New Escape Time Koch Curve in Complex Plane

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ABSTRACT
Von Koch curves allow numerous variations and have inspired many researchers and fractal artists to produce amazing pieces of art. In this paper we present a new algorithm for plotting the Koch curve using complex variables. Further we have applied various coloring algorithms to generate complex Koch fractals.

Keywords
Koch curves, Fractal Coloring, Escape Time Algorithm.

1. INTRODUCTION

A fractal is an object or quantity that displays self-similarity on all scales. The theory behind fractals started taking shape as early as in the 17th Century, when the concept of self similarity was first brought up by the philosopher and mathematician Lebinz.[14] In 1872, Weierstrass discovered a curve which was continuous everywhere and differentiable nowhere[24]. Encouraged by Weierstrass's example, Helge Von Koch developed one of history's first fractals by defining a simple geometric algorithm for manipulating a line [1].

In 1915, Wallow Sierpinski created a triangle and a carpet. Originally these were described as curves rather than the 2D shapes. In 1938, Paul Pierre Lévy described a new fractal curve known as the the Lévy C curve. Georg Cantor provided new fractals known as Cantor sets which are examples of subsets of the real line with unusual properties [10].

Although much work was done in the past, it was Mandelbrot who brought the Fractals in front of the world [14]. He expressed the Fractals as unions of rescaled copies of themselves.[13]. The Cantor set, Von Koch snowflake curve and the Sierpinski gasket are some of the most famous examples of such sets. Mandelbrot used the Koch curves as the base for his discovery on fractals [7].

The Koch curves were introduced by Swedish mathematician Helge Von Koch in 1904 and are famous example of fractal geometry. These are deterministic fractals which exhibit self similarity [11].

Fractals capture iterations as well as algebraic conditions of angle and scale to original size. Unlike function-based curves of the form y = f(x), a Koch curve is nowhere continuous, and therefore has no tangent anywhere along its length. As the domain of a smooth curve narrows, it progressively resembles a straight line. The Koch Curve has a geometric property like that of f(x) = |x| at x = 0, where no tangent is defined.[15] Because of its unique construction the Koch curve has a noninteger dimensionality. In the Koch curve, a fractal pattern of 60-degree-to-base line segments one-third the length of the previous line is repeated. The portion of the base line under a newly formed triangle is deleted. Such geometric manipulation continued indefinitely, forms the Koch Curve. [23]

The Von Koch Curve shows the self-similar property of a fractal. The same pattern appears everywhere along the curve in different scale, from visible to infinitesimal. The Koch curve has a Hausdorff dimension of ***log***(4)/log(3), i.e. approximately 1.2619 and is enclosed in a finite area [8].

In their paper [22] they have suggested two other different methods of generating the Koch curves. The **attractive method** is based on a characterization of Koch curves as the smallest nonempty sets closed with respect to a union of similarities on the plane. This characterization was first studied by Hutchinson [23]. The **repelling method** is in principle dual to the previous one, but involves a nontrivial problem of selecting the appropriate transformation to be applied at each iteration step.

Koch curves don't exist in mathematical isolation, but can be combined to form the famous Koch Snowflake [6]. The Koch snowflake is a fractal which is created by repeating the process of the generating a Koch curve, infinite number of times on the three sides of an equilateral triangle. The first iteration looks similar to a Star of David with internal line segments missing. Subsequent iterations on each side of the Koch Snowflake reveal the "snowflake" resemblance.

The perimeter of the Koch Snowflake lengthens infinitely while its area is finite [12]. The Koch Snowflake has perimeter that increases by 4/3 of the previous perimeter for each iteration and an area that is $8/5^{th}$ of the original triangle. The Koch Snowflake exhibits the scale invariance symmetry.

2. ESCAPE TIME ALGORITHM

The simplest algorithm for generating a representation of the Mandelbrot and Julia sets in a complex plane is known as the "escape time algorithm". A repeating calculation is performed for each point based on the behavior of that calculation, a color is chosen for that pixel [5].

The ***x*** and ***y*** coordinates of each point are used as starting values in iterating the calculation. The result of each iteration is used as the starting values for the next. The values are checked during each iteration to see if they have reached a critical 'escape' condition also known as 'bailout' [6]. The calculation stops upon reaching this critical condition and the pixel is then drawn and then next ***x***, ***y*** point is examined. For some starting values, escape occurs quickly, after only a small number of iterations. For some values very close to but not in the set, it may take hundreds or thousands of iterations to escape. For values within the Mandelbrot set, escape will never occur. For higher number of iterations, the time taken to calculate the fractal image will be longer but the final image will be very clear and detailed. Escape conditions can be simple or complex. Because no complex number with a real or imaginary part greater than 2 can be part of the set, a common bailout is to escape when either coefficient exceeds 2 [16].

