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Design of Improved Fractional Order Integrators using Indirect Discretization Method

Maneesha Gupta [1], Richa Yadav [2]
Advanced Electronics Lab, Division of Electronics and Communication Engineering, Netaji Subhas Institute of Technology, Sector-3, Dwarka, New Delhi 110075, India.

ABSTRACT

This paper has adopted rational approximation based on Regular Newton method, for frequency domain fitting of transfer functions of fractional order integrators (FOIs). Further, different discretized mathematical models of one-half, one-third and one-fourth order integrators based on Al-Alaoui operator and New optimized four segment operator have been also developed using these rational approximations by indirect discretization technique. All the proposed models of FOIs are found to be stable when investigated for stability. Simulation results of magnitude responses, phase responses and absolute magnitude errors show that the proposed FOIs obtained by approximations based on Regular Newton method, clearly outperform the other existing approximation techniques which have been used for designing fractional order operators. Results of absolute magnitude errors for all proposed fractional order integrators have been reported to be as low as 0.01, in range $0.35 \leq \omega \leq 1\ \pi$ radians of full band of normalized frequency. Among the proposed FOIs, the one-half, one-third and one-fourth order models based on Al-Alaoui operator (for 2nd iterations) are noticeable with tremendously improved results with absolute magnitude errors of $\leq$ 0.004 in complete normalized frequency range.

General Terms

Fractional order integrators, indirect discretization

Keywords

Regular Newton method, Al-Alaoui operator, New optimized four segment operator.

1. INTRODUCTION

Fractional calculus (FC) has been significantly applied in different domains of science and engineering since its initial traces in discussions between Leibnitz and Bernoulli. In the initial growing stage of FC, the main efforts of mathematicians were focused on drawing a parallel extension of integer order system into system of any arbitrary order, by generalizing the fundamental definitions of difference formulae of derivatives and integrals into their fractional forms. The frequency response of ideal fractional order differentiators/ integrators is

$$H(j\omega) = (j\omega)^{\pm\alpha} \qquad (1)$$

Where α is the order of operator (we use '$+\alpha$' for differentiator, '$-\alpha$' for integrator) and ω is the angular frequency in radians.

Two discretization techniques which mainly deal with the discretization of a fractional order (f-o) operator $s^{\pm\alpha}$ are, indirect discretization and direct discretization. Direct discretization approach expands the generating functions (either of a simple first order [1-7] or higher order [8-11]), by directly applying either of the existing expansion techniques namely continued fraction expansion (CFE) [8], power series expansion (PSE), Taylor series expansion (TSE) [9] or numerical integration formulae. Whereas, indirect discretization technique [12-15] discretizes the transfer functions of lower and higher orders, which have been already fitted in frequency domain by suitable rational approximations. This formulation of a rational approximation with accurate but speedy convergence has always been the biggest barrier in using indirect discretization in spite of its well systematic and simple nature. Till a few years back, the dominating area of formulation of approximations was centralized around the estimation of responses of impedances for any non-integer order. Since 1960s [16-20], different well established mathematical concepts have been explored for the formulation of RC networks for realizations of capacitors, RC impedances and RC admittances for non-integer orders. Various Taylor series based methods [21-22] also helped in finding the solutions of fractional order (f-o) systems for realizing them in their physical forms with finite memory. Researchers started taking interest for designing rational approximations for fractional order differentiators (FODs) and fractional order integrators (FOIs) in the beginning of last decade and gave many techniques namely Prony's method [23], Ostaloup approximation [24] and Laguerre approximation [25], but still there is much scope of doing work in design of different suitable approximation techniques.

The main objective of this brief is to address the design method of continuous-time f-o operators by indirect discretization of the rational approximations which were obtained by first and second iterations of well known regular Newton method [26-27] for different $(1/n)^{th}$ roots such as ½, 1/3 and ¼. Here, authors have proved that these rational approximations originally designed for approximating fractional capacitors can be effectively used in frequency domain fitting of operators in indirect discretization scheme for designing mathematical models of FOIs. In this paper, we have proposed one-half, one-third and one-fourth order integrators based on Al-Alaoui operator [28] and New optimized four segment operator [28]. Proposed FOIs effectively approximate their respective ideal responses with absolute magnitude error of the order of 0.01 in range $0.35 \leq \omega \leq 1\ \pi$ radians of complete range of normalized frequency and also outperform their respective existing models.

