Common Fixed Point Theorem in Sequentially Compact Intuitionistic Fuzzy Metric Spaces under Implicit Relations

Seema Mehra
Department of Mathematics,
M.D.University, Rohtak(INDIA)

ABSTRACT

The aim of this paper is to introduce the notion of sequentially compact intuitionistic fuzzy metric spaces and prove a common fixed point theorem for pairs of weakly compatible self mappings in this newly defined space.

Mathematics Subject Classification

47H10, 54H25

Keywords

Intuitionistic fuzzy metric space, sequentially compact intuitionistic fuzzy metric space, compatible mappings, weakly compatible mappings, common fixed point

1. INTRODUCTION

As a generalization of fuzzy sets introduced by Zadeh [17], Atanassov [2] introduced the concept of intuitionistic fuzzy sets. Recently, using the idea of intuitionistic fuzzy sets, In 2004, Park[12] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et al.[1] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek[10] . In 2006, Turkoglu[16] proved Jungck's[6] common fixed point theorem in the setting of intuitionistic fuzzy metric spaces for commuting mappings. Jungck and Rhoades [6] gave more generalized concept weak compatibility than compatibility. Recently, many authors have studied fixed point theory in intuitionistic fuzzy metric spaces ([1],[12],[14-16]).Recently, Rao, K.P.R., Rao, K.R.K. and Rao, T. Ranga [13] introduced the concept of sequentially compact fuzzy metric space. Using this concept, we introduce the notion of sequentially compact intuitionistic fuzzy metric spaces and prove a common fixed point theorem for pairs of weakly compatible self mappings in this newly defined space.

2. PRELIMINARIES

Definition 1[11] : A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) * is associative and commutative,

(2) * is continuous,

(3) $a * 1 = a$ for all $a \in [0, 1]$,

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

Example 1 : Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 2[11] : A binary operation $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-conorm if it satisfies the following conditions :

(1) ◊ is associative and commutative,

(2) ◊ is continuous,

(3) $a \Diamond 0 = a$ for all $a \in [0, 1]$,

(4) $a \Diamond b \le c \Diamond d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

Example 2 : Two typical examples of continuous t-conorm are $a \Diamond b = \min(a+b, 1)$ and $a \Diamond b = \max(a, b)$.

***Remark* 1.** The concept of triangular norms (*t*-norms) and triangular conforms (*t*-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively. These concepts were originally introduced by Menger [11] in his study of statistical metric spaces. Several examples for these concepts were proposed by many authors ([7-10]).

Definition 3[1] : A 5-tuple $(X,M,N, *, \Diamond)$ is called a intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t-norm, ◊ a continuous t-conorm and M,N are fuzzy sets on $X^2 \times [0,\infty)$, satisfying the following conditions : for each $x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, t) + N(x, y, t) \le 1$,

(ii) $M(x, y, 0) = 0$,

(iii) $M(x, y, t) = 1$ if and only if $x = y$,

(iv) $M(x, y, t) = M(y, x, t)$,

(v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(vi) $M(x, y, .) : [0,\infty) \to [0, 1]$ is left continuous,

(vii) $\lim_{t \to \infty} M(x, y, t) = 1$,

(viii) $N(x, y, 0) = 1$,

(ix) $N(x, y, t) = 0$ if and only if $x = y$,

(x) $N(x, y, t) = N(y, x, t)$,

(xi) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$,

(xii) $N(x, y, .) : [0,\infty) \to [0, 1]$ is right continuous,

(xiii) $\lim_{t \to \infty} N(x, y, t) = 0.$

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions M(x, y, t) and N(x, y, t) denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2 : Every fuzzy metric space (X,M, *) is an intuitionistic fuzzy metric space of the form (X,M, 1-M, *, ◊) such that t-norm * and t-conorm ◊ are associated [12], i.e. $x \lozenge y = 1 - [(1 - x) * (1 - y)]$ for any $x, y \in X$.

Remark 3 : In intuitionistic fuzzy metric space X,

M(x, y, .) is non-decreasing and N(x, y, .) is non-increasing for all

$x, y \in X$.

Example 3 : (Induced intuitionistic fuzzy metric).

Let (X, d) be a metric space. Denote a * b = ab and a ◊ b = min(a + b, 1) for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times [0,\infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}$$

$$N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in R^+$. Then $(X,M_d,N_d, *, \lozenge)$ is an intuitionistic fuzzy metric space.

Definition 4 : Let (X,M, N, *, ◊) be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all t > 0 and p > 0,

$$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.$$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all t > 0

$$\lim_{n \to \infty} M(x_n, x, t) = 1 \text{ and } \lim_{n \to \infty} N(x_n, x, t) = 0.$$

Since * and ◊ are continuous, the limit is uniquely determined from (v) and (xi), respectively.

