specific dictionary [3]. The algorithm attempts to find the best basis vectors (atoms) iteratively such that in each iteration the error in representation is reduced. This achieved by selection of the atom from the dictionary which has the largest absolute projection on the error vector. This essentially implies to the atom that adds the maximum information and hence maximally reduces the error in reconstruction. In the linear equation $y = DX$, a signal vector y is given and a dictionary D. The algorithm attempts to find the code vector X in the first three steps of the algorithm OMP. After that we use KSVD algorithm to update the dictionary D.A full description of the algorithm is given in algorithm 1:

Algorithm 1 OMP

Input:

- signal y and matrix D.
- stopping criterion e.g. until a level of accuracy is reached.

Output:

- Approximation vector c.

Algorithm

1. Start by setting the residual $r0 = y$, the time $t = 0$ and index set $V0 = \emptyset$
2. Let $v_t = i$ where d_i gives the solution of max $< r_t, d_K >$,where d_K are the row vectors of D
3. Update the set v_t with $v_t : V_t = V_{t-1} \cup \{ v_t \}$
4. Solve the least-squares problem

$$\min_{c \in \mathbb{C}^{V_t}} ||y - \sum_{j=1}^{t} c(v_j) d_{v_j} ||$$

5. Calculate the new residual using c

$$r_t = r_{t-1} - \sum_{j=1}^{t} c(v_j) d_{v_j}$$

6. Set $t \leftarrow t + 1$
7. Check stopping criterion if the criterion has not been satisfied then return to step 2.

3. K-SVD ALGORITHM

We now turn to the process of updating the dictionary [3] together with the nonzero coefficients X. Assume that both X and D are fixed and we put in question only one column in the dictionary d_k and the coefficients that correspond to it, the k- th row in X. denoted as x_T^k (this is not the vector x_k which is the k-th column in X). The objective function is defined in eq.(4)

$$\min_{D,X}\{||Y - DX||_F^2 \} \text{ subject to } \forall i, \|x_i\|_0 \le T_0 \quad (4)$$

, the penalty term can be rewritten as:

$$||Y - DX||_F^2 = ||Y - \sum_{j=1}^{K} d_j x_T^j||_F^2$$

$$= \left\| \left(Y - \sum_{j \neq k} d_j x_T^j \right) - d_k x_T^k \right\|_F^2$$

$$= \|E_k - d_k x_T^k\|_F^2 \quad (5)$$

Here, it would be tempting to suggest the use of the SVD to find alternative d_k and x_T^k. The SVD finds the closest rank-1 matrix (in Frobenius norm) that approximates error E_k , and this will effectively minimize the error as defined in eq. (5). However, such a step will be a mistake, because the new vector x_T^k is very likely to be filled, since in such an update of d_k we do not enforce the sparsity constraint. ω_k represents the group of indices pointing to examples $\{y_j\}$ that use the atom d_k, i.e., where $x_T^k(i)$ is nonzero. Thus

$$\omega_k = \{ i \mid 1 \le i \le K, x_T^k(i) \neq 0 \}. \quad (6)$$

Taking the restricted matrix E_k^R, SVD decomposes it to $E_k^R = U\Delta V^T$. We define the solution for d_k as the first column of U, and the coefficient vector x_T^k as the first column of V multiplied by $\Delta(1,1)$. Note that, in this solution, we necessarily have that the columns of D remain normalized and the support of all representations either stays the same or gets smaller by possible nulling of terms. A full description of the K-SVD algorithm [1] is given in algorithm2

Algorithm 2 K-SVD

Task : Find the best dictionary to represent the data samples $\{y_i\}_{i=1}^{N}$ as Sparse compositions ,by solving

$$min_{D,X}\{\left\|\boldsymbol{Y} - \boldsymbol{DX}\right\|_F^2 \} \text{ subject to} \forall i, \|x_i\|_0 \le T_0.$$

Initialization : set the dictionary matrix $\boldsymbol{D}^{(0)} \in R^{n \times K}$ with ℓ^2 normalized columns . Set $J = 1$

Repeat until convergence (stopping rule):

