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Complex Nature of Fractal Geometry

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ABSTRACT
The Mann iteration process and Ishikawa Iteration process are generally used to approximate the fixed point. There are a lot of work is done by researchers and still researches are being conducted to study and reveal the new concepts unexplored. Recently, Negi, Rana and Chauhan have explored the study of complex dynamics on various functions using Mann and Ishikawa Iterative processes. In this paper we have reviewed the recent work done work on the Mann iteration. This review contains a wide variety of existing iteration schemes as its special cases.

Keywords

Relative Superior Mandelbrot Set, Complex Dynamics, Relative Superior Julia Set, Ishikawa Iteration.

1. INTRODUCTION
In the study of fractals there is a most popular name is Benoit Mandelbrot, scientist and mathematician at IBM, is often characterized as the father of fractal geometry. Mandelbrot coined the word fractal in the late 1970s [13].

"*Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."* ***(Benoit Mandelbrot)***

Mandelbrot began to study an equation that later became known as the Mandelbrot Set which is the most popular and indeed beautiful object in fractal geometry under computer magnification, reveals an endless succession of repeating patterns, which impact mathematic [6], arts and other sciences. Earlier, a French mathematician Gaston Julia (1893-1978) introduced a set named Julia set. The set of points with chaotic orbits is called the *Julia set* for a given function. The Julia set define the boundary between the prisoner set and the escape set [4], Julia set is the place where all of the chaotic behavior of a complex function occurs [6]. Both the Mandelbrot and Julia Set fractals require the use of complex numbers to compute the basic Mandelbrot (or Julia) set one uses the equation $f_c(z) = z^n + c$, where both z and c are complex numbers. The Mandelbrot sets and its generalizations have been extensively studied by using Peano–Picard iterations (generally it is called Picard or function iterations). The Mann iteration scheme was introduced in 1953 by W.R. Mann [11] and has been applied extensively to fixed point problems. On applying Mann Iterates on Mandelbrot with complex polynomial equation $Q_c(z) = z^n + c$ results superior Mandelbrot set [14]. Iterative techniques for approximating fixed points of non expansive mappings have been studied by many researchers and various authors have worked to study [21-22].

2. PRELIMINARIES
2.1 Iterative Procedures/ Superior Iterations
Fractal is defined by an iterative function over the complex numbers. Given a function $f(x)$, and a starting value $\boldsymbol{x}_0$, one can construct a new value $x_1 = f(x_0)$. With some persistence, the next value x_2 is obtained by another application of $f : x_2 = f(x_1) = f(f(x_0))$. This is an *iterative* process that, generally speaking, generates a sequence $x_0, x_1, ...x_k$ where x_k is the k^{th} iterate obtained by applying the function f to $x_0 k$ times. The sequence is known as an *orbit* of its starting point x_0. The iterative procedures are one way to achieve the self-similarity exhibited by fractals [19]. One-step feedback machines are characterized by Peano–Picard iterations also called *function iterations* represented by the formula $x_{n+1} = f(x_n)$, where f can be any function of x . It requires one number as input and returns a new number. The function iteration is a very useful mathematical tool and has been developed particularly for the numerical solution of complex problems. Whereas, in two-step iterative process, the output is computed by the formula $x_{n+1} = g(x_n, x_{n-1})$, which requires two numbers as input and returns a new number [27]. The two-step feedback machine and superior iterates are characterized by Rani [24] to compute and analyze the fractals models.

Definition 2.1.1: Superior Iterates [10]

Let X be a non-empty set of real numbers and $f : X \to X$ for an $x_0 \in X$ construct a sequence x_n in the following manner.

$$
\begin{aligned}
x_1 &= \beta_1 f(x_0) + (1-\beta_1)x_0 \\
x_2 &= \beta_2 f(x_1) + (1-\beta_2)x_1 \\
\vdots &\quad \vdots \qquad \vdots \\
x_n &= \beta_n f(x_n) + (1-\beta_n)x_{n-1} \\
\vdots &\quad \vdots \qquad \vdots
\end{aligned}
$$

Where $0 < \beta_n \leq 1$ and $\{\beta_1\}$ is constructed this way is called superior sequence of iterates

Definition 2.1.2: Ishikawa Iteration [20]: Let X be a subset of real or complex numbers and $f : X \to X$ for $x_0 \in X$, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n = s'_n f(x_n) + (1-s'_n)x_n$$

$$x_{n+1} = s_n f(y_n) + (1-s_n)x_n$$

Where $0 \leq s'_n \leq 1$, $0 \leq s_n \leq 1$ and s'_n & s_n are both convergent to non zero number.

