International Journal of Computer Applications (0975 – 8887)
Volume 57– No.9, November 2012

Common Fixed Point Theorem in D-Metric Space via Altering Distances between Points

Savita Rathee
Department of Mathematics
Maharshi Dayanand University Rohtak

Asha Rani
Department of Mathematics
BMIET Raipur
Sonepat

ABSTRACT

In this paper, we obtain a fixed point theorem for weakly compatible mappings by altering distances between the points via ϕ - contractive condition in D-metric spaces. Our work include the results of Bansal, Chugh and Kumar [1], Veerapandi and Chandersekher Rao [14], and Dhage [4]. An example is given at the end to prove the validity of the theorem.

Keywords

Fixed points, D-Metric spaces

1. INTRODUCTION

In the theory of fixed points, the idea of obtaining fixed point theorems for self maps for a metric space by altering the distances between the points with the use of certain continuous control function is an interesting aspect. Number of fixed point theorems for self map in metric spaces by altering distances have been improved by Khan, Swaleh and Sessa [10], Sastry and Babu [12], and Pant, Jha and Pande [11].

The presence of control function creates certain difficulties in proving the existence of fixed point under contractive conditions. In view of these difficulties, known fixed point theorems either employ a stronger contractive condition like the Banach contractive condition e.g. in Sastry et.al [12] or assume the existence of a convergent sequence of iterates e.g., in Khan et al. [10] and Sastry and Babu [12].

Motivated by the measure of nearness, between two or more objects with respect to a specific property or characteristic, called the parameter of the nearness, Dhage [2] in 1984 in his Ph.D. thesis introduced the concept of a D-metric space by which it has been possible to determine the geometrical nearness i.e., the distance between two or more points of the set under consideration. Geometrically, a D-metric D(x, y, z) represent the parameter of the triangle with vertices x, y and z.

A few details, along with specific examples of a D-metric space, appear in [4]. In paper [5], Dhage proved some fixed point theorems of self maps of a D-metric space satisfying certain contractive conditions.

A number of fixed point theorems have been proved for 2-metric spaces. However, Hsiao [6] showed that all such theorems are trivial in the sense that the iterations of f are all collinear. The situation for D-metric spaces is quite different.

Jungck [7] and Sessa [13] introduced the concept of commuting and weakly commuting mappings respectively. In 1986, Jungck [8] introduced the concept of compatible mappings. In 1998, Jungck et.al [9] introduced the concept of weakly compatible mappings, without appeal to continuity and proved some fixed point theorems for these mappings. Commuting map $\Rightarrow$ weakly commuting maps $\Rightarrow$ weakly compatible, but the converse may not be true.

The general procedure for proving fixed point theorems in a D-metric space consists of the following three steps:

(1) Construction of a sequence $x_{n+1} = fx_n$, $x \geq 0$, which is shown to be D-Cauchy.

(2) By applying certain completeness conditions, $\{x_n\}$ is shown to be convergent, and

(3) The limit point of $\{x_n\}$ is shown to be a fixed point of the map f under certain conditions prescribed on f.

In this paper we hypothesize the above procedure and prove common fixed point theorem for weakly compatible mappings in D-metric space by altering distances between the point under a ϕ -contractive condition which includes the fixed point theorems of Bansal, Chugh and Kumar [1], Veerapandi and Chandersekhar [14] and Dhage [4].

2. PRELIMINARIES

Dhage [4] introduced the following D-metric space.

Definition 2.1. Let X be any set. A D-metric for X is a function D: $X \times X \times X \rightarrow R$ such that

(i) $D(x, y, z) \geq 0$ for all x, y, z ε X, and equality holds if and only if x = y = z.

(ii) D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x) for all x, y, z ε X.

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all x, y, z, a ε X .

If D is a D-metric for X, then the ordered pair (X, D) is called a D-metric space or the set X together with D-metric is called a D-metric space.

Definition 2.2. A sequence $\{x_n\}$ of points of a D-metric space X converges to a point x ε X if for an arbitrary $\in > 0$, there exists a positive integer n_0 such that for all $n > m > n_0$. $D(x_m, x_n, x) < \in$.

