g*b-Homeomorphisms and Contra g*b-continuous Maps in Topological Spaces

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ABSTRACT

In this paper, we first introduce a new class of closed maps called g*b -closed map and gb-closed map. Also, we introduce a new class of homeomorphisms called g*b -homeomorphism, gb-homeomorphism and we investigate a new generalization of contra-continuity called contra- g*b -continuity.

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1. Introduction

Malghan[8] introduced the concept of generalized closed maps in topological spaces. Biswas[1], Mashour[9], Sundaram[13], Crossley and Hildebrand [2], and Devi[3] have introduced and studied semi-open maps, α-open maps, and generalized open maps respectively.

Several topologists have generalized homeomorphisms in topological spaces. Biswas[1], Crossley and Hilde- brand[2], Sundaram[13] have introduced and studied semi-homeomorphism and some what homeomorphism and generalized homeomorphism and gc-homeomorphism respectively.

The notion of contra-continuity was introduced and in- vestigated by Donchev[4]. Donchev and Noiri[5], S.Jafari and T.Noiri[7,6] have introduced and investigated contra-semi-continuous functions, contra-pre-continuous functions and contra-α-continuous functions between topological spaces.

Throughout this paper (X, τ) and (Y, σ)(or simply X and Y) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, cl(A) and int(A) represents the closure of A and interior of A respectively.

2. Preliminaries

We recall the following definitions.

Definition 2.1[12]: A subset A of a topological space (X, τ) is called a generalized b-closed set (briefly gb-closed) if $bcl(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X

Definition 2.2[14]: A subset A of a topological space (X, τ) is called a g*b-closed set if $bcl(A) \subseteq U$, whenever $A \subseteq U$ and U is g-open in X.

Definition 2.3[12]: A map $f : X \to Y$ is called gb-continuous if the inverse image of every closed set in Y is gb-closed in X.

Definition 2.4[15]: A map $f : X \to Y$ is called g∗b-continuous if the inverse image of every closed set in Y is g∗b-closed in X.

Definition 2.5[4]: A map $f : X \to Y$ is called contra-continuous if the inverse image of every open set in Y is closed in X.

Definition 2.6[7]: A map $f : X \to Y$ is called contra-pre-continuous if the inverse image of every open set in Y is preclosed in X.

Definition 2.7[5]: A map $f : X \to Y$ is called contra-semi-continuous if the inverse image of every open set in Y is semi-closed in X.

Definition 2.8[6]: A map $f : X \to Y$ is called contra-α-continuous if the inverse image of every open1

set in Y is α-closed in
X.

Definition 2.9[10]: A map $f : X \to Y$ is called contra-b-continuous if the inverse image of every open set in Y is b-closed in X.

Definition 2.10[11]: A map $f : X \to Y$ is called almost-contra continuous if the inverse image of every regular open set in Y is open in X.

3. g*b -Closed Maps

In this section we introduce the concept of g*b -

closed map, g^*b -open map and gb-closed map in topological spaces.

Definition 3.1: A map $f : X \to Y$ is called g^*b - closed map if for each closed set F of X, f (F) is g^*b -closed set in Y.

Definition 3.2: A map $f : X \to Y$ is called g^*b - open map if for each open set F of X, f (F) is g^*b -open set in Y.

Definition 3.3: A map $f : X \to Y$ is called gb-closed map if for each closed set F of X, f (F) is gb-closed set in Y.

Theorem 3.4: Every closed map is a g^*b -closed map.

Proof: Let $f : X \to Y$ be an closed map. Let F be any closed set in X. Then f (F) is an closed set in Y. Since every closed set is g^*b-closed, f (F) is a g^*b-closed set. Therefore f is a g^*b -closed map.

Remark 3.5: The converse of the theorem 3.4 need not be true as seen from the following example.

Example 3.6: Let X = Y = {a, b, c} with topologies τ = {X, φ, {a}} and σ = {Y, φ, {b}, {a, b}}. Let $f : X \to Y$ be the identity map. Then f is g^*b-closed but not closed, since the image of the closed set {b, c} in X is {b, c} which is not closed in Y.

Theorem 3.7: Every g^*b -closed map is a gb-closed map.

Proof: Let $f : X \to Y$ be a g^*b -closed map. Let F be a closed set in X. Then f (F) is a g^*b -closed set in Y. Since every g^*b-closed set is gb-closed, f (F) is a gb-closed set. Therefore f is a gb-closed map.