Definition 5: An intuitionistic fuzzy metric space (X,M,N, *, ◊) is said to be complete if every Cauchy sequence is convergent.

Definition 6 : Let A and B be mappings from an intuitionistic fuzzy metric space (X,M, N, *, ◊) in to itself. The mappings A and B are said to be compatible if

$$\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1,$$

$$\lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0$$

for all t > 0, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \text{ for some } z \in X.$$

Definition 7 : Self mappings A and B of an intuitionistic fuzzy metric space (X,M, N, *, ◊) is said to be weakly compatible if they commute at their coincidence point, that is,

Ax = Bx implies that ABx = BAx for some $x \in X$.

It is easy to see that if self mappings A and B of an intuitionistic fuzzy metric space (X,M, N, *, ◊) is compatible then they are weakly compatible.

The following example shows that the converse of above statement does

not hold.

Example 4 : Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space, Where X = [0, 2] with t-norm and t-conorm defined by a * b =min{a, b} and a ◊ b = max {a, b},for all $a, b \in [0,1]$ and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)} \text{ for}$$

all t > 0

and $M_d(x, y, t) = 0$ and $N_d(x, y, t) = 1$ for t = 0,for all $x, y \in X$.

Define self maps A and B on X as follows:

$$Ax = \begin{cases} 2 & \text{if } 0 \le x \le 1 \\ \dfrac{x}{2} & \text{if } 1 < x \le 2 \end{cases}$$

$$Bx = \begin{cases} 2 & \text{if } x = 1 \\ \dfrac{x+3}{5} & \text{otherwise} \end{cases}$$

And $x_n = 2 - \frac{1}{(2n)}$. Then we have

A(1) = B(1) = 2 and A(2) = B(2) = 1.

Also AB(1) = BA(1) = 2. Thus (A, B) is weak compatible.Again

$Ax_n = 1 - \frac{1}{4n}$, $Bx_n = 1 - \frac{1}{10n}$

Thus $Ax_n = 1$, $Bx_n = 1$

Hence z = 1.

Further $ABx_n = 2$, $BAx_n = \frac{4}{5} - \frac{1}{20n}$

Now $\lim_{n \to \infty}$ M(ABx_n, BAx_n, t) = $\lim_{n\to\infty}$ M(2, $\frac{4}{5} - \frac{1}{20n}$, t) = $\frac{t}{t + \frac{6}{5}} \neq 1$,

$\lim_{n \to \infty}$ N(ABx_n, BAx_n, t) = $\lim_{n\to\infty}$ N(2, $\frac{4}{5} - \frac{1}{20n}$, t) = $\frac{\frac{6}{5}}{t + \frac{6}{5}} \neq 0$, $t > 0$

Hence (A, B) is not compatible.

Definition 8. (X,M,N, *, ◊) is said to be sequentially compact intuitionistic fuzzy metric space if every sequence in X has a convergent sub sequence in it.

Let Φ be the set of all functions ϕ, ϕ' : $[0, 1]^5 \to [0, 1]$ such that

(i) ϕ, ϕ' are non decreasing and non increasing in all coordinates respectively,

(ii) $\phi(t_1, t_2, t_3, t_4, t_5)$, $\phi'(t_1, t_2, t_3, t_4, t_5)$ are continuous in t_4 and t_5 and

(iii) $\phi(t, t, t, t, t) > t$, $\phi'(t, t, t, t, t) < t$ for every $t \in [0, 1)$.

All material on each page should fit within a rectangle of 18 x 23.5 cm (7" x 9.25"), centered on the page, beginning 2.54 cm (1") from the top of the page and ending with 2.54 cm (1") from the bottom. The right and left margins should be 1.9 cm (.75"). The text should be in two 8.45 cm (3.33") columns with a .83 cm (.33") gutter.

3. MAIN RESULTS

Here afterwards, assume that (X,M,N, *, ◊) be a sequentially compact intuitionistic fuzzy metric space with $t * t \geq t$, $s \lozenge s \leq s$ $\forall t, s \in [0, 1]$.

Theorem 3.1 Let P, Q, A, B, Sand T be self-mappings on (X,M,N, *, ◊) such that

(i) $P(X) \subset ST(X)$ and $Q(X) \subset AB(X)$,

(ii) P and AB are continuous or Q and ST are continuous,

(iii) AB = BA, ST = TS, PB = BP, TQ = QT,

(iv) the pairs (P, AB) and (Q, ST) are weakly compatible,

(v) $[1 + aM(ABx, STy, t)] * M(Px, Qy, t)$

$$\geq a[M(ABx, STy, t) * M(Px, ABx, t) * M(Qy, STy, t) * M(Px, STy, \alpha t) * M(Qy, ABx, (2 - \alpha)t)] +$$

$$\phi\begin{pmatrix} M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, t) \\ M(Px, STy, \alpha t), M(Qy, ABx, (2-\alpha)t) \end{pmatrix}$$

