New Sierpenski Curves in Complex Plane

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ABSTRACT
The Sierpinski triangle also known as Sierpinski gasket is one of the most interesting and the simplest fractal shapes in existence. There are many different and easy ways to generate a Sierpinski triangle. In this paper we have presented a new algorithm for generating the sierpinski gasket using complex variables.

Keywords
Sierpinski Gasket, Fractal Coloring, Complex variables

1. INTRODUCTION
There is a lot of history behind Fractals even though the word fractal did not exist until the 1970s [1]. The Cantor Set discovered in 1872 by Georg Cantor [9], Sierpinski Gasket in 1916 by Wallow Sierpinski who created a triangle and a carpet, Koch Curves in 1904 by Helge Von Koch [10], Lévy C curve in 1938 by Paul Pierre Lévy have been known for quite some time [17].

Benoit Mandelbrot also known as the father of fractal geometry, discovered a fractal which was later known as the Mandelbrot Set (M-Set) [5]. According to Mandelbrot the turning point in fractal study occurred in 1979-1980 with his research of the Fatou-Julia theory of iteration [17]. Later Mandelbrot connected the mathematical monsters of old together into the category of fractal geometry [14].

Since then Fractals have been an area which is being used an applied in various areas like study of turbulence in flows, Biosensor interactions, spread of forest fires and epidemics, used in fractal image compression, to produce footage for films such as Dante's Peak and Star Trek II: The Wrath of Khan [4], used to identify cancerous tumors [4]. Further its representation visually has made it easier for the students to understand than Euclidean geometry.

A number of techniques have been developed to generate the fractal shapes and used to produce fascinating images. Two techniques namely Koch construction, and function iteration in the complex domain have been popularized by Mandelbrot's book, have gained popularity. The construction of Koch curves is a language-theoretic approach and is generated by a rewriting system defined in the domain of geometric shapes[10].The method of function iteration analyzes the sequences of numbers $\{X_n\}$ generated by the formula $x_{n+1} = f(x_n)$ where f is a complex function. The colour of each point represents how quickly the values reach the escape point. Often black is used to show values that fail to escape before the iteration limit, and gradually brighter colors are used for points that escape. This gives a visual representation of how many cycles were required before reaching the escape condition.

The Mandelbrot Set μ for the quadratic is defined as the collection of all c Є C (where C is the complex plane) for which the orbit of the point 0 is bounded, [14]

i.e.,

$$\mu = \{c \in C : \{f^{(n)}(0)_{n=0}^{\infty} \text{ is bounded}\}$$

The distance between a point and the origin may be no greater than 2 in order to be included in the Mandelbrot Set i.e.

"If | c | > 2 and | z | ≥ | c |, then the orbit of z escapes to ∞."

The Mandelbrot set can also be generated by using the complex variables. Consider the complex z-plane. The real z are on the x-axis while the imaginary are on the y-axis.

So for any complex number

$$z = x + yi$$

$$z_n = x_n + y_n i$$

$$z_{n+1} = z_n^2 + c \text{ where } c = a + ib$$

$$z_{n+1} = (x_n + y_n i)^2 + a + ib$$

2. SIERPINSKI GASKET
In 1916, Sierpinski Gasket was introduced by Polish Mathematician Waclaw Sierpinski [21] who was a professor at Lvov and Warsaw. One of the moons craters is named after him. The name "Sierpinski gasket" was given by B. Mandelbrot [18]. Originally constructed as a curve, this is one of the basic examples of self-similar sets. Both the Sierpinski triangle and the Sierpinski carpet have equivalent repetitive tiling arrangement.

The basic concept of constructing a sierpinski fractal is to subdivide an equilateral triangle into four separate equilateral triangles, each of ¼ the initial size. The next triangles then can be again subdivided and so on for an infinite amount time[15].

Sierpinski gasket also known as the Sierpinski triangle is a set of points in the plane which remain if one carries out the process infinitely often. Each of the three parts in the kth step is a scaled down version in sierpinski gasket we can find copies of whole new at every point. The gasket is composed from small but exact copies of itself that's why it is called strictly self similar. It plays an important role in the theory of curves [21]. The Sierpinski carpet and Sierpinski gasket are both constructed by removal of open sets from a closed set, so they are closed. Thus they are closed, bounded subsets of Euclidean space, and hence compact. Both the Sierpinski carpet and Sierpinski gasket are path connected [13].

The Sierpinski triangle is one of the most famous fractals and the Hausdorff dimension and measure are the most important characteristics of a fractal sets. The Sierpinski triangle is defined by an iterated function system, which satisfies a technical condition called the open set condition [16].

Thus it follows from Hutchinson's Theorem [11] that the Hausdorff dimension of sierpinski gasket is equal to log 3/log 2.