Remark 3.8: The converse of the theorem 3.7 need not be true as seen from the following example.

Example 3.9: Let X = Y = {a, b, c} with topologies τ = {X, φ, {c}} and σ = {Y, φ, {a}}. Let $f : X \to Y$ be the identity map. Then f is gb-closed but not g^*b-closed, since the image of the closed set {a, b} in X is {a, b} which is not g^*b -closed in Y.

Theorem 3.10: If $f : X \to Y$ is closed and $h : Y \to Z$ is g^*b -closed, then $h \circ f : X \to Z$ is g^*b -closed.

Proof: Let V be any closed set in X. Since $f : X \to Y$, f (V) is closed in Y and since $h : Y \to Z$ is g^*b -closed,

h(f (V)) is g^*b -closed in Z . Therefore $h \circ f : X \to Z$ is g^*b –closed map.

Remark 3.11: If $f : X \to Y$ is g^*b -closed and

$h : Y \to Z$ is closed, then their composition need not be a g^*b -closed map as seen from the following example.

Example 3.12: Let X = Y = Z = {a, b, c} with topologies τ = {X, φ, {a}, {b, c}}, σ = {Y, φ, {a, c}} and η = {Z, φ, {a}, {c},{a, c}}. Define a map $f : X \to Y$ by f (a) = f (b) = b and f (c) = a and a map $g : Y \to Z$ is defined by g(a) = b, g(b) = a and g(c) = c. Then f is a g^*b -closed map and g is a closed map. But their composition $g \circ f : X \to Z$ is not a g^*b -closed map, since for the closed set {b, c} in X , $g \circ f$ ({b, c}) = {a, b} which is not g^*b -closed in Z .

Theorem 3.13: A map $f : X \to Y$ is g^*b-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a g^*b - open set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof: Suppose f is g^*b -closed. Let S be a subset of Y and U is an open set of X such that $f^{-1}(S) \subseteq U$. Then

$V = Y - f(X - U)$ is g^*b -open set containing S such that $f^{-1}(V) \subseteq U$.

Conversely suppose that F is a closed set in X .

Then $f^{-1}(Y - f(F)) = X - F$ and X_F is open. By hypothesis, there is a g*b -open set V of Y such that $Y_f(F) \subseteq Y$ and $f^{-1}(V) \subseteq X - F$. Therefore $F \subseteq X - f^{-1}(V)$. Hence $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$, which implies $f(F) = Y - V$. Since $Y - V$ is g*b -closed,

f (F) is g*b -closed and therefore f is g*b -closed map.

Theorem 3.14: If $f : X \to Y$ is g-closed, $g : Y \to Z$ be g*b -closed and Y is $T_{1/2}$ -space then their composition

$g \circ f : X \to Z$ is g*b -closed map.

Proof: Let A be a closed set of X. Since f is g-closed, f (A) is g-closed in Y . Since Y is $T_{1/2}$ -space, f (A) is closed in Y . Since g is g*b -closed, g(f (A)) is g*b -closed in Z and $g(f(A)) = g \circ f(A)$. Therefore $g \circ f$ is g*b -closed.

Theorem 3.15: If $f : X \to Y$ is g*b -closed and

$A = f^{-1}(B)$ for some closed set B of Y , then

$f_A : A \to Y$ is g*b -closed.

Proof: Let F be a closed set in A. Then there is a closed set H in X such that $F = A \cap H$. Then $f_A(F) = f(A \cap H) = f(H) \cap B$. Since f is g*b -closed, f (H) is g*b -closed in Y . So $f(H) \cap B$ is g*b-closed, since the intersection of a g*b-closed set and a closed set is a g*b- closed set. Hence f_A is g*b -closed.

Remark 3.16: If B is not closed in Y then the theorem 3.15 may not hold as seen from the following example.

Example 3.17: Let X = Y = {a, b, c} with topologies

$\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a\}\}$. Let $f : X \to Y$ be the identity map. Take $B = \{a\}$ is not closed in A. Then $A = f^{-1}(B) = f^{-1}(\{a\}) = \{a\}$ and {a} is closed in A. But $f_A(\{a\}) = \{a\}$ is not g*b-closed in Y . Therefore {a} is also not g*b -closed in B.