$[1 + aN(ABx, STy, t)] \lozenge N(Px, Qy, t)$

$$\leq a[N(ABx, STy, t) \lozenge N(Px, ABx, t) \lozenge N(Qy, STy, t) \lozenge N(Px, STy, \alpha t) \lozenge N(Qy, ABx, (2 - \alpha)t)] +$$

$$\phi'\begin{pmatrix} N(ABx, STy, t), N(Px, ABx, t), N(Qy, STy, t) \\ N(Px, STy, \alpha t), N(Qy, ABx, (2-\alpha)t) \end{pmatrix}.$$

for every $x, y \in X$, for all $t > 0$ and for every $\alpha \in (0, 2)$, where ϕ, $\phi' \in \Phi$ and $a \in \mathbf{R}$, Then P, Q, A, B, S and T have a unique common fixed point p in X .

Proof. Suppose P and AB are continuous.

For every t > 0, let m = sup { M(Px, ABx, t) : $x \in X$ },

and n = inf {N(Px, ABx, t) : $x \in X$ },

Since P and AB are continuous on sequentially compact intuitionistic fuzzy metric space,there exists $u \in X$ such that m = M(Pu, ABu, t), n = N(Pu, ABu, t).

Since $P(X) \subset ST(X)$, there exists $v \in X$ such that Pu = STv...... (vi)

Suppose m < 1 and $n \geq 0$.

Putting x = u, y = v, $\alpha = 1 - q_1$, $q_1 \in (0, 1)$ in (v) we have

$[1 + aM(ABu, STv, t)] * M(Pu, Qv, t)$

$$\geq a[M(ABu, STv, t) * M(Pu, ABu, t) * M(Qv, STv, t) * M(Pu, STv, (1 - q_1)t) * M(Qv, ABu, (1 + q_1)t)] +$$

$$\phi\begin{pmatrix} M(ABu, STv, t), M(Pu, ABu, t), M(Qv, STv, t) \\ M(Pu, STv, (1-q_1)t), M(Qv, ABu, (1+q_1)t) \end{pmatrix},$$

$[1 + aN(ABu, STv, t)] \lozenge N(Pu, Qv, t)$

$$\leq a[N(ABu, STv, t) \lozenge N(Pu, ABu, t) \lozenge N(Qv, STv, t) \lozenge N(Pu, STv, (1- q_1)t) \lozenge N(Qv, ABu, (1 + q_1)t)]+$$

$$\phi'\begin{pmatrix} N(ABu, STv, t), N(Pu, ABu, t), N(Qv, STv, t) \\ N(Pu, STv, (1-q_1)t), N(Qv, ABu, (1+q_1)t) \end{pmatrix}$$

$[1 + am] * M(STv, Qv, t) \geq a[m * m * M(Qv, STv, t) * 1 * m * M(Qv, STv, q_1t)] + \phi(m, m, M(Qv, STv, t), 1, m * M(Qv, STv, q_1t))$, [1 + a n] ◊ N(STv, Qv, t) $\le$ a[n ◊ n ◊ N(Qv, STv, t) ◊ 0 ◊ n ◊ N(Qv, STv, q_1t)] + ϕ' (n, n,N(Qv, STv, t), 0,n ◊ N(Qv, STv, q_1t))

Letting $q_1 \to 1$, we have

[1 + am] * M(STv, Qv, t) ≥ a[m * M(STv, Qv, t)]+

ϕ (m, m,M(Qv, STv, t), 1,M(Qv, STv, t) * m),

[1 + a n] ◊ N(STv, Qv, t) $\le$ a[n ◊ n ◊ N(Qv, STv, t) ◊ 0 ◊ n ◊ N(Qv, STv, q_1t)] + ϕ' (n, n,N(Qv, STv, t), 0,n ◊ N(Qv, STv, q_1t))

M(STv, Qv, t) ≥ φ(m, m,M(Qv, STv, t), 1,M(Qv, STv, t) * m),

N(STv, Qv, t) $\le$ ϕ' (n, n,N(Qv, STv, t), 0,N(Qv, STv, t) ◊ n),

If m ≥ M(Qv, STv, t) = r and n $\le$ N(Qv, STv, t) = r then using t * t ≥ t , s ◊ s $\le$ s , we have

r = M(Qv, STv, t) ≥ ϕ (r, r, r, r, r) > r, r = N(Qv, STv, t) $\le$ ϕ' (r, r, r, r, r) < r.

It is a contradiction. Hence we have

m<M(Qv, STv, t) and n > N(Qv, STv, t) ......(vii).

