Some Domination Parameters of Direct Product Graphs of Cayley Graphs with Arithmetic Graphs

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ABSTRACT

Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern.

Nathanson [1] paved the way for the emergence of a new class of graphs, namely Arithmetic Graphs by introducing the concepts of Number Theory, particularly, the Theory of Congruences in Graph Theory. Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph. Inspired by the interplay between Number Theory and Graph Theory several researchers in recent times are carrying out extensive studies on various Arithmetic graphs in which adjacency between vertices is defined through various arithmetic functions.

In this paper, we consider direct product graphs of Cayley graphs with Arithmetic graphs and present some domination parameters of these graphs.

Keywords

Dominating set, Total Dominating set, Euler Totient Cayley Graph, Arithmetic V_n graph , Direct Product Graph.

AMS (MOS) Subject Classification: 6905c

1. INTRODUCTION

Domination in graphs has been an extensively research branch of graph theory. (For more details refer [2, 3]). Dominating sets play an important role in practical applications, such as allocation of re-hydrants or serving sites of other supplies, modeling of relations in human groups or animal biotopes. They have applications in diverse areas such as logistics and networks design, mobile computing, resource allocation and telecommunication. Cayley graphs are excellent models for interconnection networks, investigated in connection with parallel processing and distributed computation.

In this section we present necessary definitions, observations and some useful results that we need for next sections.

DOMINATING SET

Let G be a graph with vertex set V. A subset D of V is said to be a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D.

The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

TOTAL DOMINATING SET

Let G be a graph without isolated vertices. Then a total dominating set T is a subset of vertex set V such that every vertex of V is adjacent to some vertex in T.

The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$.

DIRECT PRODUCT GRAPHS

In the literature, the direct product is also called as the tensor product, categorical product, cardinal product, relational product, Kronecker product, weak direct product, or conjunction. As an operation on binary relations, the tensor product was introduced by Alfred North Whitehead and Bertrand Russell in their Principia Mathematica [4]. It is also equivalent to the Kronecker product of the adjacency matrices of the graphs given by Weichsel [5].

If a graph can be represented as a direct product, then there may be multiple different representations (direct products do not satisfy unique factorization) but each representation has the same number of irreducible factors. Wilfried Imrich [6] gives a polynomial time algorithm for recognizing tensor product graphs and finding a factorization of any such graph.

This product is commutative and associative in a natural way (refer [7] for a detailed description on product graphs).

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \dots, u_l\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$ respectively. Then the direct product of these two graphs denoted by $G_1 \times G_2$ is defined to be a graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 such that any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if u_1u_2 is an edge of G_1 and v_1v_2 is an edge of G_2.

The cross symbol $\times$, shows visually the two edges resulting from the direct product of two edges.

Now we consider the direct product graph of Euler totient Cayley graphs with Arithmetic V_n graphs. The properties of these graphs are presented in [8]. We briefly present Euler totient Cayley graph and Arithmetic V_n graph.

EULER TOTIENT CAYLEY GRAPH $G(Z_n, \varphi)$ AND ITS PROPERTIES

Madhavi [9] introduced the concept of Euler totient Cayley graphs and studied some of its properties.

For any positive integer n, let $Z_n = \{0,1,2, \dots\dots n-1\}$. Then $(Z_n, \oplus)$, where, $\oplus$ is addition modulo n, is an abelian group of order n. The number of positive integers less than n and relatively prime to n is denoted by $\varphi(n)$ and is called Euler totient function. Let S denote the set of all positive integers less than n and relatively prime to n.

That is $S = \{r/ 1 \le r < n$ and $\text{GCD}\,(r,n) = 1\}$. Then $|S| = \varphi(n)$.

Now we define Euler totient Cayley graph as follows.

For each positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all integers less than n and relatively prime to n. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2, \dots . n-1\}$ and the edge set is $E = \{(x,y)/x - y \in S$ or $y - x \in S\}$.

