b^*-Continuous Functions in Topological Spaces

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ABSTRACT

The aim of this paper is to introduce and study b^*-continuous functions in topological spaces. Also we investigate topological properties of b^*-open map and closed map in topological spaces.

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1. Introduction

Levine[4, 5] introduced the concepts of semi-open sets and semi-continuous in a topological space and investi- gated some of their properties. Strong forms of stronger and weaker forms of continuous map have been in- troduced and investigated by several mathematicians. Ekici [3] introduced and studied b-continuous functions in topological spaces. In this paper we introduce a new class of function called b^*-continuous functions. Moreover we obtain basic properties and preservation theorem of b^*-continuous functions.

2. Preliminaries

Before entering into our work we recall the following definitions

Definition 2.1 [2]: A function $f : X \to Y$ is said to be generalized continuous (g-continuous) if $f^{-1}(V)$ is g-open in X for each open set V of Y.

Definition 2.2 [3]: A function f: $X \to Y$ is said to be b-continuous if for each $x \in X$ and for each open set of V of Y containing $f(x)$, there exists $U \in bO(X, x)$ such that $f(U) \subseteq V$.

Definition 2.3 [1]: A subset A of a topological space

(X, τ) is called a b-open set if $A \subseteq cl(int(A)) \cup int(cl(A))$ and b-closed set if $cl(int(A)) \cup int(cl(A)) \subseteq A$.

Definition 2.4 [5]: A subset A of a topological space (X, τ) is called a generalized closed set(briefly g-closed) if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X.

Definition 2.5 [6]: A subset A of a topological space (X, τ) is called a b^* closed set if $int(cl(A)) \subseteq U$, whenever $A \subseteq U$ and U is b-open.

Definition 2.7 [4]: A subset A of a topological space (X, τ) is called a semi-open set if $A \subseteq cl(int(A))$ and semi closed set if $int(cl(A)) \subseteq A$.

3. Some basic properties of b^*-continuous functions

In this section we introduce the concept of b^*-continuous functions in topological space.

Definition 3.1: A map $f : X \to Y$ from a topo- logical space X into a topological space Y is called b^*-continuous map if the inverse image of every closed set in Y is b^*-closed in X.

Theorem 3.2: If a map $f : X \to Y$ from a topo- logical space X into a topological space Y is continuous then it is b^*-continuous but not conversely.

Proof: Let $f : X \to Y$ be continuous. Let M be any closed in Y. Then the inverse image $f^{-1}(M)$ is closed in Y. Since every closed set is b^*-closed, $f^{-1}(M)$ is b^*-closed in X. Therefore, f is b^*-continuous.

Remark 3.3: The converse of the above theorem need not be true as seen from the following example.

Example 3.4: Let $X=\{a,b,c\}$ with the topology

$\tau = \{X, \varphi, \{c\}, \{a, c\}\}$, $Y=\{a,b,c\}$, $\sigma = \{X, \varphi, \{c\}\}$. A function $f : X \to Y$ is defined by $f(a) = c$, $f(b) = b$, $f(c) = a$. Then, f is b^*-continuous. But f is not continuous since $f^{-1}\{c\} = \{a\}$ is not open in X.

Theorem 3.5: Every b-continuous is b^*-continuous but not conversely.

Remark 3.6: The converse of the above theorem need not be true as seen from the following example.

Example 3.7: Let $X=\{d, e, h\}$ with the topology $\tau = \{X, \varphi, \{h\}, \{d, h\}\}$, $Y=\{d, e, h\}$, $\sigma = \{X, \varphi, \{h\}\}$. A function $f : X \rightarrow Y$ is defined by $f(d) = h$, $f(e) = e$, $f(h) = d$. Then, f is b^*-continuous. But f is not b-continuous since $f^{-1}\{h\} = \{d\}$ is not open in X.

Theorem 3.8: Let $f : X \rightarrow Y$ be a single valued function, where X and Y are topological spaces. Then the following are equivalent.