Definition 2.1.3: Mann Iteration [11] Mann iteration is the dynamical system defined for a continuous function $f:[0,1]\to[0,1]$, $x_n=\frac{1}{n}\sum_{k=0}^{n-1} f(x_k)$ with $x_0\in[0,1]$. In other words $x_k=\frac{(k-1)x_{k-1}+f(x_{k-1})}{k}$. We observe that this iteration always converges to fixed point of (f).

Definition 2.1.4: Mandelbrot Set [4]

Mandelbrot set M for the quadratic $Q_c(z)=z^n+c$ is defined as the collection of all $c\in C$ for which the orbit of point 0 is bounded, that is $M=\{c\in C:\{Q_c^n(0)\};n=0,1,2,3...\, is\, bounded\}$. An equivalent formulation is $M=\{c\in C\{Q_c^n(0)\ does\ not\ tends\ to\ \infty\ as\ n\to\infty\}\}$ we choose the initial point 0, as 0is the only critical point of Q_c.

Definition 2.1.5: Julia Set [5]

The set of points K whose orbits are bounded under the function iteration function of $Q_c(z)$ is called the Julia set. We choose the initial point 0, as 0 is the only critical point of $Q_c(z)$.

Definition2.1.6: Relative Superior Orbit [5]

The sequence x_n and y_n constructed above is called Ishikawa sequence of iteration or relative superior sequence of iterates. We denote it by $RSO(x_0,s_n,s'_n,t)$

Notice that $RSO(x_0,s_n,s'_n,t)$ with $s'_n=1$ is $SO(x_0,s_n,t)$ i.e. Mann's orbit [21] and if we place $s_n=s'_n=1$ then $s_n=s'_n=1$ reduces to $O(x_0,t)$. We remark that Ishikawa orbit $RSO(x_0,s_n,s'_n,t)$ with $s'_n=\frac{1}{2}$ is relative superior orbit.

Definition 2.1.7: Relative Superior Mandelbrot Set [4]

Now we define Mandelbrot set for the function with respect to Ishikawa iterates. We call them a Relative Superior Mandelbrot sets. Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z)=z^n+c$, where $n=1,2,3,...$ is defined as the collection of $c\in C$ for which the orbit of 0 is bounded i.e. $RSM=\{c\in C:\{Q_c^n(0)\}:k=0,1,2,3,...\}$ is bounded.

In functional dynamics, we can categorize the resultant points in two different categories. Points that leave the interval after a finite number of iterations are named as stable set of infinity. Whereas the points that never leave the interval after any number of iterations are called bounded orbits [5].

Definition 2.1.8: Relative Superior Julia Sets [10]

The set of Points RSK whose orbits are bounded under relative superior iteration of function $Q_c(z)$ is called Relative Superior Julia Sets. Relative Superior Julia Set of Q is boundary of Julia set RSK

In 2006 Negi and Rani [15] have presented the study of Mandelbrot set and superior Mandelbrot set and their midgets for the complex valued quadratics function $Q_c(z)=z^n+c, n\geq 2$ and higher degree polynomial of the same family. Here to mention that the midgets are the small mini Mandelbrot set like images they are found surroundings of the superior Mandelbrot set [23-24] & fig [1]). In the process of generation and study of superior Mandelbrot set for various values of n they have followed the Devany's nomenclature [6] and occasionally Philp [18] (see fig. [1]) they have considered two cases i.e. ***s*** = 1 (special case) and $0<s\leq 1$ (general case) [15]. Their study shows that the nature of midgets of Mandelbrot set of n^{th} order were the mini Mandelbrot set of the same order, whereas the midgets of superior Mandelbrot were found to be of order two irrespective of the power of the generating function, i.e. the order of the midgets does not depend on the degree of the polynomial after these observations they find some of the super Mandelbrot sets are effetely different from the usual Mandelbrot sets. There study shows that the two types of Mandelbrot sets are different in many cases.