The domination parameters of these graphs are studied by Uma Maheswari [8] and we present some of the results which we need without proofs and can be found in [10].

Theorem 1.1: If n is a prime, then the domination number of $G(Z_n, \varphi)$ is 1.

Theorem 1.2: If n is power of a prime, then the domination number of $G(Z_n, \varphi)$ is 2

Theorem 1.3: The domination number of $G(Z_n, \varphi)$ is 2, if $n = 2p$ where p is an odd prime.

Theorem 1.4: Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are primes and $\alpha_1, \alpha_2, \dots . \alpha_k$ are integers ≥ 1. Then the domination number of $G(Z_n, \varphi)$ is given by $\gamma\big(G(Z_n, \varphi)\big) = \lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with n.

Theorem 1.5: If n is a prime**,** then the total domination number of $G(Z_n, \varphi)$ is 2.

Theorem 1.6: If n is power of a prime, then the total domination number of $G(Z_n, \varphi)$ is 2.

Theorem 1.7: The total domination number of $G(Z_n, \varphi)$ is 4, if $n = 2p$, where p is an odd prime.

Theorem 1.8: Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_k^{\alpha_k}$ where $p_1, p_2, \dots p_k$ are primes and $\alpha_1, \alpha_2, \dots . \alpha_k$ are integers ≥ 1. Then the total domination number of $G(Z_n, \varphi)$ is given by $\gamma_t\big(G(Z_n, \varphi)\big) = \lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V each of which shares a prime factor with n.

ARITHMETIC V_n GRAPH

Vasumathi [11] introduced the concept of Arithmetic V_n graphs and studied some of its properties.

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if GCD $(u,v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence we consider Arithmetic graph V_n without vertex 1 as the contribution of this isolated vertex is nothing when the properties of these graphs and enumeration of some domination parameters are studied.

Clearly, V_n graph is a connected graph. Because if n is a prime, then V_n graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacency in V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n.

The domination parameters of these graphs are studied by S.Uma Maheswari [8] and we present some of the results which we need without proofs and can be found in [12].

Theorem 1.9: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are primes and $\alpha_1, \alpha_2, \dots . \alpha_k$ are integers ≥ 1, then the domination number of $G(V_n)$ is given by

$$\gamma\big(G(V_n)\big) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one i} \\ k & \text{Otherwise.} \end{cases}$$

where k is the core of n.

Theorem 1.10: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_k^{\alpha_k}$ where $\alpha_i \ge 1$, $\forall i$. Then the total domination number of $G(V_n)$ is k, where k is the core of n

2. DOMINATION IN DIRECT PRODUCT GRAPH $G_1 \times G_2$

In this section we discuss the dominating sets of the direct product graph of Euler totient Cayley graph G_1 and Arithmetic V_n graph G_2.

Theorem 2.1: If n is a prime, then the domination number of $G_1 \times G_2$ is n.

Proof: Suppose n is a prime. Then $G_1 \times G_2$ is a completely disconnected graph on n vertices [13]. Since there is no edge between these n vertices, all these vertices form a dominating set. Hence its domination number is n. ∎

Theorem 2.2: If n is power of a prime, then the domination number of $G_1 \times G_2$ is 4.

Proof: Let $n = p^{\alpha}$. Consider the graph $G_1 \times G_2$.

Let $V(G_1) = \{0,1,2,3,\dots\dots, p^{\alpha} - 1\} = V_1$

$V(G_2) = \{p, p^2, \dots \dots p^{\alpha}\} = V_2$ and

$V(G_1 \times G_2) = V_1 \times V_2 = V$ be the set of vertices of G_1, G_2 and $G_1 \times G_2$ respectively. Since n is power of a prime, we have $\gamma(G_1) = 2$ (Theorem 1.2).

