Strongly b*- Continuous Functions in Topological Spaces

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ABSTRACT

In this paper, we present and study a new generation of strongly b*-continuous functions. Furthermore, we obtain basic properties and preservation theorems of strongly b*- continuous functions and relationships between them. Also we studied the strongly b*- open and closed maps.

General Terms

2000 Mathematics Subject Classification: 54C05, 54C10.

Keywords

strongly b*- continuous functions, strongly b*-open maps and closed maps.

1. INTRODUCTION

Levine[11] introduced the concept of generalized closed sets in topological spaces and a class of topological spaces called $T_{1/2}$ - spaces. Dunham[7] and Dunham and Levine [8] further studied some properties of generalized closed sets and $T_{1/2}$- spaces. Strong forms of continuous maps have been introduced and investigated by several mathematicians. strongly continuous maps, perfectly continuous maps, completely continuous maps, clopen continuous maps were introduced by Levine[13], Noiri[18], Munshi and Bassan[15] and Reilly and Vamanamurthy[20] respectively. Semi continuous functions have been studied by several authors. Dontchev[5], Ganster and Reilly[6] introduced contra-continuous functions and regular set - connected functions. Erdal Ekici [9] introduced and studied a new class of functions called almost contra-pre- continuous functions which generalize classes of regular set - connected [6], contra- pre continuous [11], contra continuous [5], almost s - continuous [17] and perfectly continuous functions [18]. In this paper, we introduce and study the strongly b* - continuous functions in topological spaces. Also we studied the strongly b*- open and closed maps.

2. PRELIMINARIES

In this section, we begin by recalling some definitions

Definition 2.1[21]: A map f: X $\rightarrow$ Y from a topological space X into a topological space Y is called semi- generalized continuous (sg-continuous) if $f^{-1}(V)$ is sg- closed in X for every closed set V of Y.

Definition 2.2[3]: A map f: X $\rightarrow$ Y is semi-continuous if and only if for every closed set B of Y, $f^{-1}(B)$ is semi-closed in X.

Definition 2.3[2]: A function f: X $\rightarrow$ Y is said to be generalized continuous (g-continuous) if $f^{-1}(V)$ is g-open in X for each open set V of Y.

Definition 2.4[10]: A function f: X $\rightarrow$ Y is said to be b-continuous if for each $x \in X$ and for each open set of V of Y containing f(x), there exists $U \in bO(X, x)$ such that $f(U) \subseteq V$.

Definition 2.5[22]: A function f: X $\rightarrow$ Y is said to be w-continuous if $f^{-1}(V)$ is w - open in X for each open set V of Y.

Definition: 2.6 [14]: A function f: X $\rightarrow$ Y is said to be α- continuous if $f^{-1}(V)$ is α- open in X for each open set V of Y.

Definition: 2.7 [16]: Let X and Y be topological spaces. A map f: X $\rightarrow$ Y is said to be weakly generalized continuous (wg-continuous) if the inverse image of every open set in Y is wg-open in X.

Definition 2.8:[4] A function f: X $\rightarrow$ Y is said to be αg- continuous if $f^{-1}(V)$is αg - open in X for each open set V of Y.

Definition 2.9[1]: A map f: X $\rightarrow$ Y is semi pre-continuous if and only if for every closed set B of Y, f^{-1}(B)is semi pre-closed in X.

Definition 2.10[19]: A subset A of a topological space (X, τ) is called a strongly b*- closed set (briefly sb*- closed) if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is b open in X.

3. STRONGLY b* - CONTINUOUS FUNCTIONS

In this section, we introduce the new class of definition sb*-continuous function in topological space. Also we discuss some of its properties.

Definition 3.1: Let X and Y be topological spaces. A map f: X $\rightarrow$ Y is called strongly b* - continuous (sb*- continuous) if the inverse image of every open set in Y is sb* - open in X.

Theorem 3.2: If a map f: X $\rightarrow$ Y is continuous then it is sb* - continuous but not conversely.

Proof: Let f: X $\rightarrow$ Y be continuous. Let F be any open set in Y. The inverse image of F is open in X. Since every open set is sb*-open set, inverse image of F is sb*- open set in X. Therefore f is sb* - continuous.

Remark 3.3: The converse of the above theorem need not be true as seen from the following example.

Example 3.4: Consider X={1, 2, 3} with τ = {X, φ, {1,3}}, Y = {a, b, c} and σ = {Y, φ, {b}, {a, c}}. Let f:(X, τ) $\to$ (Y, σ) be defined by f(1)=a, f(3)=b, f(2)=c. Then f is sb*-continuous. But f is not continuous since for the open set U = {a,c} in Y, f^{-1}(U) = {1,2} is not open in X.