- Sparse Coding Stage: Use any pursuit algorithm to compute the Representation vectors x_i for each example y_i , by approximating the solution of

  $$min_{x_i} \left\{ \left\|\boldsymbol{y_i} - \boldsymbol{D}\boldsymbol{x_i}\right\|_F^2 \right\} \text{ subject to } \|\boldsymbol{x}_i\|_0 \le T_0.$$

- Codebook Update Stage : For each column $k = 1,2, \dots, K \ in \ \boldsymbol{D}^{(J-1)}$ Update it by
  1. Define the group of examples that use this atom ,

     $$\omega_k = \{ i \mid 1 \le i \le N, \boldsymbol{x_T^k}(i) \neq 0 \}.$$

  2. Compute the overall representation error matrix , $\boldsymbol{E}_k$,by

     $$\boldsymbol{E}_k = \boldsymbol{Y} - \sum_{j \neq k} d_j \boldsymbol{x}_T^j \ .$$

  3. Restrict $\boldsymbol{E}_k$ by choosing only the

columns corresponding to ω_k,and Obtain $\boldsymbol{E_k^R}$.

4. Apply SVD decomposition $\boldsymbol{E_k^R} = \boldsymbol{U}\,\Delta\,\boldsymbol{V^T}$. Choose the updated dictionary column d_k to be the first column of $\mathbf{U}$. Update the coefficient vector $\boldsymbol{x_R^k}$ to be the first column of $\mathbf{V}$ multiplied by $\Delta(1,1)$.

- Set $J = J + 1$.

4. GABOR WAVELETS DICTIONARY (GW)

Wavelet transform could extract both the time (spatial) and frequency information from a given signal. Among kinds of wavelet transforms, the Gabor wavelet transform [5,9] has some impressive mathematical and biological properties and has been used frequently on researches of image processing, where the equation of Gabor wavelet is given by (7) :

$$\psi(\mathrm{x}, y, \rho, \theta) = \frac{\omega_0}{\sqrt{2\pi\kappa}} e^{-\frac{\rho_0}{8\kappa^2}\left(4(\mathrm{x}\cos\theta + y\sin\theta)^2 + (\mathrm{y}\cos\theta - \mathrm{x}\sin\theta)^2\right)} \cdot \left[e^{\omega_0 i(\mathrm{x}\cos\theta + y\sin\theta)} - e^{-\frac{\kappa^2}{2}}\right] \qquad (7)$$

Where ρ_0 is the radial frequency in radians per unit length and θ is the wavelet orientation in radians. The Gabor wavelet is centered at (x=0, y=0) and the normalization factor is such that $< \psi, \psi >= 1$ i.e., normalized by l_2 norm, κ is a constant.

Among various wavelet bases, Gabor functions provide the optimal resolution in both the time (spatial) and frequency domains, and the Gabor wavelet transform seems to be the optimal basis to extract local features for several reasons [9]:

- ***Biological motivation***: The simple cells of the visual cortex of mammalian brains are best modeled as a family of self-similar 2D Gabor wavelets.
- ***Mathematical and empirical motivation:*** Gabor wavelet transform has both the multi-resolution and multi-orientation properties and are optimal for measuring local spatial frequencies. Besides, it has been found to yield distortion tolerance space for pattern recognition tasks.

Depending on these advantages of Gabor wavelet transform, it has been used in many image analysis applications, such as face recognition, texture classification, facial expression classification, and some other excellent researches. Therefore, we used Gabor Wavelet dictionary for best sparse representation with image denoising.

5. APPLICATIONS AND RESULTS

In this work, we used an overcomplete Gabor Wavelet dictionary as an initial dictionary of size 64x256 in which each basis was arranged as an atom in the dictionary. The dictionary was learned by alternating between sparse coding with the current dictionary and dictionary updated with the current sparse representation. For doing this, we used the K-SVD algorithm. We evaluated the performance of our method by calculating the PSNR and compare our results with the K-SVD methods using DCT dictionary, which showed that our method gave better results over the K-SVD especially with high noise. This is illustrated in figure 1 which contains (a) the noised Lena image ($\boldsymbol{\sigma}$=25, PSNR=28.6692dB), (b) the original image and (c) the result of our method and (d) the result of K-SVD based on DCT dictionary. In figure 2, we applied our algorithm on another example with different noise on Barbara image. In table1, more results with different noise.