Remark 3.18: The Composition of two g*b-closed maps need not be g*b -closed map in general and this is shown by the following example.

Example 3.19: Let X = Y = Z = {a, b, c} with topologies $\tau = \{X, \varphi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}\}$ and $\eta = \{Z, \varphi, \{b\}, \{a, b\}\}$. Define $f : X \to Y$ by $f(a) = a, f(b) = b, f(c) = c$ and $g : Y \to Z$ be the identity map. Then f and g are g*b -closed maps, but their composition $g \circ f : X \to Z$ is not g*b-closed map, because F= {b} is closed in X , but $g \circ f(F) =$

$g \circ f(\{b\}) = g(\{b\}) = \{b\}$ which is not g*b -closed in Z .

4. g*b-Homeomorphisms

Definition 4.1: A bijection $f : X \to Y$ is called g∗b-homeomorphism if f is both g*b -continuous and g∗b-closed.

Definition 4.2: A bijection $f : X \to Y$ is called gb- homeomorphism if f is both gb-continuous and gb-closed.

Theorem 4.3: Every homeomorphism is a g*b-homeomorphism.

Proof: Let $f : X \to Y$ be a homeomorphism. Then f is continuous and closed. Since every continuous function is g*b -continuous and every closed map is g*b -closed, f is g*b -continuous and g*b -closed. Hence f is a g*b -homeomorphism.

Remark 4.4: The converse of the theorem 4.3 need not be true as seen from the following example.

Example 4.5: Let X = Y = {a, b, c} with topologies $\tau = \{X, \varphi, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{a, c\}\}$. Let $f : X \to Y$ be the identity map. Then f is g*b-homeomorphism but not a homeomorphism, since the inverse image of $\{b, c\}$ in Y is not closed in X .

Theorem 4.6: Every g*b -homeomorphism is a gb-homeomorphism.

Proof: Let $f : X \to Y$ be a g*b -homeomorphism. Then f is g*b -continuous and g*b -closed. Since every g*b -continuous function is gb-continuous and every g*b b-closed map is gb-closed, f is gb-continuous and gb-closed. Hence f is a gb-homeomorphism.

Remark 4.7: The converse of the theorem 4.6 need not be true as seen from the following example.

Example 4.8: Let X = Y = $\{a, b, c\}$ with topologies $\tau = \{X, \varphi, \{a\}\}$ and $\sigma = \{Y, \varphi, \{b\}\}$. Let $f : X \to Y$ be the identity map. Then f is gb-homeomorphism but not a g*b-homeomorphism, since the inverse image of $\{a, c\}$ in Y is not g*b-closed in X .

Theorem 4.9: For any bijection $f : X \to Y$ the following statements are equivalent.

(a) Its inverse map $f^{-1} : Y \to X$ is g*b -continuous.

(b) f is a g*b -open map.

(c) f is a g*b -closed map.

Proof: (a) ⇒ (b)

Let G be any open set in X . Since f^{-1} is g*b -continuous, the inverse image of G under f^{-1}, namely f (G) is g*b -open in Y and so f is a g*b-open map.

(b) ⇒ (c)

Let F be any closed set in X . Then F^c open in X . Since f is g*b -open, $f(F^c)$ is g*b-open in Y . But

$f(F^c) = Y - f(F)$ and so $f(F)$ is g*b -closed in Y . Therefore f is a g*b -closed map.

(c) ⇒ (a)

Let F be any closed set in X . Then the inverse image of F under f^{-1}, namely $f(F)$ is g*b -closed in Y since f is a g*b -closed map. Therefore f^{-1} is g*b -continuous.

Theorem 4.10: Let $f : X \to Y$ be a bijective and g*b -continuous map. Then, the following statements are equivalent.

(a) f is a g*b -open map

(b) f is a g*b -homeomorphism.

(c) f is a g*b -closed map.

Proof: (a) ⇒ (b)

Given $f : X \to Y$ be a bijective, g*b -continuous and g*b -open. Then by definition, f is a g*b -homeomorphism.

(b) ⇒ (c)

Given f is g*b -open and bijective. By theorem 4.9, f is a g*b -closed map.

(c) ⇒ (a)

Given f is g∗b-closed and bijective. By theorem 4.9, f is a g*b -open map.