The color of each point represents how quickly the values reached the escape point. Often black is used to show values that fail to escape before the iteration limit, and gradually brighter colors are used for points that escape. This gives a visual representation of how many cycles were required before reaching the escape condition [19].

Koch Curve [22] :

$$T(f)(z) = \int_T \frac{f()d\mu\ (C)}{C-z}, z \in ð \qquad \dots (1)$$

Where $\boldsymbol{T} \in \boldsymbol{C}$ is a Von Koch curve, μ a finite measure of ***T***, and f:$\boldsymbol{T} \rightarrow \boldsymbol{C}$ is essentially bounded.

3. FRACTAL COLORING

Fractals have been investigated for their visual qualities as art, their relationship to explain natural processes, music, medicine, and in mathematics [21]. We can see structures resembling Fractals in the leaves, in the course of a river, in the shape of a broccoli, in our arterial system, and in clouds. Fractal art is a new way of looking at space and various forms. With the introduction of computers and all of its associated peripherals a whole new vision of the world has been found [3]. Fractal Art is a subclass of two-dimensional visual art, and is in many respects similar to photography. Fractal art can be created easily by using many coloring algorithms like divergence algorithm, convergence algorithms decomposition algorithm, orbit trap etc [2].

In this paper we have used escape time algorithm for generating the complex Koch curves. We have used circles instead of triangle as the building block for the Koch curve. We have further applied the orbit trap algorithm to create fractal art and have created beautiful variants of the Koch curve Fractals.

In the Orbit Trap function, a region on the complex plane is chosen (which can be a point, a line, a spiral or etc.), then the behavior of the ***z*** values with respect to this region is investigated. If the orbit of a point falls into the chosen region then that point is taken as trapped. Then the iteration ends and the point is colored based on the distance to the center of the chosen region.

4. ALGORITHM FOR COMPLEX KOCH CURVE

Initialize ***z*** = coordinate (***x***, ***y***), float theta = atan2(z), Boolean bail = false,

Initialize float mag=1.7, integer n, integer i = 0

```
loop:
if i > 1
if (orbit is escaped)  OR  (orbit is  both bounded as well
as escaped  AND  i >1)
if  mod of  z is less than 1
assign value true to bail
endif
elseif (orbit is bounded)  OR  (orbit is  both escaped as
well as escaped  AND i=2)
if (mod of z > 1)
assign value true to bail
endif
endif
assign atan2(z) to theta
if  rounded value of (n/(2*3.14)*theta = 0) AND (i > 2)
if (theta > 0)
assign value 2*3.14/n to theta2
else
assign value -2*3.14/n to theta2
endif
endif
assign  (z*exp(-1i*theta2)) to z
if orbit is escaped  then  assign  (- z + (1 + 1/mag^2)) to z
elseif orbit is  bounded  then  assign  (- z + (1 -
1/mag^2)) to z
elseif (orbit is bounded as well as escaped and i =2 ) then
assign (z - (1 - 1/mag^2)) to z
elseif  (orbit is bounded as well as escaped and i >2) then
assign (z + (1 + 1/mag^2)) to z
endif
assign mag*z to z
endif
end loop
```

5. ESCAPE TIME COMPLEX KOCH CURVES USING FRACTAL ART

The Koch curve fractals generated by our method exhibits n-way rotational symmetry and scale variant symmetry. When n is "***odd***" we observe the circular Koch fractals whereas when ***n*** is "even" we get Koch curves of square, hexagon and octagon shapes. The sides of the Koch curves are directly proportional to the order of ***n***. We also observe that there exists broccoli or tree like structure buried inside some variants of the Koch curve fractals.

We have generated the following figures of complex koch curves for two cases:

a. When the orbit of the points is bounded by the iterations

b. When the orbit of the points escapes.

6. CONCLUSIONS AND REMARKS

In this paper we have used escape time algorithm to generate the images that gives a new aspect to the Koch curve Fractals. We have applied the various coloring functions and schemes to the fractals and have generated some beautiful artistic variants of the Koch curves fractals. We also observe that the resulting Koch curves using our method are not as strictly self-similar than the Koch curves generated by generator iterations or IFS iterations.