Let $D_1 = \{u_{d_1}, u_{d_2}\}$ be a dominating set of G_1 with minimum cardinality 2. Now in $G_1 \times G_2$, consider the set of vertices $D = \{(u_{d_1}, p), (u_{d_1}, p^j), (u_{d_2}, p), (u_{d_2}, p^j)\}$, $j \ne 1$.

Let (u,v) be any vertex of $V - D$ in $G_1 \times G_2$. Then the vertex u in G_1 is adjacent to either u_{d_1} or u_{d_2} in D_1. The vertex v in G_2 is adjacent to either p or p^j according to $\text{GCD}(v,p) = p$ or GCD $(v, p^j) = p$. Thus by the definition of direct product, the vertex (u,v) in $V - D$ is adjacent to either (u_{d_1}, p) or (u_{d_1}, p^j); or (u_{d_2}, p) or (u_{d_2}, p^j) in D. Thus D becomes a dominating set of $G_1 \times G_2$.

We now show that D is minimal. That is deletion of any vertex in D does not make D, a dominating set any more. Suppose the vertex (u_{d_1}, p) is deleted from D. We know that the degree of a vertex in G_1 is $\varphi(n)$. So let u_{d_1} be adjacent to the vertices $u_1, u_2, \dots . u_{\varphi(n)}$ in G_1.

Consider the set $S = \{(u_1, p^k), (u_2, p^k), \dots, (u_{\varphi(n)}, p^k)\}$ in $G_1 \times G_2$. Since $deg_{G_1}(u_{d_2})$ is $\varphi(n)$, all the vertices $u_1, u_2, \dots, u_{\varphi(n)}$ are not dominated by u_{d_2}. Otherwise the vertices of $V_1 - \{u_1, u_2, \dots u_{\varphi(n)}\}$ in G_1 are dominated neither by u_{d_1} nor by u_{d_2}. So all the vertices in S are not dominated by (u_{d_2}, p) and (u_{d_2}, p^j). Since GCD $(p^j, p^k) \neq p$, all the vertices in S are not dominated by (u_{d_1}, p^j). Thus no vertex in $D - \{(u_{d_1}, p)\}$ can dominate the vertices of S. Thus $D - \{(u_{d_1}, p)\}$ is not a dominating set of $G_1 \times G_2$. Similar is the case with the deletion of any other vertex in D. Thus D becomes a minimal dominating set of $G_1 \times G_2$.

Further if we form a dominating set of $G_1 \times G_2$ in any other manner, then the order of such a set is not smaller than that of D. This is because of the properties of prime numbers. Thus the domination number of $G_1 \times G_2$ is 4.

That is $\gamma(G_1 \times G_2) = |D| = 4$. ■

Theorem 2.3: The domination number of $G_1 \times G_2$ is 6, if $n = 2p$ where p is an odd prime.

Proof: Let $n = 2p$, p is an odd prime. Consider the graph $G_1 \times G_2$.

Let $V(G_1) = \{0, 1, 2, \dots, 2p - 1\} = V_1$

$V(G_2) = \{2, p, 2p\} = V_2$ and

$V(G_1 \times G_2) = V_1 \times V_2 = V$

be the set of vertices of the graphs G_1, G_2 and $G_1 \times G_2$ respectively.

Consider a dominating set D_1 of G_1 with cardinality 2, given by $D_1 = \{u_{d_1}, u_{d_2}\}$ where $|u_{d_1} - u_{d_2}| = p$,

(Theorem 1.2). To obtain a dominating set of $G_1 \times G_2$, consider the set D of vertices in $G_1 \times G_2$ given by $D = \{(u_{d_1}, 2), (u_{d_1}, p), (u_{d_1}, 2p), (u_{d_2}, 2), (u_{d_2}, p), (u_{d_2}, 2p)\}$ We now prove that D is a dominating set of $G_1 \times G_2$ as follows.