Definition of Sierpinski Triangle [8]

The Sierpinski Triangle ***S*** is defined to be the intersection $\bigcap_{k=0}^{\infty} S_k$ where ***S*** is a closed set containing all points inside S_k ($k\geq1$) and on the boundary of an equilateral triangle, and is obtained by removing all points inside the medial triangles of each "basic triangle" of S_{k-1}

Fractals have been investigated for their visual qualities as art, their relationship to explain natural processes, music, medicine, and in mathematics [19]. Fractal Art is a subclass of two-dimensional visual art, and is in many respects similar to photography .Fractal art can be created easily by using many coloring algorithms like divergence algorithm, convergence algorithms decomposition algorithm, orbit trap etc [2].

In our paper we present an algorithm for generating the images of the complex Sierpinski gasket. By making slight changes in the variables of the formula and using coloring algorithm we get variants of the Sierpinski gasket which we have presented in our paper.

3. ALGORITHM FOR COMPLEX SERPINSKI GASKET

```
initialize z =  coordinate (x, y),
initialize float x = real(z),   float y = imag(z),
float a = pi/3,
initialize  float theta = tan(a),  boolean  bail = false,
int i = 0
loop:
if i = 1
if (theta*x + theta < y ) OR (-theta*x + theta < y )
OR (y < 0)
endif
assign  -2*z to z
assign  (z + 1i*theta) to z
assign  z - offset to z
else
assign z + offset to z
assign real(z) to x
assign imag(z) to y
if (-theta*x + theta >= y)  AND  (theta*x + theta >= y )
AND (y >= 0)
then assign value true to  bail
elseif (theta*x + theta < y)
then assign = 2*z + 4 - 1i*theta to z
elseif (-theta*x + theta < y)
then  assign 2*z - 4 - 1i*theta to z
elseif (y < 0)
then  assign  2*z + 2+ 1i* theta to z
endif
assign  z - offset to z
endif
endloop
```

4. GEOMETRY OF COMPLEX SIERPINSKI GASKET

The fractal generated by our method shows Koch curve like structure due to which we name it as Koch Sierpinski Gasket. We observe that by simply playing with the formula we can generate many variants of the Sierpinski gasket and that all the variants are composed from small but exact copies of itself. The variants of the Sierpinski gasket exhibit triangular similarity. The variants of the Sierpinski gasket also exhibit various kinds of transformations like congruence, rotational, translational, reflections, symmetry etc.[8]

We observe that Fig. 4 and Fig. 6 show the effect of transformation. These figures exhibit the property of reflection transformations. In Fig 9 we present a figure which is similar to a Sierpinski carpet except that instead of circles or square we have a triangle.

5. CONCLUSIONS AND REMARKS

In this paper we have presented a new algorithm to generate the images that gives a new aspect to the Sierpinski Gasket. We have applied the various coloring functions and schemes to the fractals and have generated some beautiful artistic variants of the Sierpinski gasket. We also observe that one of the resulting Sierpinski gasket looks like the Koch snowflakes.

This paper also gives a view that by simply playing with the formula we can generate many variants of the sierpinski gasket and that all the variants are composed from small but exact copies of itself.

6. REFERENCES

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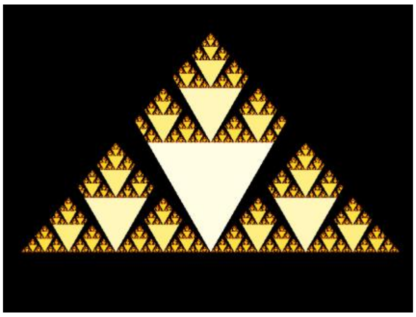

Fig 1. Koch Sierpinski Curve generated by our method

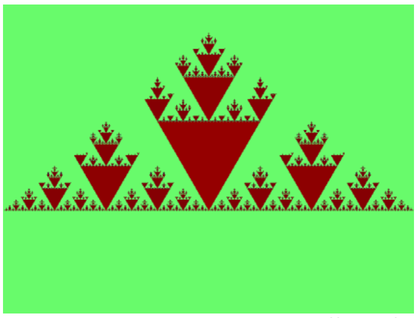

Fig 2: Koch sierpinski Curve with a different formula

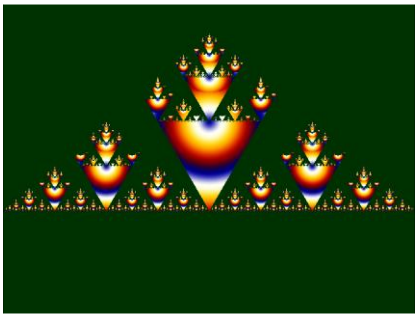

Fig 3: Figure obtained after applying Coloring algorithm to Fig 2.

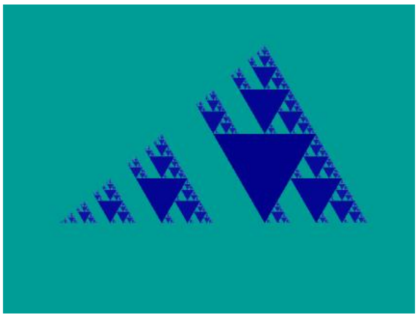

Fig 4. Sierpinski Gasket using a different variable in our algorithm

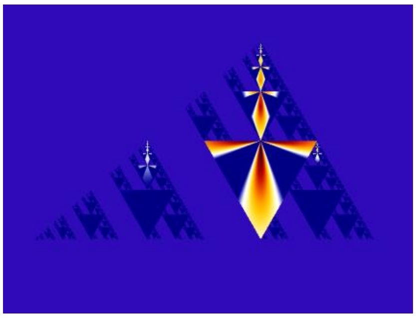

Fig 5.: Colouring algorithm applied to Fig 4.