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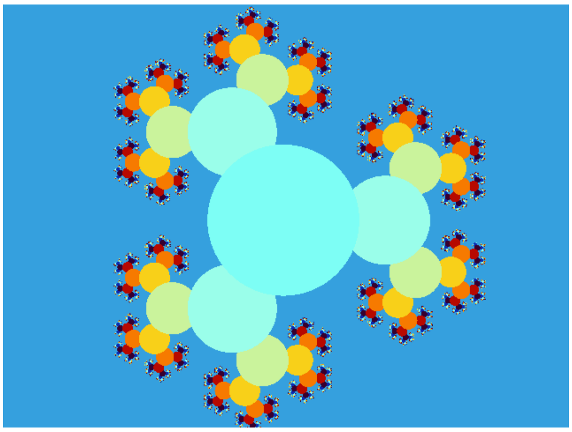

Fig 1. Koch curves**:** orbit=escapes, n=3, magnification scale=1.7

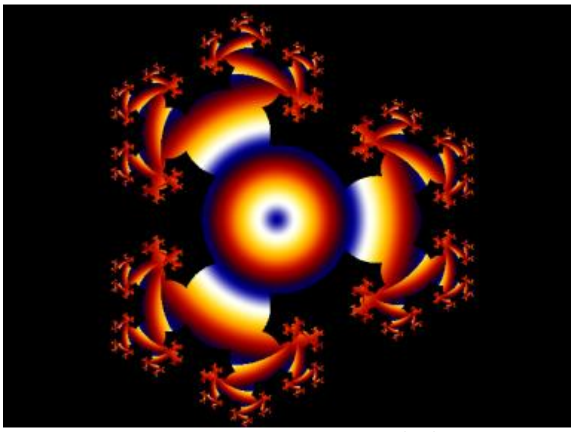

Fig 2. Diving Frogs: orbit=escapes, n=3, Transfer function =Linear, Trap shape= ring apples, Trap coloring =magnitude

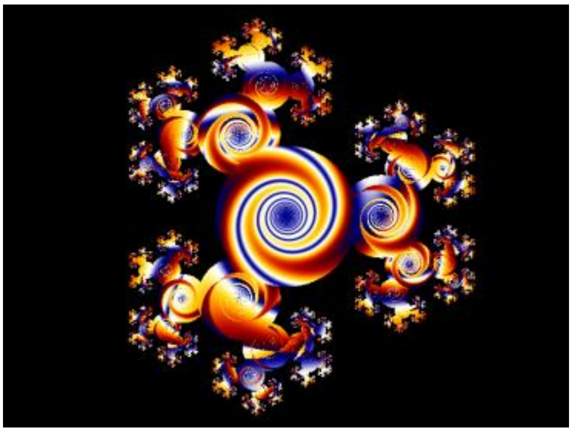

Fig 3: Chakra: orbit= escapes, n=3, Transfer function =Linear, Trap shape= spiral, diameter=2,Trap coloring =real, Trap mode=closest

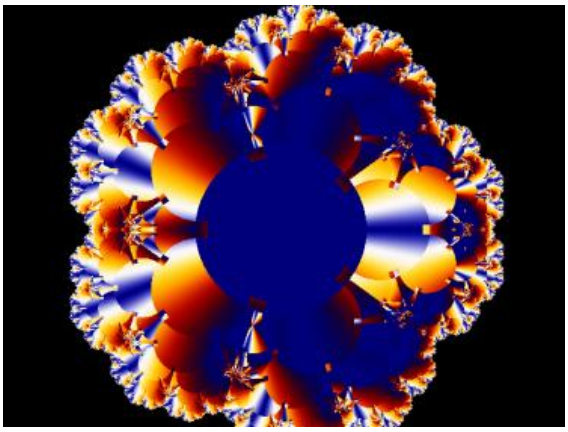

Fig 4: Blue sapphire: orbit= escapes, n=5, Transfer function = Sin, Trap shape= grid apples, Diameter =3, Order=3, Frequency=1, Trap coloring =angle to origin, Trap mode=sum