This paper is organized as follows: section 2 describes the formulations of different adopted rational approximations based on Regular Newton method. These rational approximations have been discretized by using Al-Alaoui operator and New optimized four segment operator using them as s-to-z transformations and transfer functions of all the proposed one-half, one-third and one-fourth order integrators based on these two operators have been discussed in section 3. Section 4 deals with the simulation plots for magnitude responses, phase responses and absolute magnitude errors of all the proposed FOIs and their performances have been also discussed in the same section. Section 5 has drawn the conclusions.

2. RATIONAL APPROXIMATIONS BASED ON REGULAR NEWTON METHOD FOR FRACTIONAL ORDER INTEGRATORS

The rational approximations which were designed for approximating fractional capacitors in [26, 27] have been adopted here for frequency domain fitting of fractional integral operator $(1/s)^{\alpha}$. Regular Newton method is an iterative method and it presents better results with increase in order of iteration. Here, the rational approximations of one-half, one-third and one-fourth order integrators for first and second iterations of the Regular Newton method have been used and are listed in Table 1.

Table 1. Details of rational approximations based on Regular Newton method for one-half, one-third and one-fourth order FOIs

Rational Approximations based on Regular Newton Method for Proposed FOIs
Rational approximations of one-half order integrators from first and second iterations of $(s)^{1/2}$
$X_{1st\_1/2} = \left(\frac{s+3}{3s+1}\right)$
$X_{2nd\_1/2} = \left(\frac{(s^4+36s^3+126s^2+84s+9)}{(9s^4+84s^3+126s^2+36s+1)}\right)$
Rational approximations of one-third integrators from first and second iterations of $(s)^{1/3}$
$X_{1st\_1/3} = \left(\frac{s+2}{2s+1}\right)$
$X_{2nd\_1/3} = \left(\frac{s^5+24s^4+80s^3+92s^2+42s+4}{4s^5+42s^4+92s^3+80s^2+24s+1}\right)$
Rational approximations of one-fourth order integrators from first and second iterations of $(s)^{1/4}$
$X_{1st\_1/4} = \left(\frac{3s+5}{5s+3}\right)$
$X_{2nd\_1/4} = \frac{729s^6+15450s^5+58375s^4+91500s^3+69975s^2+24090s+20}{2025s^6+24090s^5+69975s^4+91500s^3+58375s^2+15450s+7}$

3. DERIVATION OF MATHEMATICAL MODELS OF PROPOSED ONE-HALF, ONE-THIRD AND ONE-FOURTH ORDER INTEGRATORS BY INDIRECT DISCRETIZATION APPROACH

In this section, the above mentioned rational approximations have been discretized by using Al-Alaoui operator $H_A(z)$ [28] and New optimized four segment operator $H_O(z)$ [28] by using them as s-to-z transformations.

3.1 Transfer functions of operators used for indirect discretization.

The transfer function of Al-Alaoui operator $H_A(z)$ [28] is

$$H_A(z) = \left(\frac{8(z-1)}{7T(z+1/7)}\right) \qquad (2)$$

The transfer function of New optimized four segment operator $H_O(z)$ [28] is given in (4). We have applied pole reflection method on one zero (1.027) which was originally lying outside the unit circle.

$$H_O(z) = \left(\frac{1.0968}{1.027}\right)\left(\frac{(z-0.9874)\left(z+\frac{1}{1.027}\right)}{(z+0.9768)(z+0.2209)}\right) = \left(\frac{1.1264z^2-0.01542z-1.083}{z^2+1.198z+0.2158}\right) \qquad (3)$$

3.2 Proposed models of FOIs and their comparisons with the existing one-half, one-third and one-fourth order integrators.