In 2006 Negi and Rani, [16] introduced a new noise criterion on superior Mandelbrot map. They inspired by noise of the Mandelbrot map two parameter deformation families. In their study on dynamic noise on the superior Mandelbrot map. They observed various categories of additive and multiplicative noise on Mandelbrot set. And found that the multiplicative noise plays a important role in the loss of symmetry and distortion of the Mandelbrot set they use a general noise with new parameter that general noise on Mandelbrot set is defined as $x_{n+1}=\lambda x_a+(1-\lambda)x_m$, $x_{n+1}=\lambda y_a+(1-\lambda)y_m$ Where $x_a,y_a\ and\ \lambda=1$ satisfy the additive noise and $x_m,y_m\ and\ \lambda=0$ satisfy the multiplicative noise [16]. With new parameter $0\leq\lambda\leq 1$.

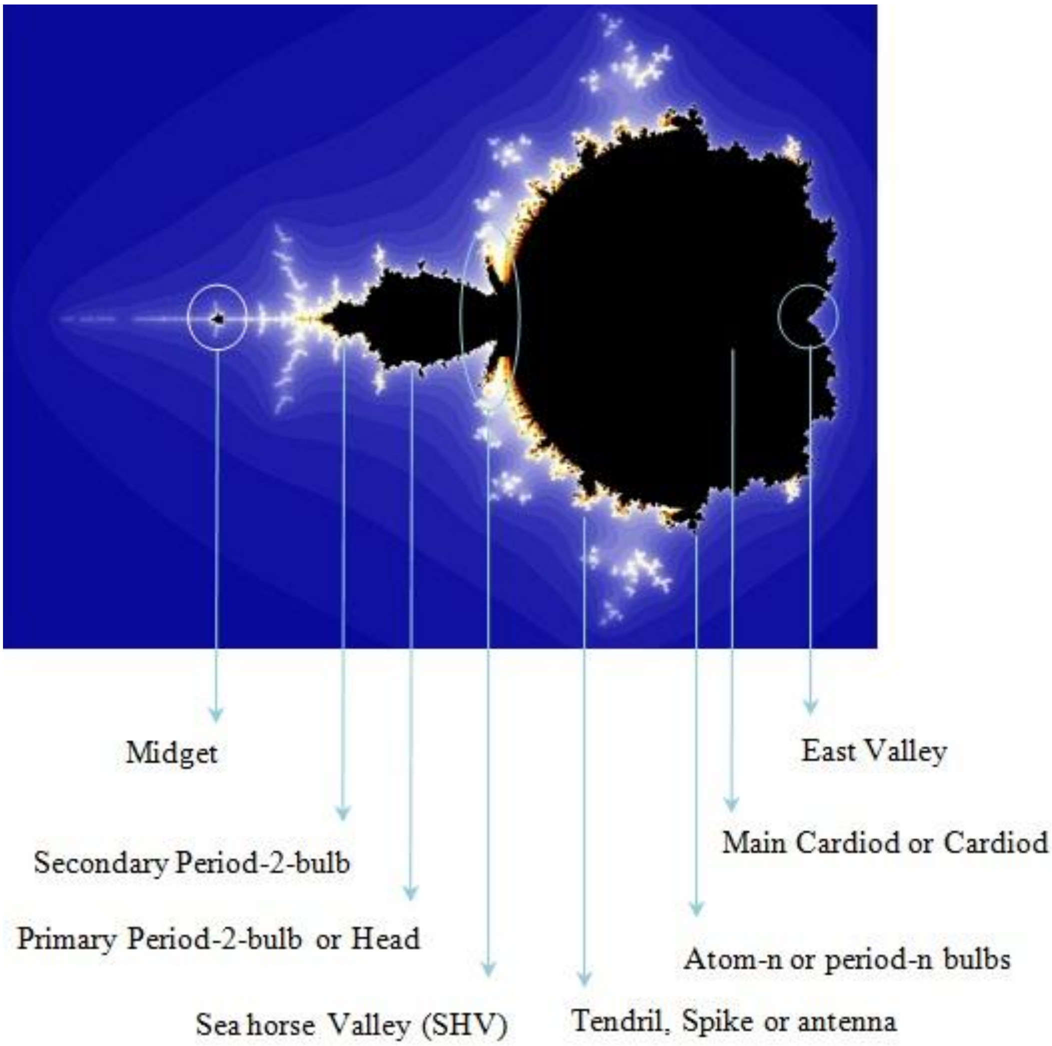

Fig 1: If necessary, the images can be extended both columns

Mandelbrot set

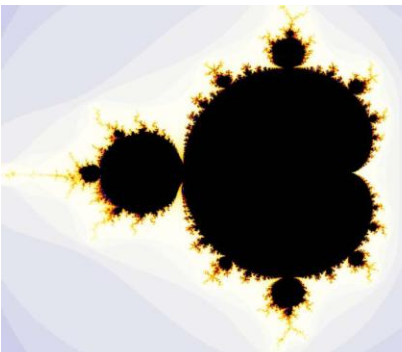

Fig.2.Mandelbrot set for low additive and low multiplicative noise (m_1, m_2, k_1, k_2) = (0.01, 0.01, 0.01, 0.01).

The Extremely Distorted Mandelbrot Set

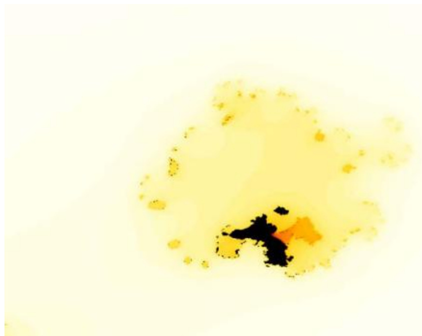

Fig.3.Mandelbrot set for high additive and high multiplicative noise (m_1, m_2, m_3,) = (0.5, 0.5, 0.5, 0.5).

Further they observed the strength of noise on Mandelbrot set. When the strength is increased the Mandelbrot set losses it summery it can be reduced considerately. After that they make a comparison of stability between superior Mandelbrot set and Mandelbrot set and found the superior