Let (u, v) be any vertex in $V - D$ in $G_1 \times G_2$. Then the vertex u in G_1 is adjacent to either u_{d_1} or u_{d_2} in D_1. Since $n = 2p$, there are no isolated vertices in G_2. Hence the vertex v in G_2 is adjacent to at least one vertex of $\{2, p, 2p\}$ in G_2. Thus by the definition of direct product, the vertex (u, v) in $V - D$ is adjacent to either $(u_{d_1}, 2)$ or (u_{d_1}, p) or $(u_{d_1}, 2p)$; $(u_{d_2}, 2)$ or (u_{d_2}, p) or $(u_{d_2}, 2p)$ in D. Since (u, v) is an arbitrary vertex in $V - D$ it follows that D is a dominating set of $G_1 \times G_2$.

We now show that D is minimal. For this consider any vertex in D, say (u_{d_1}, p). Suppose we delete this vertex from D. Here the vertex u_{d_1} is non-adjacent to u_{d_2}, because $|u_{d_1} - u_{d_2}| = p$ and hence GCD $(u_{d_1} - u_{d_2}, n) \neq 1$. This implies that the vertex (u_{d_1}, p) is non-adjacent to the vertices $(u_{d_2}, 2), (u_{d_2}, p), (u_{d_2}, 2p)$. Also (u_{d_1}, p) is non-adjacent to the vertices $(u_{d_1}, 2)$, $(u_{d_1}, 2p)$ by the definition of direct product. Thus the vertex (u_{d_1}, p) in $V - D$ is not adjacent to any vertex of $D - \{(u_{d_1}, p)\}$. That is, $D - \{(u_{d_1}, p)\}$ is not a dominating set. Similar is the case with the deletion of any other vertex in D. Hence D is a minimal dominating set of $G_1 \times G_2$. Further if we form a dominating set of $G_1 \times G_2$ in any other manner, then the order of such a set is bigger than that of D due to the properties of prime numbers.

Therefore $\gamma(G_1 \times G_2) = |D| = 6$. ■

Theorem 2.4: If $n \neq p^\alpha$, $n \neq 2p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$ where $\alpha_i \geq 1$, then the domination number of $G_1 \times G_2$ is given by $\gamma(G_1 \times G_2) = (\lambda + 1)k$, where λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n and k is the core of n.

Proof: Let $n \neq p^\alpha$, $n \neq 2p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$ where $\alpha_i \geq 1$. Then by Theorem 1.4 , $\gamma(G_1) = \lambda + 1$.

Let $D_1 = \{u_{d_1}, u_{d_2}, \dots\dots, u_{d_{\lambda+1}}\}$ be a dominating set of G_1 with minimum cardinality $\lambda + 1$. Again by Theorem 1.10, we know that $\gamma_t(G_2) = k$ where k is the core of n. Let $D_2 = \{p_1, p_2, \dots\dots p_{k-1}, p_1 p_2 \dots p_k\}$ be a total dominating set of G_2 with minimum cardinality k.

Now consider the Cartesian product of the sets D_1 and D_2 as follows.

$$D = D_1 \times D_2$$

$$= \{u_{d_1}, u_{d_2}, \dots\dots, u_{d_{\lambda+1}}\} \times \{p_1, p_2, \dots\dots p_{k-1}, p_1 p_2 \dots p_k\}$$

$$= \{(u_{d_1}, p_1), \dots, (u_{d_1}, p_{k-1}), (u_{d_1}, p_1 p_2 \dots p_k),$$
$$(u_{d_2}, p_1), \dots, (u_{d_2}, p_{k-1}), (u_{d_2}, p_1 p_2 \dots p_k),$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$(u_{d_{\lambda+1}}, p_1), \dots (u_{d_{\lambda+1}}, p_{k-1}), (u_{d_{\lambda+1}}, p_1 p_2 \dots p_k)\}.$$

Let (u, v) be any vertex of $V - D$ in $G_1 \times G_2$. Then vertex u is adjacent to some vertex u_{d_l} in D_1 and vertex v is adjacent to either p_j, for $1 \leq j \leq k - 1$ or $p_1 p_2 \dots p_k$ as D_1 and D_2 are dominating sets of G_1 and G_2 respectively. That is vertex (u, v) is adjacent to (u_{d_l}, p_j) or $(u_{d_l}, p_1 p_2 \dots p_k)$ in D.