One-half, one-third and one-fourth order integrators have been developed by substituting Al-Alaoui operator and New optimized four segment operator in place of 's' in the rational approximations of FOIs given in Table I. The transfer functions of proposed discretized mathematical models have been given in pole-zero form in Tables 2 and 3 respectively.

Table 2. Transfer functions of one-half, one-third and one-fourth order integrators based on Al-Alaoui Operator

Proposed FOIs $(1/s)^{\alpha}$ based on Al-Alaoui operator
Transfer functions of one-half order integrators (α = 1/2)
$H_{A\_1st\_1/2}(z) = \left(\frac{0.93548(z-0.1724)}{(z-0.6522)}\right)$
$H_{2nd\_1/2} = \left(\frac{0.93541(z+0.1036)(z-0.1724)(z-0.5643)(z-0.88}{(z-0.007419)(z-0.3667)(z-0.7419)(z-0.9697)}\right.$
Transfer function of one-third order integrator (α = 1/3)
$H_{A\_1st\_1/3}(z) = \left(\frac{0.95652(z-0.2727)}{(z-0.6522)}\right)$
$H_{2nd\_1/3} = \left(\frac{(0.95647(z+0.08187)(z-0.2727)(z-0.886)(z^2-1.011z+0.2}{((z-0.001648)(z-0.6522)(z-0.9527)(z^2-0.8623z+0.2018}\right.$
Transfer function of one-fourth order integrator (α = 1/4)
$H_{A\_1st\_1/4}(z) = \left(\frac{0.96721(z-0.322)}{(z-0.6066)}\right)$
$H_{2nd\_1/4} = \left(\frac{\left(\frac{0.96717(z-0.8917)(z-0.4763)(z-0.322)}{(z+0.07034)(z^2-0.9874z+0.2695)}\right)}{\left(\frac{(z-0.9436)(z-0.6066)(z-0.457)}{(z+0.005448)(z^2-0.8912z+0.2249)}\right)}\right)$

Table 3. Transfer functions of one-half, one-third and one-fourth order integrators based on New optimized four segment Operator

Proposed FOIs $(1/s)^{\alpha}$ based on New optimized four segment Operator
Transfer functions of one-half order integrators (α = ½)
$H_{O\_1st\_1/2}(z) = \left(\frac{0.94227\,(z+0.9755)\,(z-0.1082)}{(z-0.7111)\,(z+0.9741)} \right)$
$H_{O\_2nd\_1/2}(z) = \left(\frac{\begin{pmatrix} 0.94222\,(z+0.9769)\,(z+0.9755)\,(z+0.9744)\,(z+0.973 \\ (z-0.8601)\,(z-0.522)\,(z-0.1082)\,(z+0.1801) \end{pmatrix}}{\begin{pmatrix} (z-0.9549)\,(z-0.7111)\,(z-0.3128)\,(z+0.9738) \\ (z+0.9741)\,(z+0.9748)\,(z+0.9762)\,(z+0.06447) \end{pmatrix}} \right)$
Transfer function of one-third order integrator (α = 1/3)
$H_{O\_1st\_1/3}(z) = \left(\frac{0.96114\,(z+0.9751)\,(z-0.2137)}{(z-0.6154)\,(z+0.9742)} \right)$
Transfer function of one-fourth order integrator (α = 1/4)
$H_{O\_1st\_1/4}(z) = \left(\frac{0.97071(z-0.2656)\,(z+0.975)}{(z-0.5669)\,(z+0.9743)} \right)$

Second order iterations for one-third and one-fourth FOIs based on New optimized four segment operator have not been used because of large order of resulting discretized transfer function. All the proposed one-half, one-third and one-fourth FOIs based on Al-Alaoui operator and New optimized four segment operator have been observed as stable with minimum phase, as all the poles and zeros of these discretized models are interlaced along the line z Є (-1,1) inside unit circle which is the desirable condition for a better frequency domain fitted transfer function.