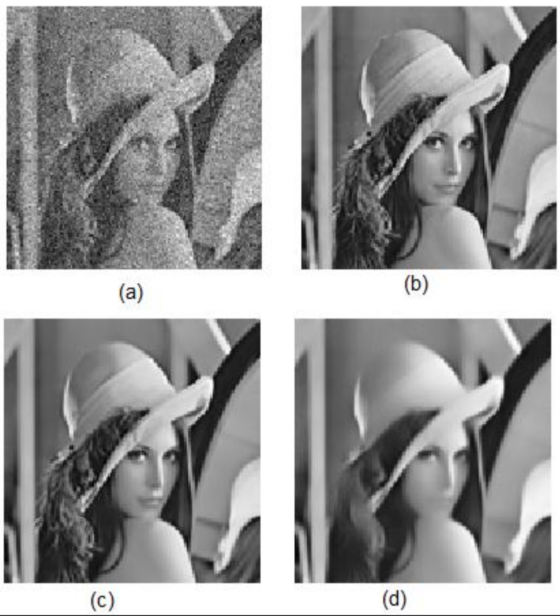

Fig1: (a) The noised image by adding Gaussian noise with $\boldsymbol{\sigma}$ =25. (b) The original image. (c) The denoised image by using K-SVD based on OMP and Gabor Wavelet dictionary and (d) the denoised image by using K-SVD based on OMP and DCT dictionary.

$\boldsymbol{\sigma}$/PSNR	**Lena**		**Barbara**	
	Omp_DCT	Omp_GW	Omp_DCT	Omp_GW
5/34.16	38.2629	39.3964	38.8966	38.9773
10/28.14	34.9692	36.9633	32.2396	33.532
15/24.61	31.6891	32.7867	28.7425	29.8403
25/20.18	28.6692	29.1952	26.8638	27.8683

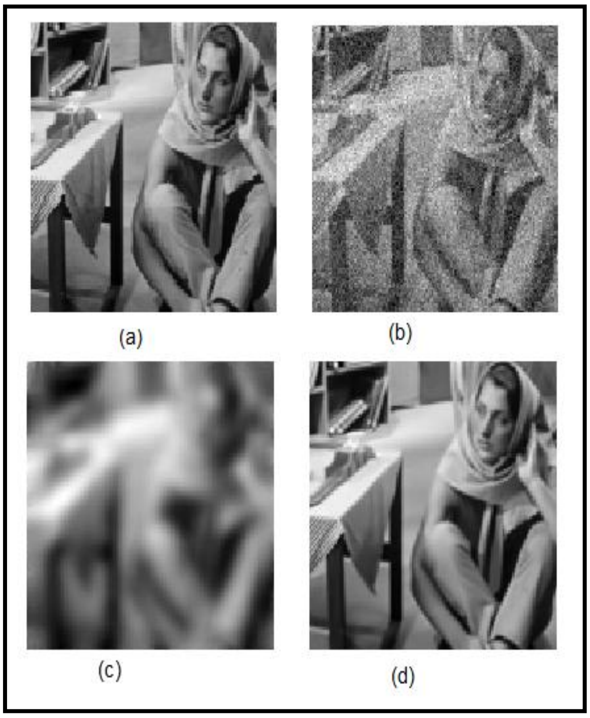

Fig2 : (a) The noised image by adding Gaussian noise with σ =25. (b) The original image. (d) denoised image by using K-SVD based on OMP and Gabor Wavelet dictionary and (c) the denoised image by using K-SVD based on OMP and DCT dictionary.

7. CONCLUSION

In this paper, we addressed the image denoising problem based on sparse coding using overcomplete dictionary. We presented K-SVD algorithm based on OMP and using Gabor wavelet dictionary for finding the sparse representation of data set. We found that Gabor wavelet dictionary is better than DCT dictionary for image denoising and sparse representation.

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