Since Q(X) $\subseteq$ AB(X), there exists w $\in$ X such that Qv = ABw......(viii)

Now M(Pw,ABw, t) ≤ m < 1 and N(Pw,ABw, t) > n $\ge$ 0 . From (ii), with α = 1+q_2 ,

$q_2 \in (0, 1)$ we have

[1 + aM(ABw, STv, t)] * M(Pw, Qv, t)

≥ a[M(ABw, STv, t) * M(Pw,ABw, t) * M(Qv, STv, t) *

M(Pw, STv, (1 + q_2)t) * M(Qv, ABw, (1 − q_2)t)] +

$$\phi\begin{pmatrix} M(ABw, STv, t), M(Pw, ABw, t), M(Qv, STv, t) \\ M(Pw, STv, (1+q_2)t), M(Qv, ABw, (1-q_2)t) \end{pmatrix}$$

,

[1 + aN(ABw, STv, t)] ◊ N(Pw, Qv, t)

$\le$ a[N(ABw, STv, t) ◊ N(Pw,ABw, t) ◊ N(Qv, STv, t) ◊

N(Pw, STv, (1 + q_2)t) ◊ N(Qv, ABw, (1 − q_2)t)] +

$$\phi'\begin{pmatrix} N(ABw, STv, t), N(Pw, ABw, t), N(Qv, STv, t) \\ N(Pw, STv, (1+q_2)t), N(Qv, ABw, (1-q_2)t) \end{pmatrix}$$

[1 + aM(Qv, STv, t)] * M(Pw,ABw, t)

≥ a[M(Qv, STv, t) * M(Pw,ABw, t) * M(Qv, STv, t) *

M(Pw,ABw, t) * M(Qv, STv, q_2 t) * 1] +

$$\phi\begin{pmatrix} M(Qv, STv, t), M(Pw, ABw, t), M(Qv, STv, t) \\ M(Pw, ABw, t) * M(Qv, STv, q_2 t), 1 \end{pmatrix}$$

,

[1 + aN(Qv, STv, t)] ◊ N(Pw,ABw, t)

$\le$ a[N(Qv, STv, t) ◊ N(Pw,ABw, t) ◊ N(Qv, STv, t) ◊

N(Pw,ABw, t) ◊ N(Qv, STv, q_2 t) ◊ 0] +

$$\phi'\begin{pmatrix} N(Qv, STv, t), N(Pw, ABw, t), N(Qv, STv, t) \\ N(Pw, ABw, t) \Diamond N(Qv, STv, q_2 t), 0 \end{pmatrix}$$

,

Letting $q_2 \to 1$, we have

[1 + aM(Qv, STv, t)] * M(Pw,ABw, t)

≥ a[M(Qv, STv, t)] * M(Pw,ABw, t)+

$$\phi\begin{pmatrix} M(Qv, STv, t), M(Pw, ABw, t), M(Qv, STv, t) \\ M(Qv, STv, t) * M(Pw, ABw, t), 1 \end{pmatrix}$$

,

[1 + aN(Qv, STv, t)] ◊ N(Pw,ABw, t)

$\le$ a[N(Qv, STv, t)] ◊ N(Pw,ABw, t)+

$$\phi'\begin{pmatrix} N(Qv, STv, t), N(Pw, ABw, t), N(Qv, STv, t) \\ N(Qv, STv, t) \Diamond N(Pw, ABw, t), 0 \end{pmatrix}$$

M(Pw,ABw, t) ≥

$$\phi\begin{pmatrix} M(Qv, STv, t), M(Pw, ABw, t), M(Qv, STv, t) \\ M(Qv, STv, t) * M(Pw, ABw, t), 1 \end{pmatrix}$$

,

N(Pw,ABw, t) $\le$.

$$\phi'\begin{pmatrix} N(Qv, STv, t), N(Pw, ABw, t), N(Qv, STv, t) \\ N(Qv, STv, t) \Diamond N(Pw, ABw, t), 0 \end{pmatrix}$$

.

If M(Qv, STv, t) ≥ M(Pw,ABw, t)

= s = N(Qv, STv, t) < N(Pw,ABw, t) then

M(Pw,ABw, t) ≥ φ(s, s, s, s, s) > s and

N(Pw,ABw, t) $\leq \phi'$ (s, s, s, s, s) < s

It is a contradiction. Hence we have

M(Qv, STv, t)<M(Pw,ABw, t),

N(Qv, STv, t)>N(Pw,ABw, t) ......(ix).

Now from definition of m, n and (ix) , (vii) we have

m ≥ M(Pw,ABw, t)>M(Qv, STv, t)> m,

n $\leq$ N(Pw,ABw, t)< N(Qv, STv, t) < n It is a contradiction. Hence m = 1and n = 0.