(i) The function f is b^*-continuous.

(ii) The inverse image of each b-open set in Y is b^*-open in X.

(iii) If $f : X \rightarrow Y$ is b^*-continuous, then

$f(cl^*(A)) \subset cl(f(A))$ for every subset A of X.

Proof: (i)Assume that $f : X \rightarrow Y$ is b^*-continu- ous. Let M be open in Y. Then M^c is closed in Y. Since Y is b^*-continuous $f^{-1}(M^c)$ is b^*-closed in X. But $f^{-1}(M^c) = X - f^{-1}(G)$. Thus $X - f^{-1}(G)$ is b^*-closed in X and so $f^{-1}(M)$ is b^*-open in X. Therefore (i) $\Rightarrow$ (ii).

conversely assume that the inverse image of each open set in Y is b^*-open in X. Let B be any closed set in Y. Then B^c is open in Y. By assumption, $f^{-1}(B^c)$ is b^*-open in X. But $f^{-1}(B^c) = X - f^{-1}(B)$. Thus $X - f^{-1}(B)$ is b^*-open in X and so $f^{-1}(B)$ is b^*-closed in X. Therefore f is b^*-continuous. Hence (ii) $\Rightarrow$ (i). Thus (i) and (ii) are equivalent.

(iii) Assume that f is b^*-continuous. Let A be any subset of X. Then $cl(f(A))$ is a closed set in Y. Since f is b^*-continuous , $f^{-1}(cl(f(A)))$ is b^*-closed in X and it contains A. But $cl^*(A)$ is the intersection of all b^*-closed sets containing A. Therefore $cl^*(A) \subseteq f^{-1}(cl(f(A)))$ and so $f(cl^*(A)) \subset cl(f(A))$.

Theorem 3.9: A map $f : X \rightarrow Y$ is b^*-continuous if and only if the inverse image of every closed set in Y is b^*-closed in X.

Proof: Let F be closed in Y. Then F^c is open in Y. Since f is b^*-continuous, $f^{-1}(F)$ is b^*-open in X. But $f^{-1}(F^c) = X - f^{-1}(F)$ and so $f^{-1}(F)$ is b^*-closed in X.

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Conversely assume that the inverse image of every closed

set in Y is b^*-closed in X. Let V be an open set in Y then V^c is closed in Y. By hypothesis, $f^{-1}(V^c) = X - f^{-1}(V)$ is b^*-closed in X and so $f^{-1}(V)$ is b^*-open in X. Thus f is b^*-continuous.

Theorem 3.10: Let X and Y be topological spaces. If a map $f : X \rightarrow Y$ is b-continuous then it is b^*-continuous.

Proof: Assume that a map $f : X \rightarrow Y$ is b-continuous. Let V be an open set in Y. Since f is b-continuous, $f^{-1}(V)$ is b-open and hence b^*-open in X. Therefore f is b^*-continuous.

Remark 3.11: The converse of the above theorem need not be true as seen from the following example.

Example 3.12: Let $X=Y=\{a,b,c\}$ with $\tau = \{X, \varphi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a, c\}\}$ and f be the identity map. Then f is b^*-continuous but not b- continuous as the inverse image of the open set $\{a,c\}$ in Y is $\{a,c\}$ in X is not b-open.

4. b^*-open map and b^*-closed map

Definition 4.1: Let x and Y be two topological spaces. A map $f : X \rightarrow Y$ is called b^*-open map, if the image of every open set in X is b^*-open in Y.

Definition 4.2: Let x and Y be two topological spaces. A map $f : X \rightarrow Y$ is called b^*-closed map, if the image of every closed set in X is b^*-closed in Y.

Theorem 4.3: Every open map is b^*-open but not conversely.

Proof: Let $f : X \rightarrow Y$ is an open map and V be an open set in X. Then $f(V)$ is open and hence b^*-open in Y. Thus f is b^*-open.

Remark 4.4: The converse of the above theorem need not be true as seen from the following example.