Definition 2.3. A sequence $\{x_n\}$ of points of a D-metric space X is Cauchy sequence if for an arbitrary $\in > 0$ there exists a positive integer n_0 such that for all $\rho > n > m \geq n_0$. $D(x_m, x_n, x_p) < \in$.

Definition 2.4. A D-metric space X is a complete D-metric space if every Cauchy sequence $\{x_n\}$ in X converges in X.

Definition 2.5. Let x_0 ε X and $\in > 0$ be given. Then we define the open ball $B(x_0, \in)$ in X centered at x_0 of radius of $\in$ by

$B(x_0, \in) = \{ y \ \varepsilon\ X | D(x_0,y,y) < \in$ if $y = x_0$ and

$\sup_{z \varepsilon X} D(x_0, y, z) < \in$ if $y \neq x_0\}$.

The collection of all open balls $\{B(x, \in): x \varepsilon X, \in >0\}$ define the topology on X denoted by τ.

Definition 2.6. Let (X, D) be a D-metric space. A pair of maps f and g is called weakly compatible pair if they commute at coincidence points i.e., fx = gx if and only if fgx = gfx .

Definition 2.7. A control function ψ is defined as $\psi : R_+ \to R_+$ which is continuous at zero, monotonically increasing, $\psi(2t) \leq 2\psi(t)$ and $\psi(t) = 0$ if and only if t = 0.

Notation 2.8. If A, B, S and T are four self mappings of (X, d) and ψ is a control function on R_+, we write

$$M_\psi(x, y, z) = \max\{\psi(D(Sx, Ty, Sz)), \psi(D(Ax, Sx, Sz)), \psi(D(By,Ty,Az)),\psi(D(Ty, Ax, Ax))\}$$

3. MAIN RESULTS

Theorem 3.1. Let (A, S) and (B, T) be weakly compatible pairs of self mappings of a complete D-metric space (X, D) and ψ be as in definition (2.7) satisfying $A(X) \subset S(X)$, $B(X) \subset T(X)$ and $\psi(D(Ax, By, Az)) \leq \phi(M_\psi(x, y, z))$, for all x, y, z in X. Whenever $M_\psi(x, y, z) > 0$ and $\phi : R_+ \to R_+$ be an upper semi-continuous function such that $\phi(t) < t$ for each t > 0 and ψ is control function. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Sx_{2n+1};\ y_{2n+1} = Bx_{2n+1} = Tx_{2n+2} \tag{3.3}$$

We claim that $\{y_n\}$ is a Cauchy sequence. We write $\alpha_n = \psi(D(y_n, y_{n+1}, y_{n+2}))$. Then, using condition (ii), it follows that $\alpha_{2n} = \psi(D(y_{2n},y_{2n+1},y_{2n+2})) = \psi(D(Ax_{2n},Bx_{2n+1},Ax_{2n+2}))$

$$\leq \phi(M_\psi(x_{2n}, x_{2n+1}, x_{2n+2}))$$

$$= \phi(\max\{\psi(D(Sx_{2n}, Tx_{2n+1}, Sx_{2n+2})), \psi(D(Ax_{2n}, Sx_{2n}, Sx_{2n+2})), \psi(D(Bx_{2n+1}, Tx_{2n+1}, Ax_{2n+2})), \psi(D(Tx_{2n+1}, Ax_{2n}, x_{2n}))\})$$

$$=\phi(\max\{\psi(D(y_{2n-1}, y_{2n}, y_{2n+1})),\psi(D(y_{2n}, y_{2n-1}, y_{2n+1})),\psi(D(y_{2n+1},y_{2n},y_{2n+2})), \psi(D(y_{2n}, y_{2n}, y_{2n}))\})$$