Mandelbrot set is a extension of the Mandelbrot set also analysis that the value of λ get various instructing results. In Same year 2006 [13]they study on Complex Carotid-Kundalini function Negi and Rani generated different filled superior C-K Julia sets and analysis their characteristics. They are inspired by cooper study on c-k function give the interesting results on Julia sets. Then they apply superior iterations on C-K functions and find some instructing results then after compare with the cooper results. They observed when superior iterates apply on the function $z_{j+1} = s\left(k_{n,c}\left(z_j\right)\right) + \left(1-s\right)z_j$, where $0 \le s \le 1$. for large value of $\Re(N)$ and $\Im(N)$ [13]. The filled superior $C-K$ Julia set is convert into single bigger component located at the center call them main body see fig [3]. And the structure is connected whereas for similar values of (N) fig [4] also found filled superior $C-K$ Julia set is disconnected. The whole study of filled superior $C-K$ Julia set enables a new ways in generation of the complex-valued $C-K$ function. After that in paper Cubic Superior Julia Sets [12] Rani introduced to cubic polynomials inspired by Bodi Banner and John Hubburd. They are the researcher first study on the iterated complex map tor cubic polynomials in Picard orbit. In the similar manner Author study cubic polynomials $Q_{a,b}(z)=z^3+az+b,$ in superior orbit, and visualized very interesting Julia sets. And also define prisoner set using cubic superior escape criterion [13].