Theorem 3.5: Let f: (X, τ) $\to$ (Y, σ) be a map from a topological space (X, τ) in to a topological space (Y, σ). The statement (a) f is sb* - continuous is equivalent to the statement (b) the inverse image of each open set in Y is sb*-open in X.

Proof: Assume that f : X $\to$ Y is sb*-continuous. Let G be open in Y. Then G^c is closed in Y. Since f is sb*-continuous, $f^{-1}(G^c)$ is sb* -closed in X. But $f^{-1}(G^c)$= X - f^{-1}(G). Thus X-f^{-1}(G) is sb*-closed in X and so f^{-1}(G) is sb*-open in X. Therefore (a) $\Longrightarrow$ (b).

Conversely, assume that the inverse image of each open set in Y is sb*- open in X. Let F be any closed set in Y. Then $f^{-1}(F^c)$ is sb* - open in X. But $f^{-1}(F^c)$ = X – f^{-1} (F). Thus X - f^{-1} (F) is sb* - open in X and so f^{-1}(F) is sb*-closed in X. Therefore f is sb*-continuous. Hence (b) $\Longrightarrow$ (a). Thus (a) and (b) are equivalent.

Theorem 3.6: Let f : X $\to$ Y be a sb* - continuous map from a topological space X in to a topological space Y and let H be a closed subset of X. Then the restriction f/H: H $\to$ Y is sb* - continuous where H is endowed with the relative topology.

Proof: Let F be any closed subset in Y. Since f is sb* - continuous, f^{-1}(F) is sb* - closed in X. Intersection of sb*-closed sets is sb* - closed set. Thus if $f^{-1}(F) \cap H = H_1$ then H_1 is sb* - closed set in X. Since $(f/H)^{-1}(F) = H_1$, it is sufficient to show that H_1 is sb* - closed set in H. Let G_1 be any open set of H such that $H_1 \subset G_1$. Let $G_1 = G \cap H$ where G is open in X. Now $H_1 \subset G \cap H \cap G$. Since H_1 is sb* - closed in X, $\overline{H_1} \subset G$. Now $cl_H(H_1) = \overline{H_1} \cap H \subset G \cap H = G_1$, where cl_H(A) is the closure of a subset A $\subset$ H in a subspace H of X. Therefore f/H is sb* - continuous.

Remark 3.7: In the above theorem, the assumption of closedness of H cannot be removed as seen from the following example.

Example 3.8: Let X = {a,b,c}, τ = {X, φ, {b}}, Y = {p,q} and σ = {Y, φ, {p}}. Let f:(X,τ) $\to$ (Y,σ) be defined by f(a)=f(c)=q, f(b)=p. Now H = {a,b} is not closed in X. Then f is sb* - continuous but the restriction f/H is not sb*-continuous. Since for the closed set F = {q} in Y, f^{-1}(F) = {a,c} and $f^{-1}(F) \cap H$ = {a} is not sb*-closed in H.

Theorem 3.9: A map f: X $\to$ Y is sb* - continuous if and only if the inverse image of every closed set in Y is sb* - closed in X.

Proof: Let F be a closed set in Y. Then F^c is open in Y. Since f is sb* -continuous, $f^{-1}(F^c)$ is sb* - open in X. But $f^{-1}(F^c)$ = X - f^{-1}(F) and so f^{-1}(F) is sb* - closed in X.

Conversely, let the inverse image of every closed set in Y is sb* - closed set in X. Let V be an open set in Y and V^c is closed in Y. Now by the assumption $f^{-1}(V^c)$ = X - f^{-1}(V) is sb* - closed set in Y. Therefore f^{-1}(V) is sb* - open in X. Then f is sb* - continuous.

Theorem 3.10: If a function f: X $\to$ Y is sb* -continuous then it is b-continuous but not conversely.

Proof: Assume that a map f: X $\to$ Y is sb* - continuous. let V be an open set in Y. Since f is sb* - continuous f^{-1} (V) is sb* -open and hence b - open in X. Therefore f is b - continuous

Remark 3.11: The converse of the above theorem need not be true as seen from the following example.

Example 3.12: Let X = Y = {a,b,c} with τ = {X, φ, {a}, {c}, {a,c}} , σ = {Y, φ, {b}, {c}, {b,c}} and f = {(a,b),(b,b),(c,c)}. Then f is b-continuous but not sb*-continuous. Since the inverse image of the open set {b} in Y is {a,b} in X is not sb* - open.

Theorem 3.13: If a map f: X $\to$ Y is α- continuous then it is sb*-continuous but not conversely.