$$=\phi(\psi(D(y_{2n-1}, y_{2n}, y_{2n+1})))$$

$$\leq \phi(\psi(D(y_{2n-1}, y_{2n}, y_{2n+1}))) = \phi(\alpha_{2n-1})$$

That is, $$\alpha_{2n} \leq \phi(\alpha_{2n-1}) < \alpha_{2n-1} \tag{3.4}$$

Similarly, $\alpha_{2n-1} < \alpha_{2n-2}$; $\alpha_{2n-2} < \alpha_{2n-3}$ and so on. Thus $\{\alpha_n\} = \{\psi(D(y_n, y_{n+1}, y_{n+2}))\}$ is a strictly decreasing sequence of positive numbers and hence converges, say to $\alpha \geq 0$. Suppose $\alpha > 0$. Then the inequality (3.2) on making $n\to\infty$ and in view of upper semi continuity of ϕ yields $\alpha \leq \phi(\alpha) < \alpha$, a contradiction. Hence $\alpha = \lim_{n\to\infty} \psi(D(y_n, y_{n+1}, y_{n+2}) = 0$. This, by monotonically increasing property of ψ implies

$$\lim_{n\to\infty} D(y_n, y_{n+1}, y_{n+2}) = 0 \tag{3.5}$$

and also $\{D(y_n, y_{n+1}, y_{n+2})\}$ is a strictly decreasing sequence of positive numbers. We now show that $\{y_n\}$ is a Cauchy sequence. But by virtue of (3.5), it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Now, for each positive integer p, we get

$$\psi(D(y_{2n}, y_{2n+1}, y_{2(n+p)+1})) = \psi(D(Ax_{2n}, Bx_{2n+1}, Ax_{2(n+p)+1}))$$

$$\leq \phi(M\psi(x_{2n}, x_{2n+1}, x_{2(n+p)+1}))$$

$$=\phi(\max\{\psi(D(Sx_{2n},Tx_{2n+1},Sx_{2(n+p)+1})),\psi(D(Ax_{2n},Sx_{2n},Sx_{2(n+p)+1})), \psi(D(Bx_{2n+1}, Tx_{2n+1}, Ax_{2(n+p)+1})), \psi(D(Tx_{2n+1}, Ax_{2n}, Ax_{2n}))\})$$

$$= \phi(\max\{\psi(D(y_{2n-1}, y_{2n}, y_{2(n+p)})), \psi(D(y_{2n}, y_{2n-1}, y_{2(n+p)}))$$

$$\psi(D(y_{2n+1}, y_{2n}, y_{2(n+p)+1})), \psi(D(y_{2n}, y_{2n}, y_{2n}))\})$$

$$= \phi(\psi(D(y_{2n-1}, y_{2n}, y_{2(n+p)})))$$

$$< \psi(D(y_{2n-1}, y_{2n}, y_{2(n+p)})) \tag{3.6}$$

Also,

$$\psi(D(y_{2n-1}, y_{2n}, y_{2(n+p)})) = \psi(D(Ax_{2n-1}, Bx_{2n}, Ax_{2(n+p)}))$$

$$\leq \phi(M\ \psi(x_{2n-1}, x_{2n}, x_{2(n+p)}))$$

$$= \phi(\max\{\psi(D(Sx_{2n-1}, Tx_{2n}, Tx_{2(n+p)})), \psi(D(Ax_{2n-1}, Sx_{2n-1}, Sx_{2(n+p)})),$$

$$\psi(D(Bx_{2n}, Tx_{2n}, Ax_{2(n+p)})), \psi(D(Tx_{2n}, Ax_{2n-1}, Ax_{2n-1}))\})$$

$$= \phi(\max\{\psi(D(y_{2n-2}, y_{2n-1}, y_{2(n+p)-1})), \psi(D(y_{2n-1}, y_{2n-2}, y_{2(n+p)-1})),$$

$$\psi(D(y_{2n}, y_{2n-1}, y_{2(n+p)})), \psi(D(y_{2n-1}, y_{2n-1}, y_{2n-1}))\})$$

$$= \phi(\psi(D(y_{2n-2}, y_{2n-1}, y_{2(n+p)-1})))$$

$$< \psi(D(y_{2n-2}, y_{2n-1}, y_{2(n+p)-1})).$$

This shows that $\{\psi(D(y_{2n}, y_{2n+1}, y_{2(n+p)+1}))\}$ is a decreasing sequence in R and hence converges, say, to $r \geq 0$. Suppose that r > 0. Then the inequality (3.6) on making $n\to\infty$ and in view of upper semi-continuity of ϕ yields $r \leq \phi(r) < r$, which is a contradiction. Hence r = 0. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there is a point z in X such that $y_n \to z$ as $n\to\infty$. Hence from (3.1), we have

$y_{2n} = Ax_{2n} = Sx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2} \to z$.