Example 4.5: Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \varphi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}\}$. Let a map $f : X \rightarrow Y$ be defined by $f(a) = a = f(b)$, $f(c) = c$. Then this function is b^*-open but not open as the image of the open set $\{a,c\}$ in X is $\{a,c\}$ is not open in Y.

Theorem 4.6: Every closed map is b^*-closed but not conversely.

Proof: Let $f : X \rightarrow Y$ be closed map and V be an closed set in X. Then $f(V)$ is closed and hence b^*-closed in Y. Thus f is b^*-closed.

Remark 4.7: The converse of the above theorem need not be true as seen from the following example.

Example 4.8: Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \varphi, \{a\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{a, b\}\}$ and a map $f : X \rightarrow Y$ be defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Thus function f is b^*-closed but not closed as $f(\{b, c\}) = \{a, c\}$ is not closed in Y.

Theorem 4.9: A map $f : X \rightarrow Y$ is b^*-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a b^*-open set V of Y such that $S \subseteq U$ and $f^{-1}(V) \subseteq U$.

Proof: Suppose f is b^*-closed. Let S be a subset of Y and U be an open set of X such that $f^{-1}(S) \subseteq U$. $U = Y - f(X - V)$ is a b^*-open set containing S such that $f^{-1}(V) \subseteq U$.

For the converse, suppose that F is a closed set of X. Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open. By hypothesis, there is a b^*- is b^*-closed map.

Theorem 4.10: If a map $f : X \rightarrow Y$ is continu ous and b^*-closed, A is b^*-closed set of X then $f(A)$ is b^*-closed in Y.

Proof: Let $f(A) \subseteq O$ where O is an open set of Y. Since f is continuous $f^{-1}(O)$ is an open set containing A. Hence $cl(int(A)) \subseteq f^{-1}(O)$ as A is b^*-closed. Since f is b^*-closed $f(cl(int(A)))$ is a b^*-closed set contained in the open set O, which implies $cl(int(f(cl(A)))) \subseteq O$ and hence $cl(int(A)) \subseteq O$. So $f(A)$ is b^*-closed in Y.

Corollary 4.11: If a map $f : X \rightarrow Y$ is continuous and closed and A is b^*-closed, then $f(A)$ is b^*-closed in Y.

Corollary 4.12: If a map $f : X \rightarrow Y$ is b^*-closed and A is a closed set of X then $f_A : A \rightarrow Y$ is b^*-closed.

Corollary 4.13: If a map $f : X \rightarrow Y$ is b^*-closed and continuous and A is b^*-closed set of X, then $f_A : A \rightarrow Y$ is continuous and b^*-closed.

Proof: Let F be a closed set of A then F is b^*- closed set of X. From theorem [4.10] it follows that $f_A(F) = f(F)$ is b^*-closed set of Y. Hence f_A is b^*-closed

and also continuous.

Theorem 4.14: If a map $f : X \rightarrow Y$ is open, continuous, b^*-closed, and surjection, where X is regular then Y is regular.

Proof: Let V be an open set containing a point x of X, such that $f(x) = P$. Since X is regular and f is continuous, there is an open set V such that $x \in V \subseteq f^{-1}(V)$. Here $P \in f(V) \subseteq f(cl(V)) \subseteq U$. Since f is b^*-closed, $f(cl(V))$ is b^*-closed set contained in the open set U. It follows that $cl(int(f(cl(V)))) \subseteq U$ and hence $P \in f(V) \subseteq cl(f(V)) \subseteq U$ and $f(V)$ is open. Since f is open. Hence Y is regular.

Theorem 4.15: If a map $f : X \rightarrow Y$ is closed map and a map $g : Y \rightarrow \tau$ is b^*-closed then $g \circ f : X \rightarrow \tau$ is b^*-closed.

Proof: Let H be a closed set in X. Then $f(H)$ is closed and $(g \circ f)(H) = g(f(H))$ is b^*-closed as g is b^*-closed. Thus $g \circ f$ is b^*-closed.

5. References

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