3. GENERATING PROCESS

3.1 Generation Process [23]

The basic principle of generating fractals employs the iterative formula: $z_{n+1} \leftarrow f(z_n)$ where z_0 = the initial value of z, and z_i = the value of the complex quantity z at the i^{th} iteration. For example, the Mandelbrot's self-squared function for generating fractals is $f(z)=z^2+c$, where z and c are both complex quantities. We propose the use of the transformation function $z \to z^n+c,\ n\geq 2$ and $z \to (z^n+c)^{-1}$ for generating fractal images with respect to Ishikawa iterates, where z and c are the complex quantities and n is a real number. Each of these fractal images is constructed as a two-dimensional array of pixels. Each pixel is represented by a pair of (x,y) coordinates. The complex quantities z and c can be represented as:

$$z = z_x + iz_y$$

$$c = c_x + ic$$

Where $i=\sqrt{(-1)}$ and z_x, c_x are the real parts and z_y & c_y are the imaginary parts of z and c, respectively. The pixel coordinates (x,y) may be associated with (c_x,c_y) or (z_x,z_y)

Based on this concept, the fractal images can be classified as follows:

a) z -plane fractals, wherein (x,y) is a function of (z_x,z_y).

b) c -plane fractals, wherein (x,y) is a function of (c_x,c_y) in the literature, the fractals for $n=2$ in z plane are termed as the Mandelbrot set while the fractals for $n=2$ in c plane are known as Julia sets.

3.2 Generating the Fractals

Fractals have been generated from $z \to z^n+c, n\geq 2$ and $z \to (z^n+c)^{-1}, n\geq 2$ using escape-time techniques, for example by Gujar etal. [8]. we have used in this paper escape time criteria of Relative Superior Ishikawa iterates for both of these functions.

4. ESCAPE CRITERION FOR RELATIVE SUPERIOR JULIA AND MANDELBROT SETS

4.1 Escape Criterion [16]

We obtain a general escape criterion for polynomials of the form $G_c(z)=z^n+c$

Theorem

For general function

$G_c(z)=z^n+c, n=1,2,3,...$ Where $0<s\leq, 0<s'<1$ and c is the complex plane. Define $z_1=(1-s)z+sG_c(z)$

$$\vdots$$

$$z_n=(1-s)z_{n-1}+sG_c(z_{n-1})$$

The general escape criterion is

$\max\{|c|,(2/s)^{1/n+1},(2/s')^{1/n+1}\}$.

4.2 Escape Criterion for Quadratics

Suppose

That $|z|>\max\left\{|c|,\frac{2}{s},\frac{2}{s'}\right\}$ then a

And $|z_n|\to\infty$ as $n\to\infty$ .So, $|z|\geq|c|$ and $|z|>2/s$ as well as $|z|>2/s'$ shows the escape criteria for quadratics [15].

4.3 Escape Criterion for Cubic's

Suppose $|z|>\left\{\max|b|,\left(|a|+\frac{2}{s}\right)^{\frac{1}{2}},\left(|a|+\frac{2}{s'}\right)^{\frac{1}{2}}\right\}$ then

$|z_n|\to\infty$ as $n\to\infty$. This gives an escape criterion for cubic polynomials [22].

4.4 General Escape Criterion

Consider $|z|>\left\{\max|b|,\left(|a|+\frac{2}{s}\right)^{\frac{1}{n}},\left(|a|+\frac{2}{s'}\right)^{\frac{1}{n}}\right\}$ then

$|z_n|\to\infty$ as $n\to\infty$ *is* the escape criterion. (Escape Criterion derived in [22]). Note that the initial value z_0 should be infinity, since infinity is the critical point of $z\to(z^n+c)^{-1}$. However instead of starting with $z_0=\infty$, it is simpler to start with $z_1=c$, which yields the same result. (*A critical point of* $z\to F(z)+c$ *is a point where* $F'(z)=0$ .The role of critical points is explained in [2].

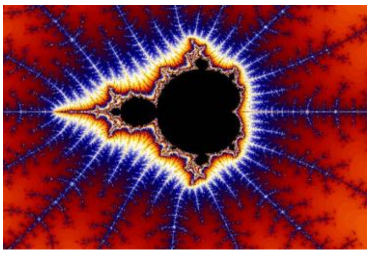

Fig. 4. Midgets of superior Mandelbrot set for n = 4 and s = 0.1

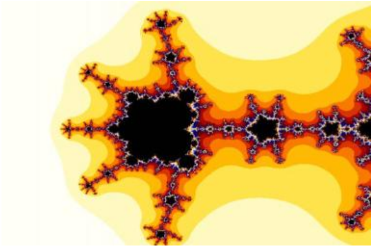

Fig. 5. Midgets of Mandelbrot set for n = 6.