Proof: Assume that f is α- continuous. Let V be an open set in Y. Since f is α- continuous, f^{-1}(V) is α-open and hence it is sb*-open in X. Then f is sb*-continuous.

Remark 3.14: The converse of the above theorem be true as seen from the following example.

Example 3.15: Let X = Y = {a,b,c} with τ = {X, φ, {b}, {a, c}} and σ = {Y, φ, {a, c}}. Consider f: X $\to$ Y which is defined as f(a) = f(b) = b, f(c) = c. This function f is sb*- continuous but not α - continuous, Since the pre image of the open set {a,c} in Y is {c}in X is not α-open.

Theorem 3.16: If a map f : X $\to$ Y is sb* - continuous then it is wg-continuous but not conversely.

Proof: Assume that a map f : X $\to$ Y is sb*- continuous. Let V be an open set in Y. Since f is sb* - continuous, f^{-1}(V) is sb*-open and hence it is wg-open in X. Then f is wg-continuous.

Remark 3.17: The converse of the above theorem need not be true as seen from the following exampl

Example 3.18: Let X = Y = {a,b,c} with τ = {X, φ, {b}} and σ = {Y, φ, {a}, {a, b}} and f be the identity map. Then f is wg-continuous but not sb*-continuous, as the inverse image of the open set {a} in Y is {a} in X is not sb*- open.

Theorem 3.19: If a map f: X $\to$ Y is w-continuous then it is sb*- continuous but not conversely.

Proof: Let f: X $\to$ Y is w-continuous and V be an open set in Y then f^{-1}(V) is w- open and hence sb* - open in X. Then f is sb* - continuous. The converse of the above theorem need not be true as seen from the following example.

Exmple 3.20: Let X = Y = {a,b,c} with τ = {X, φ, {b}} and σ = {Y, φ, {b, c}} and f be the identity map . Then f is sb* - continuous but not w-continuous, as the inverse image of the open set {b,c} in Y is {b,c} in X is not w- open.

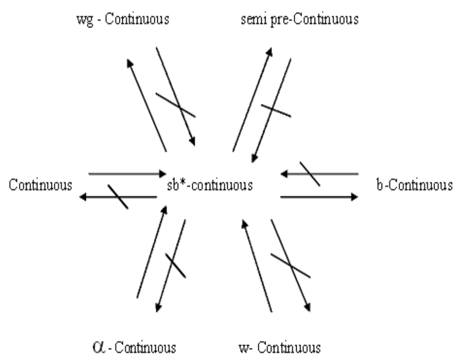
Theorem 3.21: If a map f: X $\to$ Y is sb*-continuous then it is semi pre continuous but not conversely

Proof: Let f: X $\to$ Y is sb*-continuous and V be an open set in Y then $f^{-1}(V)$ is sb*-open set and hence semi pre open set in X. Then f is semi pre continuous.

Remark 3.22: The converse of the above theorem need not be true as seen from the following example.

Example 3.23: Let X = Y = {a,b,c} with $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{b, c\}\}$ and f be the identity map. Then f is semi pre continuous but not sb* - continuous, since the inverse image of the open set {b,c} in Y is {b,c} in X is not sb*- open.

Remark 3.24: From the above results the diagram follows:



Remark 3.25: The following example shows that the g-continuous function and sb* - continuous function are independent.

Example 3.26: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{b\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b, c\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) = b, f(b)=c, f(c) = a. This function f is g- continuous but not sb* - continuous since the inverse image of the open set {a} in Y is {c} in X is not sb* - open.

Example 3.27: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a, b\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) =f(c)= b and f(b)=c. Here the inverse image of the open set {a,b} in Y is {a,c} in X which is sb*- open but not g - open. Therefore this function is sb* - continuous but not g-continuous

Remark 3.28: The following example shows that the αg- continuous function and sb* - continuous function are independent.

Example 3.29: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{b\}\}$ and $\sigma = \{Y, \varphi, \{c\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) = a, f(b)=b, f(c) = c. Here the inverse image of the open set {c} in Y is {c} in X which is αg-open set but not sb* - open. Therefore the defined function is αg - continuous but not sb*-continuous.

Example 3.30: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a, c\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) =c, f(b) = b and f(c)=a. Here the inverse image of the open set {a,c} in Y is {a,c} in X is sb*- open but not αg - open. Therefore the defined function is sb* - continuous but not αg-continuous.

Remark 3.31: The following example shows that the sb*- continuous function and sg - continuous function are independent

Example 3.32: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a, c\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) = b, f(b)=a, f(c) = c. Here the inverse image of the open set {a,c} in Y is {b,c} in X is sg-open set but not sb* - open. Therefore the defined function is sg - continuous but not sb*-continuous.