Since $A(X) \subset S(X)$, there exists a point $u \varepsilon X$ such that Su = z. Then, using (3.2), we have

$$\psi(D(Au, Bx_{2n+1}, Au) \leq \phi(M_\psi(u, x_{2n+1}, u))$$

$$= \phi(\max\{\psi(D(Su, Tx_{2n+1}, Su)), \psi(D(Au, Su, Su)),$$

$$\psi(D(Bx_{2n+1}, Tx_{2n+1}, Au)), \psi(D(Tx_{2n+1}, Au, Au))\})$$

In the limiting case, we have

$$\psi(D(Au, z, Au) = \phi(\max\{\psi(D(z, z, z), \psi(D(Au, z, z)), \psi(D(z, z, Au)),$$

$$\psi(D(z, Au, Au))\})$$

$$= \phi(\psi(D(Au, Au, z)))$$

which is a contradiction. Hence Au = z and thus Au = Su = z. Since $B(X) \subset T(X)$, there exists a point $v \in X$ such that z = Tv. Then again from (3.2)

$\psi(D(Ax_{2n}, Bv, Ax_{2n})) \le \phi(M_\psi(x_{2n}, v, x_{2n}))$

$= \phi(\max\{\psi(D(Sx_{2n}, Tv, Sx_{2n})), \psi(D(Ax_{2n}, Sx_{2n}, Sx_{2n})),$

$\psi(D(Bv, Tv, Ax_{2n})), \psi(D(Tv, Ax_{2n}, Ax_{2n}))\})$

In the limiting case, we have

$\psi(D(z,Bv,z)) = \phi(\max\{\psi(D(z,Tv,z)), \psi(D(z,z,z)),\psi(D(Bv,Tv, z)),\psi(D(Tv,z,z))\})$

$= \phi(\psi(D(Bv, z, z)))$

which is a contradiction. Hence Bv = z and thus Bv = Tv = z. Since pair of maps A and S are weakly compatible then Au = Su implies ASu = SAu i.e. Az = Sz. Now, we show that z is a fixed point of A. Then, using (3.2)

$\psi(D(Az, Bx_{2n+1}, Az)) \le \phi(M_\psi(z, x_{2n+1}, z))$

$= \phi(\max\{\psi(D(Sz, Tx_{2n+1}, Sz)), \psi(D(Az, Sz, Sz)),$

$\psi(D(Bx_{2n+1}, Tx_{2n+1},Az)), \psi(D(Tx_{2n+1}, Az, Az))\})$

In the limiting case, we have

$\psi(D(Az, z, Az)) \le \phi(\max\{\psi(D(Sz, z, Sz)), \psi(D(Az, Sz, Sz)),$

$\psi(D(z,z,Az)),\psi(D(z,Az, Az))\})$

$\le \phi(\psi(D(Az, Az, z))$ $\qquad [\because Az = Sz]$

which is a contradiction. Hence Az = z. Thus, Az = Sz = z. Similarly, pair of maps B and T are weakly compatible, we have Bz = Tz . Now, we show that z is a fixed point of B. Then, using (3.2), we have

$\psi(D(Ax_{2n}, Bz, Ax_{2n})) \le \phi(M_\psi(x_{2n}, z, x_{2n}))$

$=\phi(\max\{(\psi(D(Sx_{2n}, Tz, Sx_{2n})), \psi(D(Ax_{2n}, Sx_{2n}, Sx_{2n})),$

$\psi(D(Bz, Tz, Ax_{2n})), \psi(D(Tz, Ax_{2n}, Ax_{2n}))\})$

In the limiting case, we have

$\psi(D(z, Bz, z)) \le \phi(\psi(D(Bz, z, z))$

which is a contradiction. Hence, Bz = z. Therefore, Bz = Tz = z and Az = Bz = Tz = Sz = z.