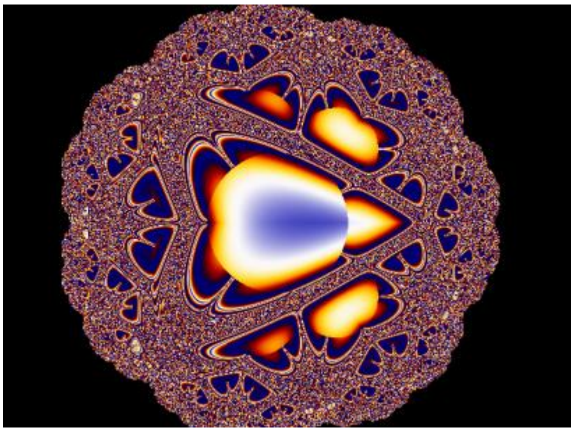

Fig 5: Pearl: orbit= escapes, n=7,Trap Shape=heart, diameter=2,order=4,trap coloring=distance, trap mode=inverted sum squared

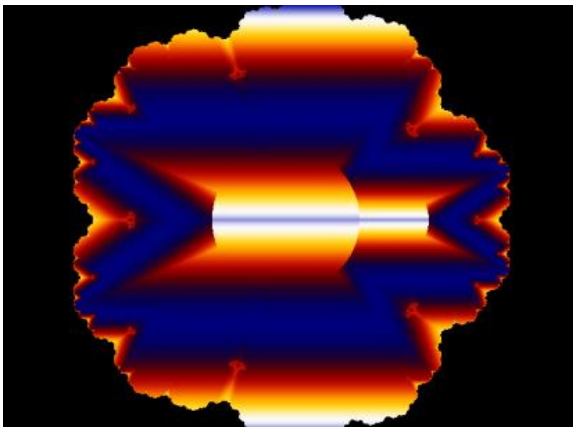

Fig 6:The Horizon: orbit= escapes, n=7,trap shape=lines, Diameter=3, Trap Coloring =distance, Trap mode=closest

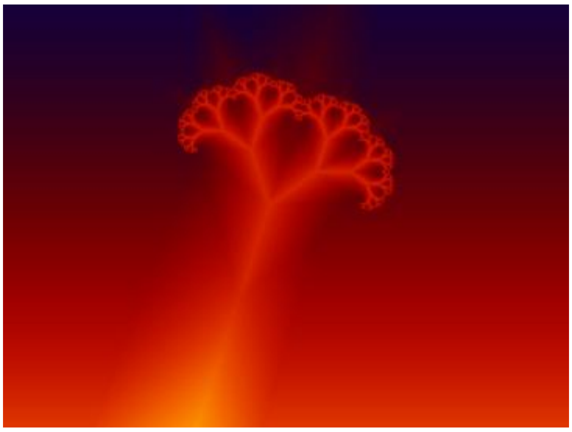

Fig 7 : Magnification of **Fig 6**. Notice the broccoli like Fractal Trees

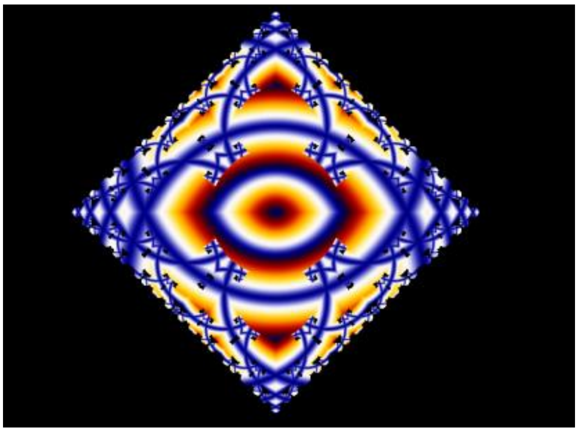

Fig 8: The Third Eye: orbit= escapes, n=7, Transfer Function=linear, Trap Shape=radial waves 2, Diameter=2, order=4,Frequency=1, Trap Coloring=distance, Trap Mode=closest. The iteration limit N= 7

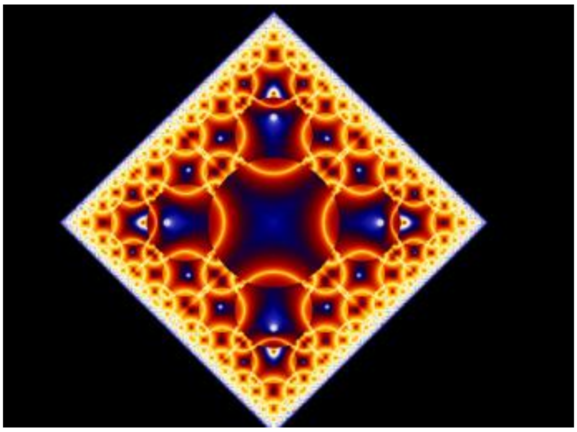

Fig 9: The Square Carpet: orbit= escapes e, n=4, Transfer function=cube root, algorithm=distance estimator, color density=16