Thus $D = D_1 \times D_2$ becomes a dominating set of $G_1 \times G_2$. We now prove that deletion of any vertex in D does not make the resulting set a dominating set any more.

Let $(u_x, v_y) \in D$. Then u_x is any one of the vertices $u_{d_1}, u_{d_2}, \dots\dots, u_{d_{\lambda+1}}$ and v_y is any one of the vertices $p_1, p_2, \dots\dots p_{k-1}, p_1 p_2 \dots p_k$. Let $u_x = u_{d_i}$, $1 \leq i \leq \lambda + 1$. Suppose u_x is adjacent to the vertices $u_1, u_2, \dots\dots, u_{\varphi(n)}$ in G_1. To select the vertex v_y, we proceed as follows.

Case 1: Suppose $v_y = p_m$ where $1 \leq m \leq k - 1$. Then v_y is adjacent to $p_m p_k$, as GCD$(p_m p_k, p_m) = p_m$. So by the definition of direct product $(u_x, v_y) = (u_{d_i}, p_m)$ is adjacent to the vertices $(u_1, p_m p_k)$,$(u_2, p_m p_k)$, ….,$(u_{\varphi(n)}, p_m p_k)$.

Suppose we delete vertex $(u_x, v_y) = (u_{d_i}, p_m)$ from D. If we consider the set of vertices $\{(u_1, p_m p_k), (u_2, p_m p_k), \dots, (u_{\varphi(n)}, p_m p_k)\}$ in $V - D$ of $G_1 \times G_2$, then no vertex in this set is adjacent to a vertex in $D - (u_x, v_y)$. This is because vertex $p_m p_k$ is not adjacent to any vertex p_j or vertex $p_1 p_2 \dots p_k$ as

GCD $(p_m p_k, p_j) = 1$ for $m \neq j$, $1 \leq m, j \leq k - 1$ and GCD $(p_m p_k, p_1 p_2 \dots p_k) = p_m p_k$. This means that $D - \{(u_x, v_y)\}$ is not a dominating set.

Case 2: Suppose $v_y = p_1p_2 \dots p_k$. Then v_y is adjacent to p_k, as GCD $(p_1p_2 \dots p_k, p_k) = p_k$. Hence in this case, by the definition of direct product $(u_x, v_y) = (u_{d_i}, p_1p_2 \dots p_k)$ is adjacent to the vertices $(u_1, p_k), (u_2, p_k), \dots, (u_{\varphi(n)}, p_k)$.

Suppose we delete vertex $(u_x, v_y) = (u_{d_i}, p_1p_2 \dots p_k)$ from D. Then the set $\{(u_1, p_k), (u_2, p_k), \dots, (u_{\varphi(n)}, p_k)\}$ in $V - D$ of $G_1 \times G_2$ is such that no vertex in this set is adjacent to a vertex of $D - (u_x, v_y)$ because

GCD $(p_k, p_j) = 1$ for $1 \le j \le k-1$. This implies that $D - (u_x, v_y)$ is not a dominating set. . Thus D becomes a minimal dominating set. Further if we form a dominating set of $G_1 \times G_2$ in any other manner, then the order of such a set is not smaller than that of D. This is due to the properties of prime numbers.

Therefore $\gamma(G_1 \times G_2) = |D| = (\lambda + 1) \cdot k$.

3. TOTAL DOMINATION IN DIRECT PRODUCT GRAPH $G_1 \times G_2$

In this section the results on the total dominating sets of direct product graph $G_1 \times G_2$ are discussed for different values of n.

$G_1 \times G_2$ is a completely disconnected graph on n vertices, if n is a prime. So there are no edges between these n vertices. Therefore total dominating set does not exist for $G_1 \times G_2$ when n is a prime.

