An Implicit Rational Method for Solution of Second Order Initial Value Problems in Ordinary Differential Equations

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ABSTRACT

In this article, we report an implicit rational method for solution of second order initial value problems in ordinary differential equation. We have presented local truncation error and stability property for the proposed method. We observed that the method has cubic rate of convergenceandA-stable. Numerical results for linear and nonlinear problems presented. These results confirm the accuracy, efficiency and effectiveness of the rational method.

Keywords

Implicit method,Rational method, Non-standard method, Initial value problems, Ordinary differential equations, Cubic order convergence.

2000 AMS Subject classification

65L 05, 65L 12

1. INTRODUCTION

The numerical solution of differential equations plays an important role in study ofmany fields of engineering and sciences.A recent research activity to develop economical methods to solve differential equations efficiently, in different way is a motivation to our work.

The higher order equation can be solved by considering an equivalent system of first order equations. But a concept to develop direct methods to solve higher order equations cannot be overemphasized in the theory of initial value problems. Much research has been done on the numerical integration of initial value problems; many researchers have done excellent works.

In recent years, researchers have applied nonstandard finite difference method and obtained competitive results to those obtained with other method. Our aim is to develop a non-classical implicit difference method which can be applied to solve higher order initial value problems. Though implicit method in general is more expensive, but in many cases they have other advantage e.g. higher orders of convergence.

In this paper, new method is proposed for the solution of second order initial value problems. The rate of convergence of the proposed method is three.

We consider numerical solution of the IVP

$$y''(x) = f(x, y) \qquad (1.1)$$

subjecttotheinitialconditions

$$y(a) = y_0 \ and \quad y'(a) = y'_0 \ , \qquad x \in [a, b] \subset \mathbb{R} ,$$

$$y(x) , f(x, y) \in \mathbb{R} \qquad (1.2)$$

Such problems have solved numerically using Runge-Kutta and single step methods[1-3].These methods are based on the polynomial function ,which are normally smooth and with sufficient continuous derivatives.Another approach to investigate the solution of such problems were and referred to as shooting method either simple or multiple [4,5]. A couple of different approach methods exist in literature to solve these initial value problems and can be found in any standard book ,of the subject.

Generally there are two types of numerical methods , namely explicit method and implicit method used to solve boundary value problems. In this paper a new single step implicit rational method is proposed to solve the initial value problem (1). To obtain the numerical solution, an implicit method requires the solution of linear or nonlinear equations. Thus implicit methods in general are more expensive, but in many cases they have advantages, e.g. higher order of convergence. The existence and uniqueness of the solution to the initial value problem is assumed.

The structure of this paper is as follows. In section 2, we discuss the steps involve in derivation of our method, to obtain numerical solution of the problem (1). In section 3,we have discussed the local truncation error, order of the method andstability property. In section 4, we have considered some model problems to test the performance of the proposedrational method and its convergence. A summary of the result and conclusion are given in last section 5.

2. DERIVATION OFTHE METHOD

In this article, it is presupposed that unique solution of the reference problem (1.1-1.2) exist. The specific assumption on $f(x, y)$,to ensure the existence and uniqueness will not be considered [4,6,7].An idea of rational methods for solution of the initial value problems and recent development can be found in the literature[8,9,10].

The first step in obtaining the proposed rational method is to partition the interval [a, b] in which the solution of the problem (1) is desired, into N, finite numbers of subinterval$[a, x_1], [x_1, x_2] \dots\dots [x_{n-1}, b]$ *by the points*

$$x_j = x_0 + j.h \quad , j = 0,1,2, \dots \quad , N \qquad (2.3)$$

where the terms in right side of expression (2) are defined

as ,

$$the\ step\ length \ \ h = \frac{(b-a)}{N} \ , \qquad x_0 = a \ \ and \ \ x_N = b.$$

Suppose we have to determine a numbery_j ,which is an approximation to the value of the theoretical solution $y(x)$of problem (1.1-1.2) at the nodal point x_j , $j = 1,2, \dots , N$.

Assuming the local assumption that no previous truncation errors have been made [11] i.e. $y_n = y(x_n)$, we are interested in obtaining an approximate value y_{n+1} for the theoretical value of the solution $y(x_{n+1})$. For that purpose ,we proposed an approximation to the theoretical solution $y(x_n + h)$ of initial value problem (1), similar to given in [12],

$$y_{n+1} = y_n + hy_n' + \frac{h^2(b_0 y_{n+1}'' + b_1 y_n'')}{(C(h) + y_n + hy_n')} \qquad (2.4)$$

where$C(h)$ is sufficiently differentiable unknown function of step length h and coefficients $b_0$and b_1 are unknown constants. These unknowns are to be determined by imposing some conditions. It is also assumed that $C(h) \neq -(y_n + y_n')$for all n .