Thus Pu = ABu......(x)

Suppose M(Qv, STv, t) < 1, N(Qv, STv, t) > 0 . Then from (ii) with α = 1 we have

[1 + aM(ABu, STv, t)] $*$ M(Pu, Qv, t)

≥ a[M(ABu, STv, t) $*$ M(Pu, ABu, t) $*$ M(Qv, STv, t) $*$ M(Pu, STv, t) $*$ M(Qv, ABu, t)] +

$$\phi\begin{pmatrix} M(ABu,STv,t), M(Pu,ABu,t), M(Qv,STv,t) \\ M(Pu,STv,t), M(Qv,ABu,t) \end{pmatrix}$$

,

[1 + aN(ABu, STv, t)] ◊ N(Pu, Qv, t)

$\leq$ a[N(ABu, STv, t) ◊ N(Pu, ABu, t) ◊ N(Qv, STv, t) ◊ N(Pu, STv, t) ◊ N(Qv, ABu, t)] +

$$\phi'\begin{pmatrix} N(ABu,STv,t), N(Pu,ABu,t), N(Qv,STv,t) \\ N(Pu,STv,t), N(Qv,ABu,t) \end{pmatrix}$$

.

[1 + a1] $*$ M(STv,Qv, t) ≥ a[1 $*$ 1 $*$ M(Qv, STv, t) $*$ 1 $*$ M(Qv, STv, t)] +

$$\phi(1,1, M(Qv,STv,t),1, M(Qv,STv,t)),$$

[1 + a1] ◊ N(STv,Qv, t) $\leq$ a[0 ◊ 0 ◊ N(Qv, STv, t) ◊ 0 ◊ N(Qv, STv, t)] +

$$\phi'(0,0, N(Qv,STv,t),0, N(Qv,STv,t)).$$

M(STv, Qv, t) ≥ ϕ (1, 1,M(Qv, STv, t), 1,M(Qv, STv, t))>M(Qv, STv, t),

N(STv,Qv, t) $\leq$ $\phi'(0,0,N(Qv,STv,t),0,N(Qv,STv,t)$ < N(Qv, STv, t).

It is a contradiction. Hence Qv = STv.....(xi)

Thus ABu = Pu = STv = Qv = p,say......(xii)

Since the pair (P, AB) is weakly compatible we have

Pp = PPu = PABu = ABPu = ABABu = ABp......(xiii).

Suppose r = M(p, Pp, t) < 1, r = N(p, Pp, t) > 0. From (ii) with α = 1, x = p, y = v we have

[1 + aM(ABp, p, t)] $*$ M(Pp, p, t)

≥ a[M(ABp, p, t) $*$ 1 $*$ 1 $*$ M(Pp, p, t) $*$ M(p, ABp, t)] +

ϕ (M(ABp, p, t), 1, 1,M(Pp, p, t),M(p, ABp, t)),

[1 + aN(ABp, p, t)] ◊ N(Pp, p, t)

$\leq$ a[N(ABp, p, t) ◊ 0 ◊ 0 ◊ N(Pp, p, t) ◊ N(p, ABp, t)] +

ϕ' (N(ABp, p, t), 0, 0,N(Pp, p, t),N(p, ABp, t)).

M(Pp, p, t) ≥ φ(r, r, r, r, r) > r and

N(Pp, p, t) $\leq \phi'$ (r, r, r, r, r) < r

It is a contradiction.

Hence Pp = p.Thus **ABp = Pp = p**....(xiv)

Since the pair (Q, ST) is weakly compatible we have Qp = STp. Using (ii) with

α = 1, x = u, y = p we can show that Qp = p.

Thus STp = p = Qp....(xv).

Suppose Bp $\neq$ p and r = M(p, Bp, t) < 1, r = N(p, Bp, t) > 0.

From (ii) with x = Bp, y = p, α = 1 we have,

[1 +aM(ABBp, STp, t)] $*$ M(PBp, Qp, t)

≥ a[M(ABBp, STp, t) $*$ M(PBp, ABBp,t) $*$ M(Qp,STp,t) $*$ M(PBp,STp,t) $*$ M(Qp, ABBp, t)] +

$$\phi\begin{pmatrix} M(ABBp,STp,t), M(PBp,ABBp,t), M(Qp,STp,t) \\ M(PBp,STp,t), M(Qp,ABBp,t) \end{pmatrix}$$

1 +aN(ABBp, STp, t)] ◊ N(PBp, Qp, t)

$\leq$ a[N(ABBp, STp, t) ◊ N(PBp, ABBp, t) ◊ N(Qp, STp, t) ◊ N(PBp,STp,t) ◊ N(Qp, ABBp, t)] +

$$\phi'\begin{pmatrix} N(ABBp,STp,t), N(PBp,ABBp,t), N(Qp,STp,t) \\ N(PBp,STp,t), N(Qp,ABBp,t) \end{pmatrix}$$
.