The proposed FOIs based on rational approximations of Regular Newton method have been compared with the existing models designed by different approximation techniques. Different models of one-half, one-third and one-fourth order integrators have been also developed using Al-Alaoui and Tustin operator for different orders, by adopting ***'CFE based direct discretization'*** and have been named here as Al-MPV and Tustin-MPV [10-11]. Another operator that used here for comparison has been designed by Krishna et. al in [12-13], is based on **'*CFE based indirect discretization*'** technique and is named here as 'Al-Krishna-Reddy'. Results of proposed models have been compared with the ideal responses and the above mentioned existing models based on different well established techniques, for validating the effectiveness of approximation technique based on Regular Newton method.

4. SIMULATION RESULTS OF COMPARISON OF PROPOSED OPERATORS WITH EXISTING ONES.

Simulations for comparisons of magnitude responses, phase responses and absolute magnitude errors of all the proposed models of one-half, one-third and one-fourth order integrators with those of existing operators of same order have been performed in MATLAB with time T as 1second and their plots have been presented in Fig. 1-9.

The simulation results of comparisons of magnitude responses, phase responses and absolute magnitude errors of proposed one-half integrators namely $H_{A\_Ist\_1/2}(z)$, $H_{A\_2nd\_1/2}(z)$, $H_{O_Ist_1/2}(z)$and $H_{O\_1/2}(z)$ with the existing [10-13] and ideal one-half integrator ($s^{-1/2}$) have been shown in Fig. 1-3 by simulating their discretized transfer functions. Fig. 4-6 show magnitude responses, absolute magnitude errors and phase responses of comparison of one-third differentiator models based on Al-Alaoui and New optimized four segment operators namely, $H_{A\_Ist\_1/3}(z)$, $H_{A\_2nd\_1/3}(z)$ and $H_{O\_Ist\_1/3}(z)$, with those obtained in [10, 12] and ideal one-third integrator ($s^{-1/3}$). Similarly, responses of proposed one-fourth integrators based on Al-Alaoui and New optimized four segment operators namely, $H_{A\_Ist\_1/4}(z)$, $H_{A\_2nd\_1/4}(z)$ and $H_{O\_Ist\_1/4}(z)$ have been compared with those developed in [10,12] and its ideal counterpart ($s^{-1/4}$), (see Fig. 7-9).

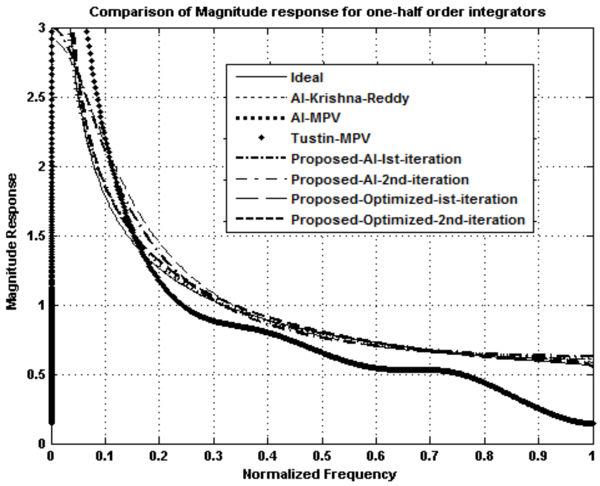

Fig. 1: Comparison of magnitude responses of proposed one-half integrators with existing one-half integrators

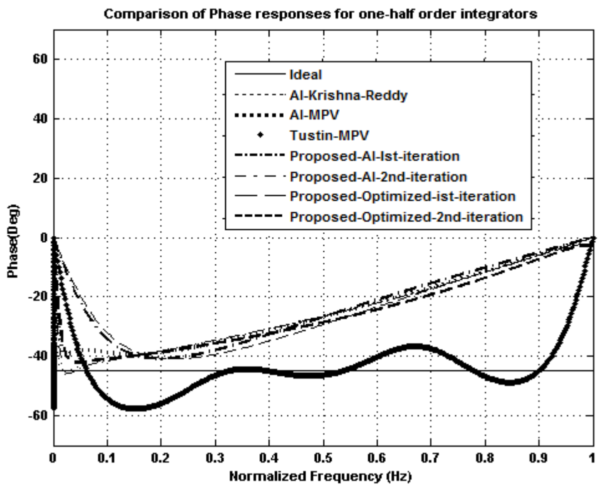

Fig. 2: Comparison of phase responses of proposed one-half integrators with existing one-half order integrators