Let define a function $F_n(a, h, y, y')$ and associate (2.4) with it as ,

$$F_n(a, h, y, y') \cong$$

$$\big(y(x_n + h) - y(x_n) - hy'(x_n)\big)\big(C(h) + y(x_n) + hy'(x_n)\big)$$

$$-h^2(b_0 y''(x_n + h) + b_1 y''(x_n)) = 0 \qquad (2.5)$$

We assume that the function $C(h)$ can be expanded in Taylor series about point$h = 0$,sowe write it as

$$C(h) = C_0 + hC_0' + \frac{h^2}{2!}C_0'' + O(h^3) \qquad (2.6)$$

Expand $F_n(a, h, y, y')$ in Taylor series about point x_n ,and using$(5) in\ it$, we obtained the following expression,

$$\left(\frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n^{(3)} + \frac{h^4}{4!}y_n^{(4)} + O(h^5)\right) \times$$

$$\left((C_0 + y_n) + h(C_0' + y_n') + \frac{h^2}{2!}C_0'' + O(h^3)\right)$$

$$-h^2\left((b_0 + b_1)y_n'' + hb_0 y_n^{(3)} + b_0\frac{h^2}{2!}y_n^{(4)} + O(h^3)\right) = 0 \qquad (2.7)$$

Imposing condition that the coefficients of$h^2, h^3 and\ h^4\ in$ (2.7)vanish ,we obtained a system of equations,

$$(C_0 + y_n - 2(b_0 + b_1))y_n'' = 0 \qquad (2.8)$$

$$(C_0' + y_n')y_n'' + \frac{1}{3}(C_0 + y_n - 6b_0)y_n^{(3)} = 0 \qquad (2.9)$$

$$C_0'' y_n'' + \frac{2}{3}(C_0' + y_n')y_n^{(3)} + \frac{1}{6}(C_0 + y_n - 12b_0)y_n^{(4)} = 0 \qquad (2.10)$$

Let us impose condition that $C_0 + y_n - 12b_0 = 0$ and $y_n'' \neq 0$. Now solve the system of equations (2.8-2.10) ,we obtained

$$b_1 = 5b_0 \qquad (2.11)$$

$$C_0 = 12b_0 - y_n \qquad (2.12)$$

$$C_0' = -y_n' - 2b_0\frac{y_n^{(3)}}{y_n''} \qquad (2.13)$$

$$C_0'' = \frac{4b_0}{3}\left(\frac{y_n^{(3)}}{y_n''}\right)^2 \qquad (2.14)$$

Substituting the values from (2.11)-(2.14) in (2.6) ,we get

$$C(h) = 12b_0 - y_n - h\left(y_n' + 2b_0\frac{y_n^{(3)}}{y_n''}\right)$$

$$+\frac{2h^2b_0}{3}\left(\frac{y_n^{(3)}}{y_n''}\right)^2 + O(h^3) \qquad (2.15)$$

Substituting the values of $b_1, C(h)$ from (2.11) and (2.15) respectively , in (2.4). After neglecting the termO(h^3),we get

$$y_{n+1} = y_n + hy_n' + \frac{3(hy_n'')^2(y_{n+1}'' + 5y_n'')}{2\left(18(y_n'')^2 - hy_n^{(3)}\left(3y_n'' - h\,y_n^{(3)}\right)\right)} \qquad (2.16)$$

where$y_n^{(3)} = \left(\frac{df}{dx}\right)_{(x_n, y_n)} = f_n'$, but

$$f_n' = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y'\right)_{(x_n, y_n)} = \left(\frac{\partial f}{\partial x}\right)_{(x_n, y_n)} + y_n'\left(\frac{\partial f}{\partial y}\right)_{(x_n, y_n)} \qquad (2.17)$$

Thus introducing the notation (2.17) in (2.16),we get our rational method as ,

$$y_{n+1} = y_n + hy_n' + \frac{3(hf_n)^2(f_{n+1} + 5f_n)}{2\big(18(f_n)^2 - hf_n'(3f_n - h\,f_n')\big)} \qquad (2.18)$$

wherey_{n+1} is an approximation to the value of the solution $y(x)$ at the point$x = x_n + h$ .Similarly we can define other terms as,$y_n = y(x_n)$, $y_n' = y'(x_n)$, $f_n = f(x_n, y_n)$, and $f_{n+1} \approx f(x_{n+1}, y_{n+1})$.