Theorem 3.1: The total domination number of $G_1 \times G_2$ is 4, if n is power of a prime.

Proof: Let $n = p^\alpha$, $\alpha > 1$. Consider the graph $G_1 \times G_2$. By Theorem 2.2 we know that $\gamma(G_1 \times G_2) = 4$ for $n = p^\alpha$. So we have $\gamma_t(G_1 \times G_2) \ge 4$. To get a total dominating set of $G_1 \times G_2$, we proceed along the lines of Theorem 2.2 and hence $T = \{(u_{d_1}, p), (u_{d_1}, p^j), (u_{d_2}, p), (u_{d_2}, p^j)\}$, j > 1 is a dominating set of $G_1 \times G_2$ with minimum cardinality.

We now show that T is a total dominating set by showing that the vertices in T dominate among themselves. Since u_{d_1}, u_{d_2} are consecutive integers, we have GCD $(u_{d_1} - u_{d_2}, n) = 1$ and hence by the definition of adjacency in G_1, vertices u_{d_1}, u_{d_2} are adjacent to each other.

Further for j > 1, we have GCD $(p, p^j) = p$. Hence by the definition of adjacency in G_2, vertices p and p^j are adjacent to each other. Therefore by the definition of direct product, $(u_{d_1}, p), (u_{d_2}, p^j)$; $(u_{d_2}, p), (u_{d_1}, p^j)$ are the pairs of adjacent vertices in T. Thus T becomes a total dominating set with minimum cardinality.

Therefore $\gamma_t(G_1 \times G_2) = 4$. ∎

Theorem 3.2: If n is not power of a single prime and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $\alpha_i \ge 1$, $k > 1$, then the total domination number of $G_1 \times G_2$ is given by $\gamma_t(G_1 \times G_2) = (\lambda + 1) \cdot k$, where λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n and k is the core of n.

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $\alpha_i \ge 1$, $k > 1$.

By Theorem 2.4 , we know that $\gamma(G_1 \times G_2) = (\lambda + 1) \cdot k$ where λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n and k is the core of n.

Therefore $\gamma_t(G_1 \times G_2) \ge (\lambda + 1) \cdot k$

Without loss of generality, we take

$$T = \{u_{d_1}, u_{d_2}, \dots, u_{d_{\lambda+1}}\} \times \{p_1, p_2, \dots, p_{k-1}, p_1p_2 \dots p_k\}$$

$$= \{(u_{d_1}, p_1), \dots, (u_{d_1}, p_{k-1}), (u_{d_1}, p_1p_2 \dots p_k), (u_{d_2}, p_1), \dots, (u_{d_2}, p_{k-1}), (u_{d_2}, p_1p_2 \dots p_k), \vdots \quad \vdots \quad \vdots \quad (u_{d_{\lambda+1}}, p_1) \dots, (u_{d_{\lambda+1}}, p_{k-1}), (u_{d_{\lambda+1}}, p_1p_2 \dots p_k)\}$$

where $u_{d_1}, u_{d_2}, \dots, u_{d_{\lambda+1}}$ are consecutive integers and $p_1, p_2, \dots, p_{k-1}, p_1p_2 \dots p_k$ are prime factors of n. Then as in Theorem 2.4, we can show that T is a dominating set of $G_1 \times G_2$ with minimum cardinality.

We now show that vertices in T are dominated by the vertices of T. Since $u_{d_1}, u_{d_2}, \dots, u_{d_{\lambda+1}}$ are consecutive integers, each vertex u_{d_i} for $1 \le i \le \lambda$ is adjacent to its succeeding vertex $u_{d_{i+1}}$, as GCD $(u_{d_{i+1}} - u_{d_i}, n) = 1$. Further GCD $(p_j, p_1p_2 \dots p_k) = p_j$, for $1 \le j \le k-1$, implies that each one of $p_1, p_2, \dots, p_{k-1}$ is adjacent to $p_1p_2 \dots p_k$. Hence by the nature of adjacency in direct product, each vertex (u_{d_i}, p_j) in T is adjacent to $(u_{d_{i+1}}, p_1p_2 \dots p_k)$ in T . This implies that the vertices in T are dominated by the vertices of T. Thus T becomes a total dominating set of $G_1 \times G_2$.