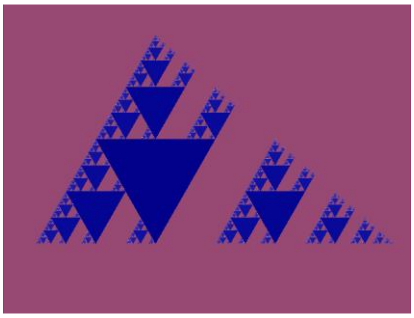

Fig6. Transformed version of Fig 4.

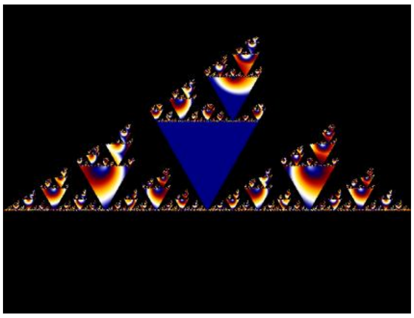

Fig 7: Figure obtained after application of Orbit Trap algorithm to our Formula.

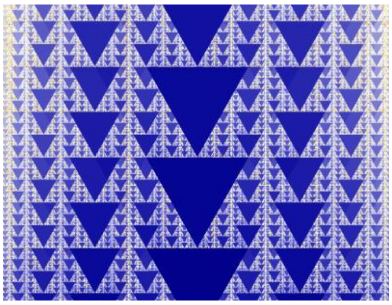

Fig 8. Triangular Sierpinski Carpet.

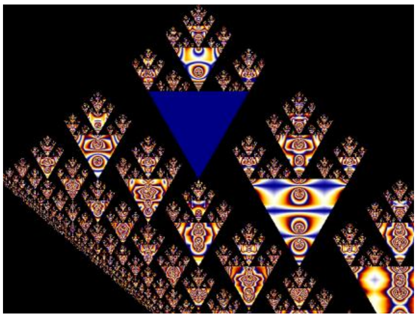

Fig 9. Left Elevated Triangular Sierpinski carpet

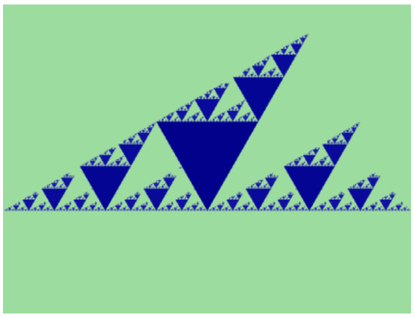

Fig 10. Triangular Sierpinski carpet using coloring algorithm

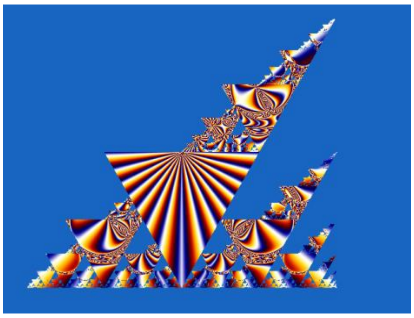

Fig 11: Right Elevated Koch Sierpinski Carpet

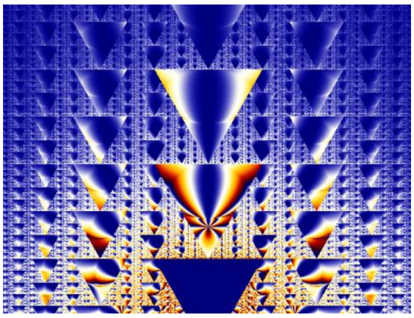

Fig 12: Orbit Trap Colouring Algorithm to Fig 8

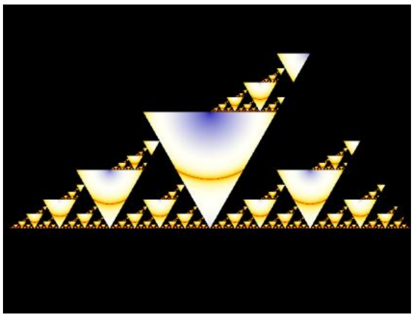

Fig 13.A different variant of our Sierpinski gasket

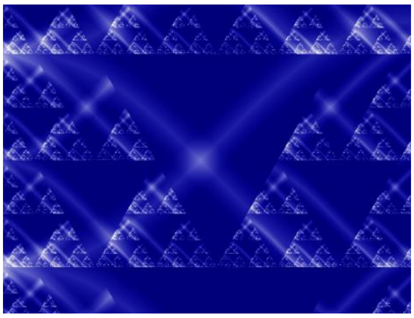

Fig14. Orbit Trap Colouring Algorithm to Fig 8 with orbit type=Point