A rational method to determiney_{n+1}' ,an approximation to the value of the derivative of the theoretical solution $y(x)$ at the point $x = x_n + h$ can be derived by same procedure as we derived fory_{n+1} .So, we write rational method directly as,

$$y_{n+1}' = y_n' + \frac{4h(f_n)^2(f_{n+1} + 2f_n)}{12(f_n)^2 - hf_n'(f_n - hf_n')} \qquad (2.19)$$

Thus we have developed an implicit single step rational method of the form

$$y_{n+1} = y_n + hy_n' + h^2G(h, x_n, y_n, y_n') \qquad (2.20)$$

and

$$y_{n+1}' = y_n' + hF(h, x_n, y_n, y_n') \qquad (2.21)$$

where$G\ and\ F$ are an increment function. These increment functions depend on$f\ and\ f'$.

The resulting system of equations (2.18,2.19) are either linear or nonlinear, depend upon if $f(x, y)$ is linear or nonlinear. Thus system of linear equation solved by direct method and system of nonlinear equation solved either by direct method or Newton-Raphson method. Computational experiments confirmed the suitability of the rational method in solving the initial value problem in ordinary differential equation of order two.

3. THE LOCAL TRUNCATIONERROR AND STABILITY PROPERTY

In this section, we consider the error associated to the proposed rational method (2.16) by imposing certain conditions. Let the local truncation error T_{n+1} , defined as

$$T_{n+1} = y(x_n + h) - y_{n+1} \quad (3.22)$$

Substituting the value of y_{n+1} from (2.16) in (3.22), so we have

$$T_{n+1} = (y(x_n + h) - y_n - hy'_n) - \frac{3(hy''_n)^2(y''_{n+1} + 5y''_n)}{2\left(18(y''_n)^2 - hy_n^{(3)}\left(3y''_n - h\,y_n^{(3)}\right)\right)} \quad (3.23)$$

Expand $y(x_n + h)$ in Taylor series about a point x_n and rewrite (3.23) as

$$T_{n+1} = \left(\frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \frac{h^4}{4!}y_n^{(4)} + \frac{h^5}{5!}y_n^{(5)}\right) - \frac{h^2}{12}(y''_{n+1} + 5y''_n)\left(1 - \left(\frac{h}{6}\frac{y'''_n}{y''_n} - \frac{h}{18}^2\left(\frac{y'''_n}{y''_n}\right)^2\right)\right)^{-1} \quad (3.24)$$

If we assume that $\left|\frac{h}{6}\frac{y'''_n}{y''_n} - \frac{h}{18}^2\left(\frac{y'''_n}{y''_n}\right)^2\right| < 1$,so we can write a binomial expansion of the term in (3.24).Write the binomial expansion and simplify the terms in equation (3.24), we get

$$T_{n+1} = \left(\frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \frac{h^4}{4!}y_n^{(4)} + \frac{h^5}{5!}y_n^{(5)}\right) - \left\{\left(\frac{h^2}{2}y''_n + \frac{h^3}{12}y'''_n + \frac{h^4}{24}y_n^{(4)} + \frac{h^5}{72}y_n^{(5)}\right)\left(1 + \frac{h}{6}\frac{y'''_n}{y''_n} - \frac{h^2}{36}\left(\frac{y'''_n}{y''_n}\right)^2 - \frac{h^3}{72}\left(\frac{y'''_n}{y''_n}\right)^3 - \dots\dots\dots\right)\right\}$$

$$T_{n+1} = -\frac{h^5}{90}y_n^{(5)} + O(h^6) \quad (3.25)$$

Thus, from (3.25) we have found that the local truncation error associated to rational method (2.16) is $O(h^5)$. So, we can conclude that the rational method (2.16) has cubic rate of convergence .Similarly we can also estimate rate of convergence for the method (2.19).

To discuss stability property of the rational method (2.16), we follow the samemethod as discussed in [13, 14].

Consider the Dahlquist test equation for stability

$$y''(x) = \lambda y(x) \quad , \ x \in [a,b] and real no. \ \lambda \geq 0 ,$$

$$subject to initial boundary conditions$$

$$y(a) = y_0 \ , \quad y'(a) = -\sqrt{\lambda}y_0 .$$

Apply the method (2.16) to this test equation, we get

$$(36 + 6h\sqrt{\lambda} - h^2\lambda)y_{n+1} = (36 - 30h\sqrt{\lambda} + 11\lambda h^2 - 2h^3\lambda\sqrt{\lambda})y_n \quad (3.26)$$

Neglecting the term $O(h^2)$in (3.26) and simplify, we get

$$y_{n+1} \cong \frac{1}{36}(36 - 30h\sqrt{\lambda})(1 + \frac{h}{6}\sqrt{\lambda})^{-1}y_n$$

$$y_{n+1} \cong (1 - h\sqrt{\lambda})y_n$$

$$y_{n+1} \cong e^{-h\sqrt{\lambda}}y_n \cong E(-h\sqrt{\lambda})y_n \quad (3.27)$$

where the stability function $E(-h\sqrt{\lambda})$ is an approximation to$e^{-h\sqrt{\lambda}}$. For the convergence of the method (2.16),

$$|E(-h\sqrt{\lambda})| < 1 \quad (3.28)$$

Solving the inequality (3.28),we get $0 < h\sqrt{\lambda} < 2$,an interval of absolute stability of the rational method (2.16).