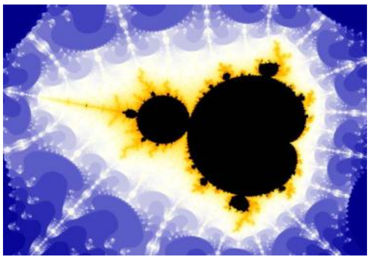

Fig. 6. Midgets of superior Mandelbrot set for n = 17 and s = 0.1

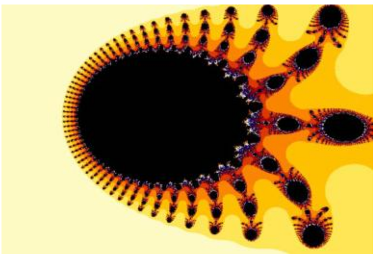

Fig. 7. Midgets of Mandelbrot set for n = 52.

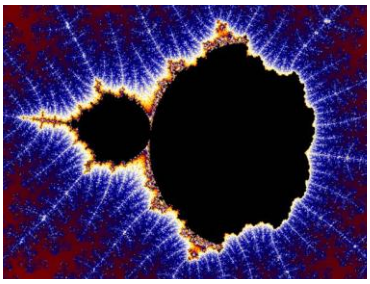

Fig. 8. Midgets of superior Mandelbrot set for n = 52 and s = 0.1

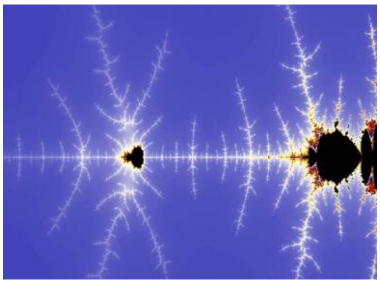

Fig. 9. Connected antenna of period-2 bulb

Table-1

Number of iteration to achieve convergence for $Q_c(z)$, taking c=(-0.4,0) for different values of s.

S.No.	Number of Iteration need		Fixed point of convergence
	Function iterates	Superior iterates	
1.0	24	24	(-0.30623,00)
0.9	-	14	(-0.30623,00)
0.8	-	8	(-0.30623,00)
0.7	-	5	(-0.30623,00)
0.6	-	4	(-0.30623,00)

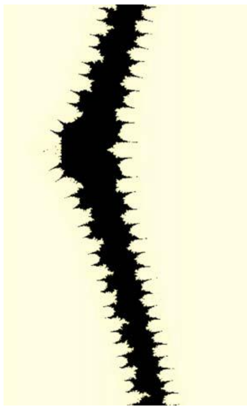

Fig. 10. Filled superior C–K Julia set for N = (0.2, 0), s = 0.5.

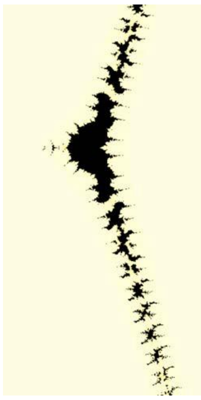

Fig. 11. Filled superior C–K Julia set for N = (1, 0), s = 0.5

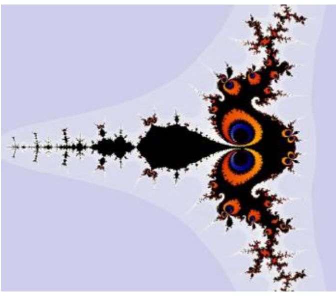

Fig. 12. Filled superior C–K Julia set for N = (9.5, 0), s = 0.09, c = (0, 0).

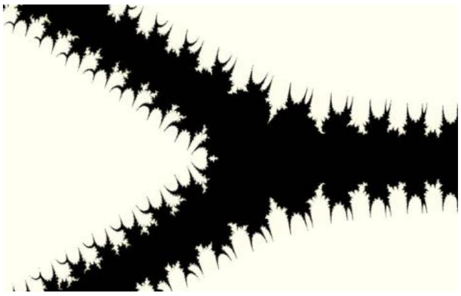

Fig. 13. Filled superior C–K Julia set for N = (0, 0.2, 0), s = 0.1, c = (0, 0).