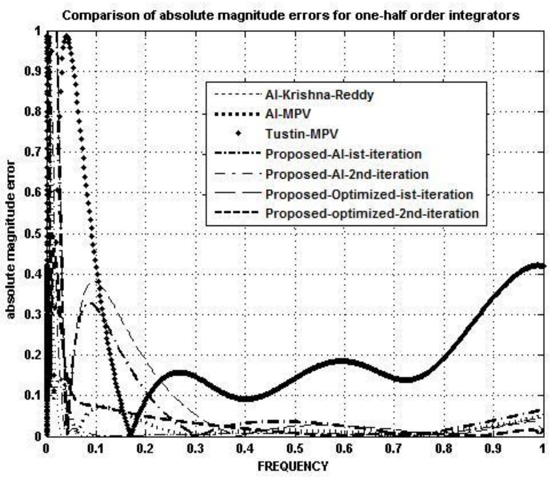

Fig. 3: Comparison of absolute magnitude errors of proposed one-half integrators with existing one-half order integrators

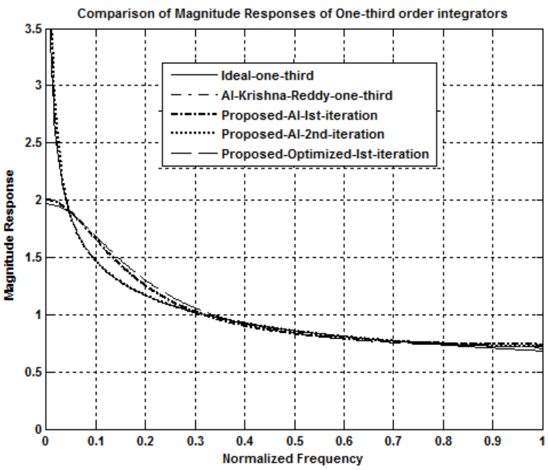

Fig. 4: Comparison of magnitude responses of proposed one-third order integrators with existing one-third order integrators

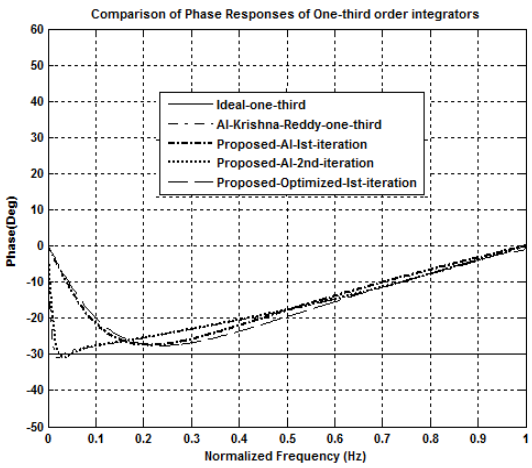

Fig. 5: Comparison of phase responses of proposed one-third order integrators with existing one-third order integrators

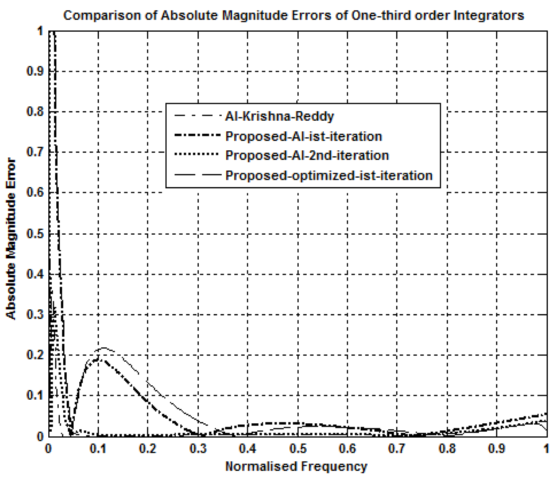

Fig. 6: Comparison of absolute magnitude errors of proposed one-third order integrators with existing one-third order integrators