Since AB = BA, PB = BP, we have

P(Bp) =B(Pp) = Bp and AB(Bp) = B(ABp) = Bp,

[1 +aM(Bp, STp, t)] ∗ M(Bp, Qp, t)

≥ a[M(Bp, STp, t) ∗ M(Bp, Bp,t) ∗ M(Qp,STp,t)

∗ M(Bp,STp,t) ∗ M(Qp, Bp, t)] +

$$\phi\begin{pmatrix} M(Bp,STp,t), M(Bp,Bp,t), M(Qp,STp,t) \\ M(Bp,STp,t), M(Qp,Bp,t) \end{pmatrix}$$
,

1 +aN(Bp, STp, t)] ◊ N(Bp, Qp, t)

≤ a[N(Bp, STp, t) ◊ N(Bp, Bp, t) ◊ N(Qp, STp, t)

◊ N(Bp,STp,t) ◊ N(Qp, Bp, t)] +

$$\phi'\begin{pmatrix} N(Bp,STp,t), N(Bp,Bp,t), N(Qp,STp,t) \\ N(Bp,STp,t), N(Qp,Bp,t) \end{pmatrix}.$$

M(Bp, p, t) ≥ a[M(Bp, Bp,t) ∗ M(p,p,t) ∗ M(Bp,p,t)

∗ M(p, Bp, t)] +

$$\phi\begin{pmatrix} M(Bp,p,t), M(Bp,Bp,t), M(p,p,t) \\ M(Bp,p,t), M(p,Bp,t) \end{pmatrix},$$

N(Bp, p, t) ≤ a[N(Bp, p, t) ◊ N(Bp, Bp, t)

◊ N(p, p, t) ◊ N(Bp,p,t) ◊ N(p, Bp, t)] +

$$\phi'\begin{pmatrix} N(Bp,p,t), N(Bp,Bp,t), N(p,p,t) \\ N(Bp,p,t), N(p,Bp,t) \end{pmatrix}.$$

M(Bp, p, t) ≥ $\phi(r,r,r,r,r)$ > r , N(Bp, p, t) ≤ ϕ' (r, r, r, r, r) < r.

It is a contradiction. Hence Bp = p.

Since p = ABp, we have **p = Ap = Bp = Pp**

Finally, we show that Tp = p by putting x = p and y = Tp, α = 1 we have,

in (v), Suppose Tp ≠ p and r = M(p, Tp, t) < 1, r = N(p, Tp, t) > 1.

If Tp ≠ p, then

[1 +aM(ABp, STTp, t)] ∗ M(Pp, QTp, t)

≥ a[M(ABp, STTp, t) ∗ M(Pp, ABp,t) ∗ M(QTp,STTp,t)

∗ M(Pp,STTp,t) ∗ M(QTp, ABp, t)] +

$$\phi\begin{pmatrix} M(ABp,STTp,t), M(Pp,ABp,t), M(QTp,STTp,t) \\ M(Pp,STTp,t), M(QTp,ABp,t) \end{pmatrix}$$

1 +aN(ABp, STTp, t)] ◊ N(Pp, QTp, t)

≤ a[N(ABp, STTp, t) ◊ N(Pp, ABp, t) ◊ N(QTp, STTp, t)

◊ N(Pp,STTp,t) ◊ N(QTp, ABp, t)] +

$$\phi'\begin{pmatrix} N(ABp,STTp,t), N(Pp,ABp,t), N(QTp,STTp,t) \\ N(Pp,STTp,t), N(QTp,ABp,t) \end{pmatrix}$$
.

Since ST = TS, TQ = QT, we have

Q(Tp) =T(Qp) = Tp and ST(Tp) = T(STp) = Tp,

[1 +aM(ABp, Tp, t)] ∗ M(Pp, Tp, t)

≥ a[M(ABp, Tp, t) ∗ M(Pp, ABp,t) ∗ M(Tp,Tp,t)

∗ M(Pp,Tp,t) ∗ M(Tp, ABp, t)] +

$$\phi\begin{pmatrix} M(ABp,Tp,t), M(Pp,ABp,t), M(Tp,Tp,t) \\ M(Pp,Tp,t), M(Tp,ABp,t) \end{pmatrix}$$

1 +aN(ABp, Tp, t)] ◊ N(Pp,Tp, t)

≤ a[N(ABp,Tp, t) ◊ N(Pp, ABp, t) ◊ N(Tp, Tp, t)

◊ N(Pp,Tp,t) ◊ N(Tp, ABp, t)] +

$$\phi'\begin{pmatrix} N(ABp,Tp,t), N(Pp,ABp,t), N(Tp,Tp,t) \\ N(Pp,Tp,t), N(Tp,ABp,t) \end{pmatrix}.$$