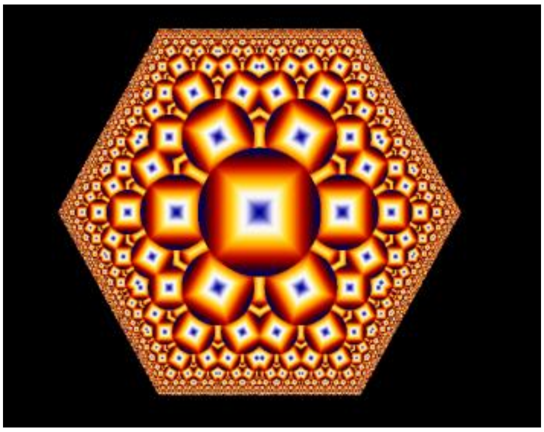

Fig 10: The Hex carpet: orbit= escapes, n=6,Trap Shape=rectangle, Trap Coloring=distance, Trap Mode-closest

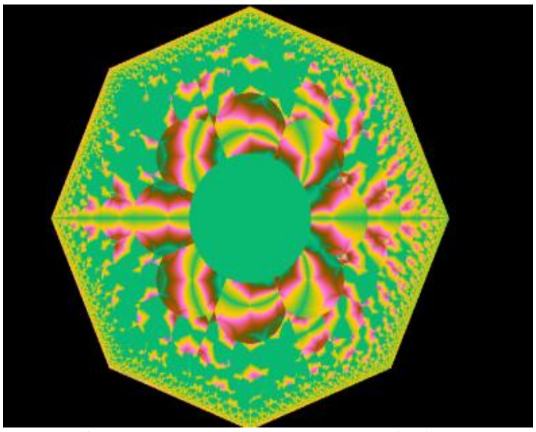

Fig 11: Core of Earth: orbit= escapes, n=8, Transfer function=cube root, algorithm=Triangle inequality Average, distance estimator, color density=16

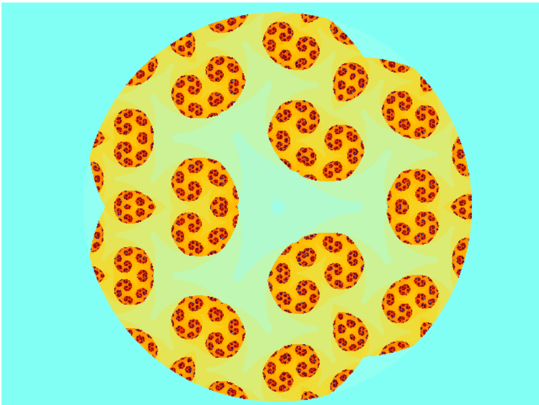

***Fig 12: Koch curves*:** orbit=bounded, n=3, magnification scale=1.5

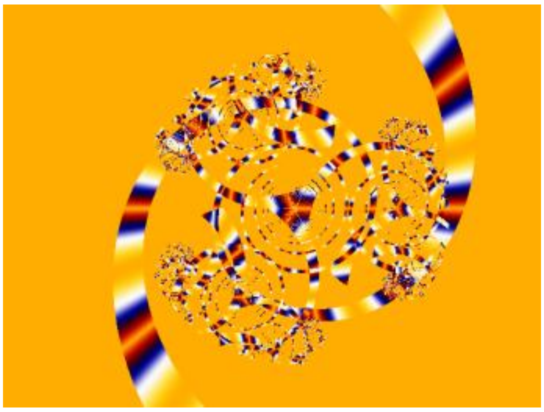

Fig 13: Chakra: orbit= bounded, n=3, Transfer Function =sin, trap mode=spiral, diameter=2, trap coloring=angle to origin 2, trap mode=sum