4. NUMERICAL EXPERIMENT

In order to illustrate the performance of the rational method (2.16) and (2.19),we have solved some model linear and nonlinear initial value problems using double precision GNU FORTRAN language .Let y_i and y'_i *are* the numbers calculated by(2.16) *and* (2.19) respectively *which are an*approximate value of the theoreticalsolution y(x) and derivative of solution*i.e.* $y'(x)$ at*the point* $x = x_i$ .Maximum absolute error is calculated in both solution and derivative of solution by

$$MAE(y) = \max_i |y(x_i) - y_i|$$

$$MAE(y') = \max_i |y'(x_i) - y'_i| \ , \ i = 1,2, \dots\dots\dots\dots\dots N.$$

Example 4.1 Consider the initial value problem [15],

$$y'' = \left(\frac{-40}{x^3} + \frac{400}{x^4}\right)y$$

The exact solution in [1,2] is $y(x) = e^{\left(\frac{-20}{x}\right)}$ .The maximum absolute error in $y(x)$ and$y'(x)$ are given Table1.

Example 4.2 Consider nonlinear initial value problem

$$y''(x) = 6y^2(x)$$

The exact solution in [0,1] is $y(x) = (1 + x)^{-2}$. The maximum absolute error in $y(x)$ and $y'(x)$ are given Table2.

Example 4.3 Consider nonlinear initial value problem

$$y''(x) = y^2(x) + \frac{1}{2}\cos(x) - sin^4(\frac{1}{2}x)$$

The exact solution in [0,1] is $y(x) = sin^2\left(\frac{1}{2}x\right)$. The maximum absolute error in$y(x)$ and $y'(x)$ are given Table3.

Example 4.4 Consider nonlinear initial value problem

$$y''(x) = y^3(x) - y(x)y'(x)$$

The exact solution in [1,2] is $y(x) = (1 + x)^{-1}$. The maximum absolute error in$y(x)$ and $y'(x)$ are given in Table4.

5. CONCLUSION

In this article, a rational method of order three for numericalsolution ofinitial value problem was described.Although stability property is not so laborious to establish .Nonethelessnumerical results for four examples were given,clearly confirm the performance of our rational method in solving initial value problems in ordinary differential equations. We are currently working to extend this idea to different class of boundary value problems.

Table 1.Maximum absolute error in$y(x) = e^{\left(\frac{-20}{x}\right)}$and $y'(x)$ for problem 1.

N	MAE	
	y	y'
8	.361871(-5)	.183845(-4)
16	.553617(-6)	.279991(-5)
32	.729924(-7)	.368564(-6)
64	.927321(-8)	.467917(-7)
128	.114596(-8)	.581349(-8)
256	.141881(-9)	.720320(-9)
512	.181899(-10)	.920161(-10)

Table 2. Maximum absolute error in $y(x) = (1 + x)^{-2}$for $and\ y'(x)$problem 2 .

N	MAE	
	y	y'
4	.22492126(-1)	.46638966(-1)
8	.33299029(-2)	.69708824(-2)
16	.43533742(-3)	.91198087(-3)
32	.54568052(-4)	.11438131(-3)
64	.64522028(-5)	.13709068(-4)
128	.11473894(-5)	.22947788(-5)

Table 3.Maximum absolute error in $y(x) = sin^2\left(\frac{1}{2}x\right)$ $and\ y'(x)$for problem 3 .

N	MAE	
	y	y'
4	.40043593(-4)	.18468499(-3)
8	.67693013(-5)	.28610229(-4)
16	.10174531(-5)	.39935112(-5)
32	.12338340(-6)	.59604645(-6)
64	.20351521(-7)	.89406967(-7)

Table 4. Maximum absolute error in $y(x) = (1 + x)^{-1} and\ y'(x)$for problem 4 .

N	MAE	
	y	y'
4	.73721014(-4)	.10743075(-3)
8	.94075995(-5)	.13493829(-4)
16	.12119611(-5)	.16846591(-5)
32	.79472862(-7)	.15729003(-6)

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