Remark 4.11: The following example shows that the composition of two g*b -homeomorphism is not a g*b -homeomorphism.

Example 4.12: Let X = Y = Z = $\{a, b, c\}$ with topologies

$\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a, b\}\}$ and $\eta = \{Z, \varphi, \{a\}\}$. Let $f : X \to Y$ and $g : Y \to Z$ be the identity maps. Then both f and g are g*b- homeomorphisms but their composition $g \circ f : X \to Z$ is not a g*b- homeomorphism, since $F = \{a, c\}$ is closed in X , but $g \circ f(F) = g \circ f(\{a, c\})$

= {a, c} which is not g*b -closed in Z.

5. Contra- g*b -Continuous Maps

In this section we introduce the concept of contra- g*b - continuous map, almost contra- g*b -continuous map and locally g*b -indiscrete space in topological spaces.

Definition 5.1: A map f : X → Y is called contra- g*b -continuous if the inverse image of every open set in Y is g*b -closed in X.

Definition 5.2: A map f : X → Y is called Almost contra- g*b -continuous if the inverse image of every regular open set in Y is g*b -closed in X.

Definition 5.3: A space X is said to be locally g*b -indiscrete if every g*b -open set of X is closed in X.

Theorem 5.4: Every contra-continuous function is contra- g*b -continuous but not conversely.

Proof: Let f : X → Y be contra-continuous. Let V be any open set in Y. Then the inverse image f^{-1} (V) is closed in X. Since every closed set is g*b -closed, f^{-1} (V) is g∗b-closed in X. Therefore f is contra- g*b -continuous.

Example 5.5: Let X = Y = {a, b, c} with topologies τ = {X, φ, {b}, {a, b}} and σ = {Y, φ, {a}}. Let f : X → Y be the identity map. Here the image of the open set is g*b -closed and hence f is contra- g*b -continuous. But f is not contra-continuous since f^{-1}{a} = {a} is not closed in X.

Theorem 5.6: Every contra-b-continuous function is contra g*b -continuous but not conversely.

Proof: Let f : X → Y be contra-b-continuous. Let V be any open set in Y. Then the inverse image f^{-1} (V) is b-closed in X. Since every b-closed set is g*b -closed,

f^{-1} (V) is g*b -closed in X. Therefore f is contra- g∗b-continuous.

Example 5.7: Let X = Y = {a, b, c} with topologies τ = {X, φ, {a}, {a,c}} and σ = {Y, φ, {c},{a, c}}. Let us define f(a) = a, f(b) = c, f(c) = b. Here the image of the open set is g*b -closed and hence f is contra g*b –continuous . But f is not contra-b-continuous since f^{-1}{a,c} = {a, b} is not b-closed in X

Theorem 5.8: Every contra-pre-continuous function is contra- g*b -continuous but not conversely.

Proof: Let f : X → Y be contra-pre-continuous. Let V be any open set in Y. Then the inverse image f^{-1} (V) is pre-closed in X. Since every pre-closed set is g*b -closed, f^{-1} (V) is g*b -closed in X. Therefore f is contra- g*b -continuous.

Example 5.9: Let X = Y = {a, b, c} with topologies τ = {X, φ, {a}, {a, c}} and σ = {Y, φ, {a, b}}. Let f : X → Y be the identity map. Here the image of the open set is g∗b-closed and hence f is contra- g*b continuous. But f is not contra-pre-continuous since f^{-1}{a, b} = {a, b} is not pre-closed in X.

Theorem 5.10: Every contra-semi-continuous function is contra- g*b -continuous but not conversely.

Proof: Let f : X → Y be contra-semi-continuous. Let V be any open set in Y. Then the inverse image f^{-1} (V) is semi-closed in X. Since every semi-closed set is g*b -closed, f^{-1} (V) is g*b -closed in X. Therefore f is contra- g*b -continuous.

Example 5.11: Let X = Y = {a, b, c} with topologies τ = {X, φ, {a}, {a, b}} and σ = {Y, φ, {b}, {a, c}}. Let f : X → Y be the identity map. Here the image of the open set is g∗b-closed and hence f is contra- g*b- continuous. But f is not contra-semi-continuous since

$f^{-1}\{a, c\} = \{a, c\}$ is not semi-closed in X.

Theorem 5.12: Every contra-α-continuous function is contra- g*b -continuous but not conversely.