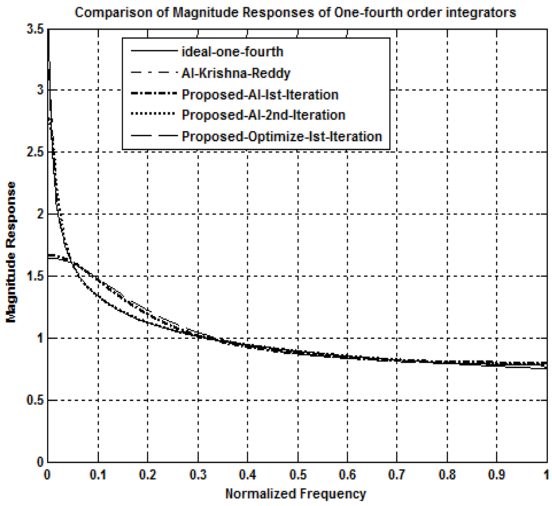

Fig. 7: Comparison of magnitude responses of proposed one-fourth order integrators with existing one-fourth order integrators

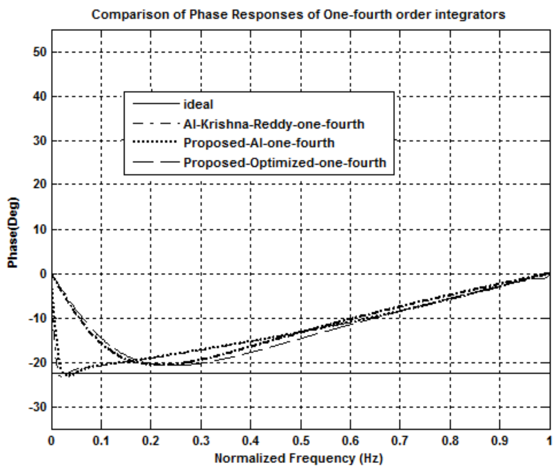

Fig. 8: Comparison of phase responses of proposed one-fourth order integrators with existing one-fourth order integrators

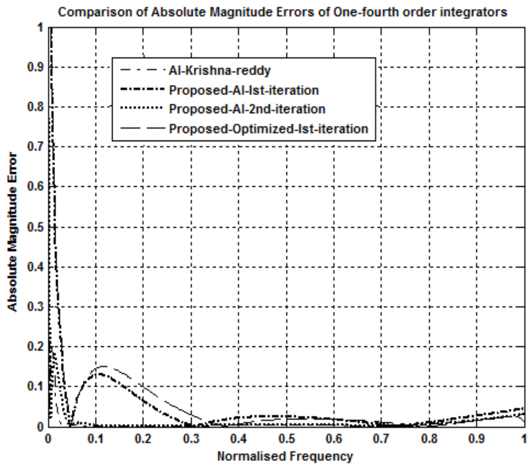

Fig. 9: Comparison of absolute magnitude errors of proposed one-fourth order integrators with existing one-third order integrators

This can be clearly observed from the above mentioned Fig. 1-9 that proposed FOIs based on novel approximation technique have linear phases and present negligible absolute magnitude errors of the order of 0.01 in range $0.35 \leq \omega \leq 1\ \pi$ radians of complete Nyquist frequency for all the three proposed orders viz. one-half, one-third and one-fourth and outperform all the existing models for these three orders. Proposed FOIs based on second iterations of Regular Newton method have shown tremendously improved results with absolute magnitude errors of ≤ 0.004 and these models also show linear variations in phase responses in full spectrum of normalized frequency.

5. CONCLUSION

We have observed from the plots of comparisons of magnitude responses, phase responses and absolute magnitude errors that the proposed approximation technique effectively approximates the proposed results based on Al-Alaoui and New optimized four segment operators, closer to the respective ideal responses and also shows superior performance as compared to existing techniques. All the proposed integrators of orders ½, 1/3 and ¼ have linear phases in complete range of normalized frequency range.

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