M(p, Tp, t) ≥ a[1 ∗ 1 ∗ M(p,Tp,t) ∗ M(Tp, p, t)] +

$$\phi\begin{pmatrix} M(p,Tp,t), M(p,p,t), M(p,p,t) \\ M(p,Tp,t), M(Tp,p,t) \end{pmatrix},$$

N(p, Tp, t) ≤ a[0 ◊ 0 ◊ N(p, Tp, t) ◊ N(Tp,p,t) +

$$\phi'\begin{pmatrix} N(p,Tp,t), N(p,p,t), N(p,p,t) \\ N(p,Tp,t), N(Tp,p,t) \end{pmatrix}.$$

M(p, Tp, t) ≥ $\phi(r,r,r,r,r)$ > r ,

N(p, Tp, t) ≤ ϕ' (r, r, r, r, r) < r.

It is a contradiction. Hence Tp = p.

Since p = STp, we have p = Sp.

By combining the above results, Ap = Bp = Sp = Tp = Pp = Qp = p,

So P, Q, A, B, S and T have a fixed common fixed point p.

Now to prove uniqueness, if possible $p_0 \neq$ p be another common fixed point P, Q, A, B, S and T. Then

Suppose $p_0 \neq$ p and r = M(p_0, p, t) < 1, r = N(p_0, p, t) > 1.

From (ii) with x = p_0 , y = p, α = 1 we have,

[1 +aM(ABp_0, STp, t)] ∗ M(Pp_0 , Qp, t)

≥ a[M(ABp_0, STp, t) ∗ M(Pp_0, ABp_0,t) ∗ M(Qp,STp,t)

∗ M(Pp_0,STp,t) ∗ M(Qp, ABp_0, t)] +

$$\phi\begin{pmatrix} M(ABp_0,STp,t), M(Pp_0,ABp_0,t), M(Qp,STp,t) \\ M(Pp_0,STp,t), M(Qp,ABp_0,t) \end{pmatrix}$$

1 +aN(ABp_0, STp, t)] ◊ N(Pp_0 , Qp, t)

≤ a[N(ABp_0, STp, t) ◊ N(Pp_0, ABp_0,t) ◊ N(Qp, STp, t)

◊ N(Pp_0,STp,t) ◊ N(Qp, ABp_0, t)] +

$$\phi'\begin{pmatrix} N(ABp_0,STp,t), N(Pp_0,ABp_0,t), N(Qp,STp,t) \\ N(Pp_0,STp,t), N(Qp,ABp_0,t) \end{pmatrix}$$

.

M(p_0 ,p, t) ≥ a[M(p_0, p_0,t) ∗ M(p,p,t) ∗ M(p_0,p,t) ∗ M(p, p_0, t)] +

$$\phi\begin{pmatrix} M(p_0,p,t), M(p_0,p_0,t), M(p,p,t) \\ M(p_0,p,t), M(p,p_0,t) \end{pmatrix}$$

N(p_0 , p, t) ≤ a[N(p_0, p_0,t) ◊ N(p, p, t)

◊ N(p_0,p,t) ◊ N(p, p_0, t)] +

$$\phi'\begin{pmatrix} N(p_0,p,t), N(p_0,p_0,t), N(p,p,t) \\ N(p_0,p,t), N(p,p_0,t) \end{pmatrix}.$$

M(p_0, p, t) ≥ $\phi(r,r,r,r,r)$ > r ,

N(p_0, p, t) ≤ ϕ' (r, r, r, r, r) < r.

It is a contradiction. Hence p_0 = p.

Hence p is a common fixed point of P, Q, A, B, S and T.

If we put B = T = I in theorem (3.1), we have the following:

Corollary 3.2. Let P, Q, A and S be self-mappings on (X,M,N, *, ◊) such that

(i) P(X) ⊂ S(X) and Q(X) ⊂ A(X),

(ii) P and A are continuous or Q and S are continuous,

(iv) the pairs (P, A) and (Q, S) are weakly compatible,

(v) [1 +aM(Ax, Sy, t)] ∗ M(Px, Qy, t)

≥ a[M(Ax, Sy, t) ∗ M(Px, Ax,t) ∗ M(Qy,Sy,t)

∗ M(Px,Sy,αt) ∗ M(Qy, Ax, (2 − α)t)] +

$$\phi\begin{pmatrix} M(Ax,Sy,t), M(Px,Ax,t), M(Qy,Sy,t) \\ M(Px,Sy,\alpha t), M(Qy,Ax,(2-\alpha)t) \end{pmatrix}$$

[1 +aN(Ax, Sy, t)] ◊ N(Px, Qy, t)

≤ a[N(Ax, Sy, t) ◊ N(Px, Ax, t) ◊ N(Qy, Sy, t)

◊ N(Px,Sy,αt) ◊ N(Qy, Ax, (2 − α)t)] +

$$\phi'\begin{pmatrix} N(Ax,Sy,t), N(Px,Ax,t), N(Qy,Sy,t) \\ N(Px,Sy,\alpha t), N(Qy,Ax,(2-\alpha)t) \end{pmatrix}.$$

for every x, y ∈ X , for all t > 0 and for every α ∈ (0, 2), where φ, ϕ' ∈ Φ and a ∈ **R**,

Then P,Q, A and S have a unique common fixed point p in X .