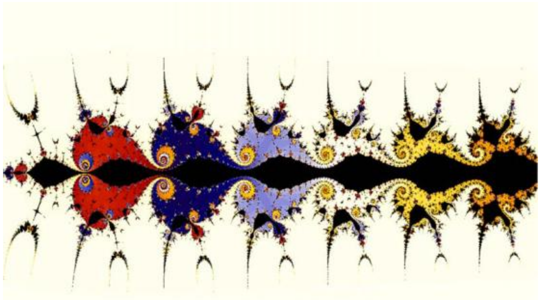

Fig. 14. Portion of filled superior C–K Julia set for N = (0, 11.5), s = 0.01, c = (0, 0).

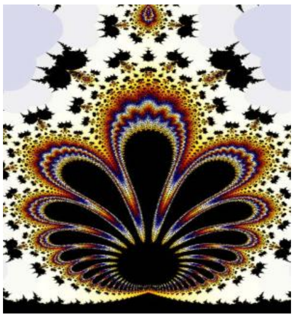

Fig. 15. Filled superior C–K Julia set for N = (0.9, 0), s = 0.1, c = (0, 0). (The above figure is zoom of the part of the C–K Julia set rotated 90_ clockwise.).

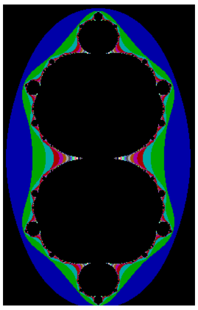

Fig. 16: Dumbbell shaped CSJ at (β, a, b) = (0.5, 1, 0)

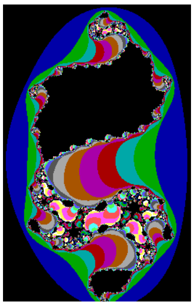

Fig. 17: *CSJ* at (*β*, *a*, *b*) = (0.5, 1, 0.5-0.1i)

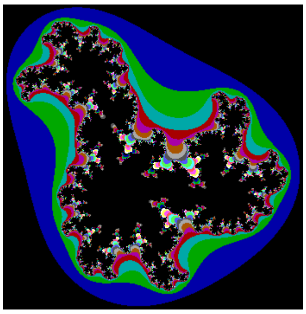

Fig. 18: *CSJ* at (*β*, *a*, *b*) = (0.5, -1-I, -1.4+0.5i)

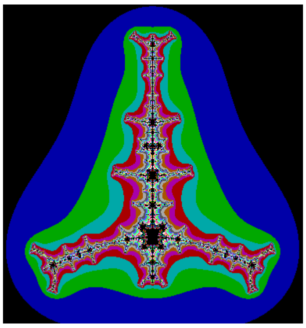

Fig. 19: *CSJ* at (*β*, *a*, *b*) = (0.5, 0, -3.55i)

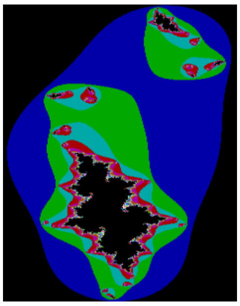

Fig. 20: ***CSJ*** **at (*β*, *a*, *b*) = (0.5, 2.5+I, 1+0.5i)**

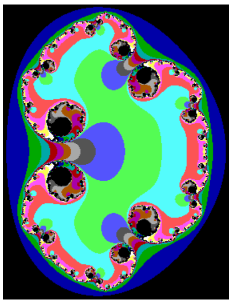

Fig. 21: ***CSJ*** **at (*β*, *a*, *b*) = (0.5, 0, -0.55)**

5. CONCLUSION

In this paper we explore the study done by researcher Negi and Rani, using the Mann iterates to analyzing the visual characteristics of the fractal images in the complex planes respectively, as well as they introduce the small mini Mandelbrot set like images, and C-K Julia set via superior iterations to describe the fractals more fascinating way and observe various remarkable fractals. Their work opens a scope of new research in the study of fractal model using the two step feedback machine.

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