Since $\gamma(G_1 \times G_2) = (\lambda + 1) \cdot k$, and $|T| = (\lambda + 1) \cdot k$, it follows that T is a total dominating set of $G_1 \times G_2$ with minimum cardinality.

Therefore $\gamma_t(G_1 \times G_2) = (\lambda + 1) \cdot k$. ∎

ILLUSTRATIONS

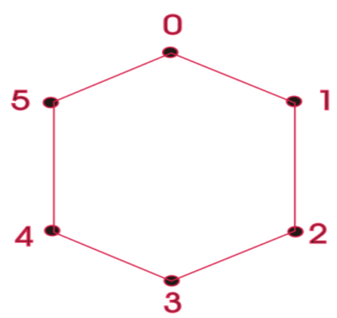

Fig. 1

$\mathbf{G_1 = G(Z_6, \varphi)}$

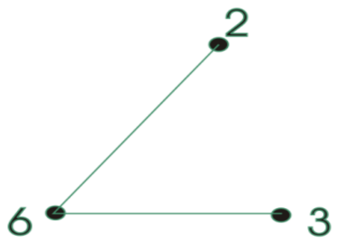

Fig. 2

$\mathbf{G_2 = G(V_6)}$

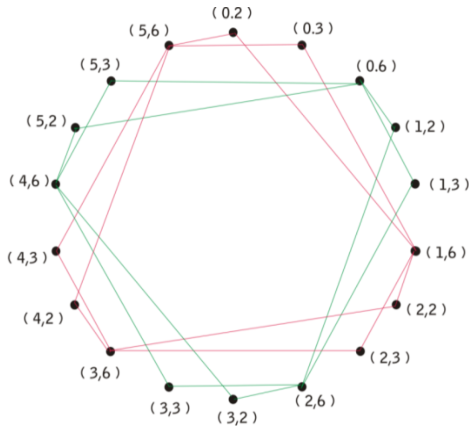

Fig. 3

$\boldsymbol{G_1} \times \boldsymbol{G_2}$ is a disconnected graph

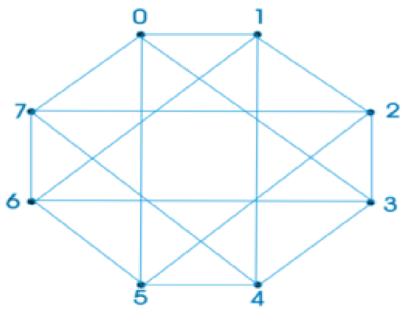

Fig. 4

$\boldsymbol{G_1 = G(Z_8, \varphi)}$

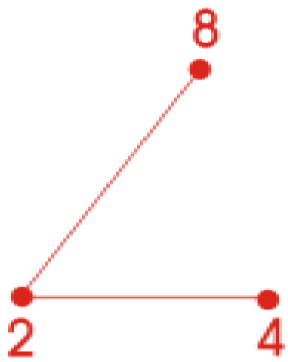

Fig. 5

$\boldsymbol{G_2 = G(V_8)}$

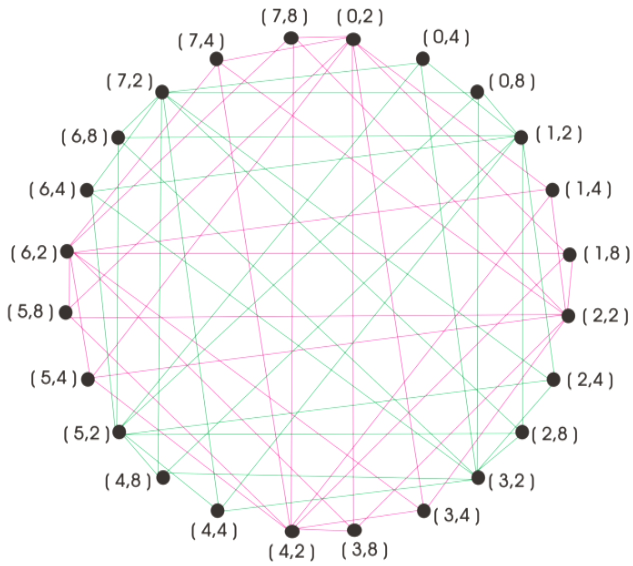

Fig. 6

$\boldsymbol{G_1} \times \boldsymbol{G_2}$ is a disconnected graph

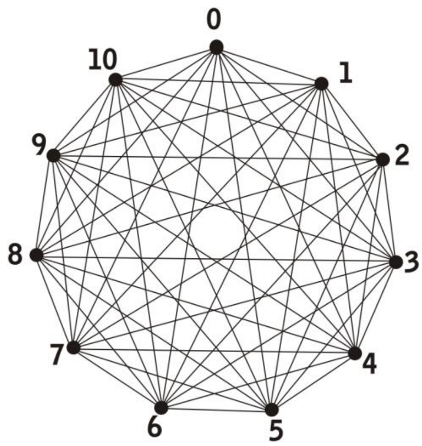

Fig. 7

$\boldsymbol{G_1 = G(Z_{11}, \varphi)}$

Fig. 8