Let z is a common fixed point for A, B, S and T. For uniqueness, let w ($w \ne z$) be another common fixed point of A, B, S and T. Then, using (3.2), we have

$\psi(D(Az, Bw, Az)) \le \phi(M_\psi(z, w, z))$

$=\phi(\max\{\psi(D(Sz, Tw, Sz)), \psi(D(Az, Sz, Sz)), \psi(D(Bw, Tw, Az)), \psi(D(Tw, Az, Az))\})$

It follows that

$\psi(D(z, w, z) \le \phi(\max\{\psi(D(z, w, z)), \psi(D(z, z, z)), \psi(D(w,w,z)), \psi(D(w,z,z))\})$

$= \phi(\psi(D(z, w, z))$, which is a contradiction. Hence w = z.

This completes the proof of the theorem.

Now we give the following example to prove the validity of our theorem.

Example 3.2: Let X = [0, 1] and A, B, S, T : $X \to X$ such that

$A(x) = \frac{x}{9}$, $S(x) = \frac{8x}{9}$, B(x) = 0 , T(x) = x for all $x \in X$. Let us define $D : X \times X \times X \to R$

By D(x, y, z) = d(x, y) + d(y, z) + d(z, x).

Then $A(X) = [0, 1/9] \subset [0, 8/9] = S(X)$ and $B(X) = \{0\} \subset [0, 1] = T(X)$

Since A(0) = S(0) = 0 and AS(0) = SA(0).

So {A, S) is weakly compatible. Similarly, the pair {B, T} is weakly compatible. Now Condition 3.2 becomes

$\psi(D(x/9, 0, z/9)) \le \phi(\max\{\psi(D (8x/9, y, 8z/9)), \psi(D(x/9, 8x/9, 8z/9)), \psi(D(0, y, z/9)), \psi(D(y, x/9, x/9))\})$

we see that condition (3.2) is satisfied and clearly 0 is the unique fixed point A, B, S and T.

4. REFERENCES

[1] Bansal, D. R, Chugh, R. and Kumar, R. 1998. Fixed points for ϕ-contractive mappings in D-metric spaces, *East Asian Math. Comm.* **1** (1998), 9-15.

[2] Dhage, B. C. 1984. A study of some fixed point theorems, Ph.D. Thesis (1984), Marathwada Univ. Aurangabad, India.

[3] Dhage, B.C. 1999. On common fixed point of coincidentally commuting mappings in D-metric spaces, *Indian J. pure Appl. Math.*, **30(3)** (1999), 395-406.

[4] Dhage, B. C. 1992. Generalized metric space and mappings with fixed point, *Bull. Cal. Math. Soc.*, **84** (1992), 329-36.

[5] Dhage, B. C. 1998. On two basic contraction mappings principles in D-metric spaces, *East Asian Math. Comm.*, (1998), 101-114.

[6] Hsiao, C. R. 1986. A. property of contractive type mappings in 2-metric spaces, *Jnanabha*, **16** (1986) 223-239.

[7] Jungck, G. 1976. Commuting mappings and fixed point, *Amer. Math. Monthly*, **83** (1976), 261-263.

[8] Jungck, G. 1986. Compatible mappings and common fixed points, *Intern. J. Math. and Math. Sci.*, **9** (1986), 771-79.

[9] Jungck, G. and Rhoades, B. E. 1998. Fixed point for set valued functions without continuity, *Indian J. pure and Applied Math.*, **29(3)** (1998), 227-238.

[10] Khan, M. S., Smaleh, S. M. and Sessa, S. 1984. *Bull. Austral. Math. S.*, **30** (1984), 1-9.

[11] Pant, R. P., Jha, K. and Pande, V. P. 2003. Common fixed points by altering distances between points, *Bull. Cal. Math. Soc.*, **95(5)**, (2003), 421-428.

[12] Sastry, K. P. R., Babu, G. V. R. 1999. Some fixed point theorems by altering distances between the points, *Int. J. pure and appl. Math.*, **30** (1999) 641.

[13] Sessa, S. 1982. On weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. Beograd*, **32(46)**, 1982, 149-153.

[14] Verrapandi, T. and Chandersekher Rao, K. 1996. Fixed points in Dhage metric spaces, *Pure Appl. Math. Sci.*, **Vol. XIII, No. 1-2**, March 1996.