If we put P = Q = f and A = S = g and B = T = I in theorem (3.1), we have the following:

Corollary 3.3. Let f, g be self-mappings on (X,M,N, *, ◊) such that

(i) f(X) ⊂ g(X),

(ii) g(X) is complete,

(iv) the pair (f, g) is weakly compatible,

(v) $[1 + aM(gx, gy, t)] * M(fx, fy, t)$

$\geq a[M(gx, gy, t) * M(fx, gx, t) * M(fy, gy, t)$

$* M(fx, gy, \alpha t) * M(fy, gx, (2 - \alpha)t)] +$

$$\phi\begin{pmatrix} M(gx, gy, t), M(fx, gx, t), M(fy, gy, t) \\ M(fx, gy, \alpha t), M(fy, gx, (2-\alpha)t) \end{pmatrix}$$

$[1 + aN(gx, gy, t)] \Diamond N(fx, fy, t)$

$\leq a[N(gx, gy, t) \Diamond N(fx, gx, t) \Diamond N(fy, gy, t)$

$\Diamond N(fx, gy, \alpha t) \Diamond N(fy, gx, (2 - \alpha)t)] +$

$$\phi'\begin{pmatrix} N(gx, gy, t), N(fx, gx, t), N(fy, gy, t) \\ N(fx, gy, \alpha t), N(fy, gx, (2-\alpha)t) \end{pmatrix}.$$

for every $x, y \in X$, for all $t > 0$ and for every $\alpha \in (0, 2)$, where $\varphi, \phi' \in \Phi$ and $a \in \mathbf{R}$,

Then f, g have a unique common fixed point p in X .

4. REFERENCES

[1] Alaca, C., Turkoglu, D. and Yildiz, C., Fixed points in intuitionistic fuzzy metric spaces,*Chaos, Solitons and Fractals,* **29(2006),** 1073- 1078.

[2] Atanassov, K., Intuitionistic fuzzy sets, *Fuzzy sets and systems* **20(1986)** 87-96.

[3] Dubois, D. and Prade, H., Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems Plenum Press, New York, 1980.

[4] George, A. and Veeramani, P. , On some results in fuzzy metric space, *Fuzzy Sets and Systems,* **64(1994),** 395-399.

[5] Grabiec, M. , Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems **27 (1988),** 385-389.

[6] Jungck, G. and Rhoades,B.E, Fixed point for set valued functions without continuity, Indian. J. Pure & Applied Math.,29(3), 1998,227-238.

[7] Klement, E.P., Operations on fuzzy sets: an axiomatic approach, Inform. Sci. 27 (1984),221-232.

[8] Klement, E.P., Mesiar, R. and Pap, E., A characterization of the ordering of continuous t-norms, Fuzzy Sets and Systems 86 (1997), 189-195.

[9] Klement, E.P., Mesiar, R. and Pap, E., Triangular Norms, Kluwer Academic Publ., Trendsin Logic 8, Dordrecht 2000.

[10] Kramosil, O. and Michalek, J. , Fuzzy metric and statistical metric spaces, *Kybernetica* , **11 (1975),** 326-334.

[11] Menger, K., Statistical metrics, Proc. Nat. Acad. Sci. 28 (1942), 535-537.

[12] Park, J.H., Intuitionistic Fuzzy Metric Spaces, Chaos, Solitons and Fractals, 22(2004) 1039-1046.

[13] Rao, K.P.R., Rao, K.R.K. and Rao, T. Ranga, Common fixed point theorems in Sequentially Compact fuzzy metric spaces, International Mathematical Forum, 2(51)(2007)2543-2549.

[14] Sedghi, S., Common fixed point theorems for multivalued contractions, International Mathematical Forum, 2(31), (2007),1499-1506.

[15] Sharma, S., Kutukcu, S., and Rathore, R.S., Common fixed point for multivalued mappings in intuitionistic fuzzy metric spaces, Korean Math.Soc.22(3)(2007),391-399.

[16] Turkoglu, D., Alaca, C., Cho, Y.J. and Yildiz, C., Common fixed point theorems in intuitionistic fuzzy metric spaces, J. Appl. Math. And Computing, 22(1-2)(2006), 411-424.

[17] Zadeh, L. A., Fuzzy sets, *Inform and Control* **8 (1965),** 338-353.