$G_2 = G(V_{11})$

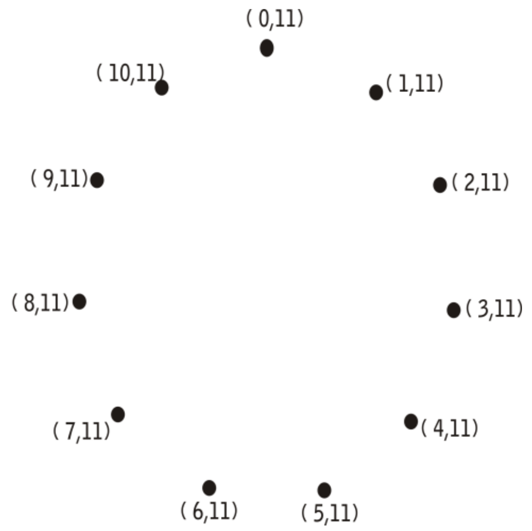

Fig. 9

$G_1 \times G_2$ is a completely disconnected graph

Table 1. Dominations in Direct Product Graph $G_1 \times G_2$

n Values	Dominating sets	$G_1 = \mathrm{G}(\mathrm{Z_n}, \varphi)$	$G_2 = G(V_n)$	$G_1 \times G_2$	Domination Number in $G_1 \times G_2$
$n = 6$	Minimum Dominating set	{0,3}	{6}	{(0,2), (0,3), (0,6), (3,2), (3,3), (3,6)}	$\gamma = 6$
	Minimum Total Dominating set	{0,1,3,4}	{2,6}	{(0,2), (0,6), (1,2), (1,6), (3,2), (3,6), (4,2), (4,6)}	$\gamma_t = 8$
$n = 8$	Minimum Dominating set	{0,1}	{2}	{(0,2), (0,8), (1,2), (1,8)}	$\gamma = 4$
	Minimum Total Dominating set	{0,1}	{2,8}	{(0,2), (0,8), (1,2), (1,8)}	$\gamma_t = 4$
$n = 11$	Minimum Dominating set	{0}	{11}	Vertex set of $G_1 \times G_2$	$\gamma = 11$
	Minimum Total Dominating set	{0,1}	Does not exist	Null graph	γ_t does not exist

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