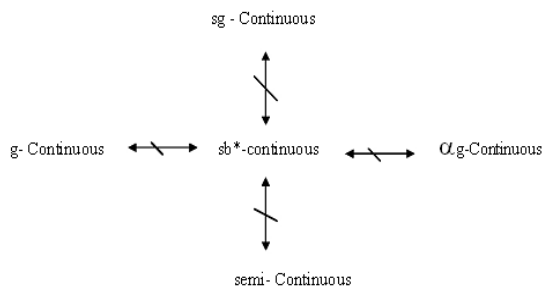
Example 3.33: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{a, b\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) =c, f(b) = b and f(c)=a. Here the inverse image of the open set {a,b} in Y is {b,c} in X is sb*- open but not sg-open. Therefore the defined function is sg - continuous but not sb*-continuous.

Remark 3.34: The following example shows that the sb*- continuous function and semi - continuous function are independent.

Example 3.35: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) = a, f(b)=c, f(c) = b. Here the inverse image of the open set {a} in Y is {a} in X which is not semi open but it is sb* - open. Therefore the defined function is sb* - continuous but not semi-continuous.

Example 3.36: Consider X = Y = {a,b,c} with $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{b, c\}\}$. Let the function f: (X, τ) $\to$ (Y, σ) be defined by f(a) =a, f(b) = c and f(c)=b. Here the inverse image of the open set {b,c} in Y is {b,c} in X which is semi- open but not sb* - open. Therefore the defined function is semi - continuous but not sb*-continuous.

Remark 3.37: From the above results the diagram follows:



4. STRONGLY b*- OPEN AND CLOSED MAPS

In this section we introduce the new concept of sb* - closed maps and studied some of their properties

Definition 4.1: Let X and Y be a topological spaces. A map f: X $\rightarrow$ Y is called strongly b* -closed (sb* - closed) map if the image of every closed set in X is sb* - closed set in Y.

Theorem 4.2: Every closed map is sb*-closed but not conversely.

Proof: Let f: X $\rightarrow$ Y be closed map and V be a closed set in X. Then f (V) is closed and hence sb*-closed in Y. Thus f is sb*- closed. The converse of the above theorem need not be true as seen from the following example.

Example 4.3: Consider X = Y = {a,b,c}, $\tau = \{X, \varphi, \{a\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{a, b\}\}$ and a map f : X $\rightarrow$ Y be defined by f(a) = a, f(b) = f(c) = b. This function f is sb*-closed but not closed as f({b,c}) = { b} is not closed in Y.

Theorem 4.4: If a map f : X $\rightarrow$ Y is continuous and sb*-closed, A is sb* - closed set of X then f(A) is sb*-closed in Y.

Proof: Let f(A)$\subseteq$ O, where O is b-open set of Y. Since f is continuous $f^{-1}(O)$ is b-open set containing A. Hence cl(int(A)) $\subseteq f^{-1}(O)$, as A is sb*-closed. Since f is sb*-closed f(cl(int(A))) is a sb*-closed set contained in the b-open set O, which implies cl(int f(A)) $\subseteq$ O. So, f(A) is sb* -open in Y.

Theorem 4.5: A map f : X $\rightarrow$ Y is sb*-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a sb*-open set V of Y such that S $\subseteq$ V and $f^{-1}(V) \subseteq$ U.

Proof: Suppose f is sb*-closed. Let S be a subset of Y and U be a open set of X such that $f^{-1}(S) \subseteq U$. V = Y-f(X-U) is a sb* - open set containing S such that $f^{-1}(V) \subseteq$ U.

For the converse, suppose that F is a closed set of X. Then $f^{-1}(Y\text{-}f(F)) \subseteq$ X-F and X-F is open. By hypothesis, there is a sb*-open set V of Y such that Y-f(F) $\subseteq$ V and $f^{-1}(V) \subseteq$ X-F. Therefore F $\subseteq$ X-$f^{-1}(V)$. Hence Y-V $\subseteq$ f(F) $\subseteq$ f(X-$f^{-1}(V)$) $\subseteq$ Y-V. Which implies f(F) = Y-V. Since Y-V is sb*-closed, f(F) is sb*-closed and thus f is sb*-closed map.

Theorem 4.7: If a map f: X $\rightarrow$ Y is closed and a map g: Y $\rightarrow$ Z is sb*-closed then g∘f : X $\rightarrow$ Z is sb*-closed.

Proof: Let V be a closed set in X. Since f: X $\rightarrow$ Y is closed, f (V) is closed set in Y. Since g: Y $\rightarrow$ Z is sb^* - closed, h (f (V)) is sb^* - closed set in Z. Therefore (g∘f): X $\rightarrow$ Z is sb^* - closed map.

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