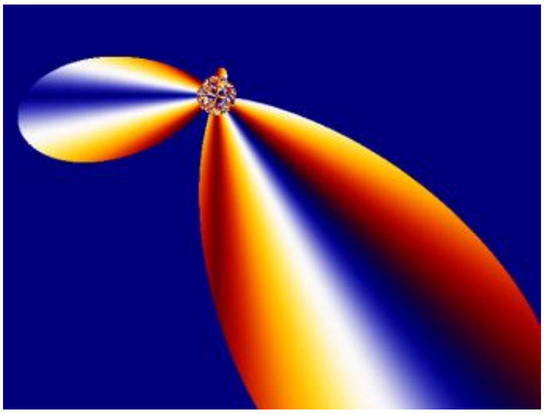

Fig: 14: The Rainbow: orbit= bounded, n=4, Transfer Function =linear, trap shape=pinch, order=3, trap coloring =angle to origin 2, trap mode=average change

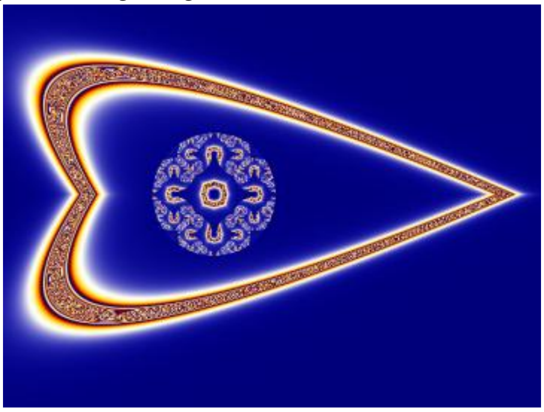

Fig 15: The Pendant: orbit= bounded, n=4, Trap =Heart ,Diameter=3, Order=3, Coloring =distance, Mode=inverted sum squared

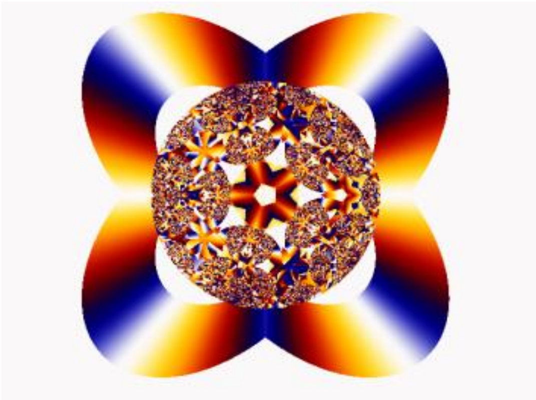

Fig: 16 : Diamond Stud: orbit=bounded n=5, trap shape=radial waves 2, diameter =1, order=3, frequency =2, trap coloring =angle to origin 2 trap mode=alternating average

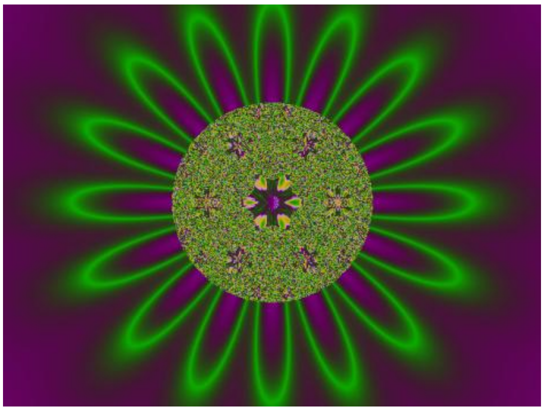

Fig 17: The Trap: orbit= bounded, n=6, Trap Shape=radial ripples, order=4, Frequency=2, Trap Coloring=magnitude, mode=inverted sum squared, Transfer Function=cube root, Color Density=0.5

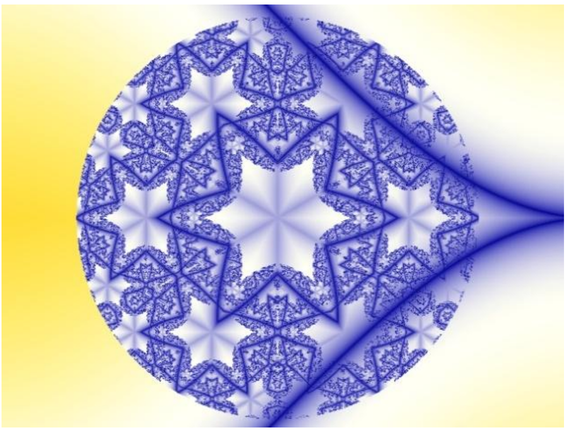

Fig 18: Complex Koch Snowflake: orbit= bounded, n=6, Trap Shape=mirrored wave, order=4, Frequency=1, Trap Coloring=distance, mode=closest, Transfer Function=cube root