Proof: Let f : X → Y be contra-α-continuous. Let V be any open set in Y. Then the inverse image $f^{-1}(V)$ is α-closed in X. Since every α-closed set is g*b -closed, $f^{-1}(V)$ is g*b -closed in X. Therefore f is contra- g*b -continuous.

Example 5.13: Let X = Y = {a, b, c} with topologies τ = {X, φ, {a}, {a, b}} and σ = {Y, φ, {b, c}}. Let us define f (a) = b, f (b) = a, f (c) = c. Here the image of the open set is g∗b-closed and hence f is contra- g*b -continuous. But f is not contra-α-continuous since $f^{-1}\{b, c\} = \{a, c\}$ is not α-closed in X.

Theorem 5.14: Every contra- g*b -continuous function is almost contra- g*b -continuous but not conversely.

Proof: The Proof follows as every regular open set is open.

Example 5.15: Let X = Y = {a, b, c} with topologies

τ = {X, φ, {a}, {b}, {a, b}} and σ = {Y, φ, {a}, {a, b}}.

Let us define f (a) = a, f (b) = b, f (c) = c. Here the image of the regular open set is g*b -closed and hence f is almost contra-g∗b-continuous. But f is not contra- g*b-continuous since $f^{-1}\{a, b\} = \{a, b\}$ is not g*b -closed in X.

Theorem 5.16: If a map f : X → Y from a topological space X into a topological space Y. The following statements are equivalent.

(a) f is almost contra- g*b-continuous.

(b) For every regular closed set F of Y, $f^{-1}(F)$ is g*b -openin X.

Proof: (a) ⇒ (b)

Let F be a regular closed set in Y, then Y − F is a regular open set in Y. By (a), $f^{-1}(Y - F) = X - f^{-1}(F)$ is g*b -closed set in X. This implies $f^{-1}(F)$ is g*b –open set in X. Therefore (b) holds. (b) ⇒ (a)

Let G be a regular open set of Y. Then Y − G is a regular closed set in Y. By (b), $f^{-1}(Y - G)$ is g*b -open set in X. This implies $X - f^{-1}(G)$ is g*b -open set in X, which implies $f^{-1}(G)$ is g*b -closed set in X. Therefore (a) holds.

Remark 5.17: The composition of two contra- g*b -continuous map need not be contra- g*b -continuous.

Let us prove the remark by the following example.

Example 5.18: Let X = Y = Z = {a, b, c} with topologies τ = {X, φ, {a}, {b}, {a, b}}, σ = {Y, φ, {a}, {a, c}} and η = {Z, φ, {a}, {b}, {a, b}, {a, c}}. Let g : (X, τ) → (Y, σ) be a map defined by g(a) = a, g(b) = b and g(c) = c. Let f : (Z, η) → (X, τ) be a map defined by f (a) = c, f (b) = a and f (c) = b. Both f and g are contra- g*b -continuous.

Define g ∘ f : (Z, η) → (Y, σ). Here {a, c} is a open set of (Y, σ). Therefore $(g \circ f)^{-1}(\{a, c\}) = \{a, b\}$ is not a g*b -closed set of (Z, η). Hence g ∘ f is not contra- g*b -continuous.

Theorem 5.19: If a map f : X → Y is g*b -irresolute map and g : Y → Z is g*b -continuous map ,then g ∘ f : X → Z is contra g*b -continuous.

Proof: Let F be an open set in Z. Then $g^{-1}(F)$ is g*b -closed in Y, because g is contra- g*b -continuous. Since f is g*b-irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is g*b -closed in X. Therefore $g \circ f$ is contra g*b -continuous.

Theorem 5.19: If a map $f : X \to Y$ is g*b -irresolute map with Y as locally g*b -indiscrete space and $g : Y \to Z$ is contra- g*b -continuous map, then $g \circ f : X \to Z$ is contra g*b -continuous.

Proof: Let F be any closed set in Z. Since g is contra- g*b -continuous, $g^{-1}(F)$ is g*b -open in Y. Since Y is locally g*b -indiscrete, $g^{-1}(F)$ is closed in Y. Hence $g^{-1}(F)$ is g*b -closed set in Y. Since f is g*b -irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is g*b-closed in X. Therefore $g \circ f$ is contra g*b -continuous.

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