# Common Fixed Point for R-Weak Commutative Mappings without Continuity in Fuzzy Metric Spaces

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## ABSTRACT
This paper deals with some results on common fixed point theorems in fuzzy metric spaces generalizing the earlier results of Pant [20], Som [28], [29] and Vasuki [30] by removing the assumption of continuity.

## General Terms
47H10, 54H25

## Keywords
Common fixed point, Fuzzy metric space, R-weak commutative mappings.

## 1. INTRODUCTION
Zadeh [31] paves the way for fuzzy mathematics by introducing the concept of fuzzy sets. Deng [7], Erceg [8], Kaleva and Seikkala [16], Kramosil and Michalek [17] have introduced the concepts of fuzzy metric spaces in different ways. Grabiec [10] followed Kramosil and Michalek [17] and obtained the fuzzy version of Banach's fixed point theorem. The most interesting references in this direction are : George and Veeramani [9], Kaleva [15], Mishra, Sharma and Singh [19], Sharma [22],[23], Sharma and Bagwan [24], Sharma and Deshpande [25],[26],[27], Cho [5] and for fuzzy mappings : Bose and Sahani [1], Lee, Cho and Jung [18], Butnariu [2], Heilpern [11], Chang [3], Chang, Cho, Lee and Lee [4]. In 1976, Jungck [12] established common fixed point theorems for commuting maps generalizing the Banach's fixed point theorem. Sessa [21] defined a generalization of commutavity, which is called weak commutativity. Further, Jungck [13] introduced more generalized commutativity, so called compatibility. Mishra, Sharma and Singh [19] introduced the concept of compatibility in fuzzy metric spaces. In 1998, Jungck and Rhoades [14] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not true. Sharma and Deshpande [26] improved the results of Mishra, Sharma and Singh [19], Cho [5], Cho Pathak, Kang & Jung [6], Sharma [22] and Sharma & Deshpande [25]. They proved common fixed point theorems for weakly compatible maps in fuzzy metric spaces without taking any mapping continuous.

## 2. PRELIMINARIES
**Definition** 2.1 [31] A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

**Definition** 2.2 [27] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if $\{[0,1], *\}$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$

whenever $a \leq c \text{ and } b \leq d$ , $a, b, c, d \in [0,1]$.

**Definition** 2.3 [17] The triplet $( X, M,* )$ is a fuzzy metric space if $X$ is an arbitrary set, $*$ is continuous $t$-norm. $M$ is a fuzzy set in $X^2 \times [0,\infty]$ satisfying the following conditions:

$(FM-1)$ $M(x, y, 0) = 0.$

$(FM-2)$ $M(x, y, t) = 1 , for\ all\ t > 0 \iff x = y$

$(FM-3)$ $M(x, y, t) = M(y, x, t).$

$(FM-4)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

$for\ all\ x, y, z \in X\ and\ t, s > 0.$

$(FM-5)$ $M(x, y, .): [0, \infty] \to [0, 1] is\ left\ continuous.$

In this paper $( X, M,* )$ will denote a fuzzy metric space in the sense of above definition with the following condition

$(FM-6)$ $\lim_{n\to\infty} M(x, y, t) = 1$ $for\ all\ x, y \in X.$

**Definition** 2.4 [17] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M,*)$ is called Cauchy sequence if

$$\lim_{n\to\infty} M\big(x_{n+p}, x_n, t\big) = 1$$

for every $t > 0$ and each $p > 0$.

**Definition** 2.5 [17] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said to be convergent to $x \in X$ if

$$\lim_{n\to\infty} M(x_n\, x , t) = 1 \text{ for each } t > 0.$$

**Definition** 2.6 [17] A fuzzy metric space $(X, M,*)$ is said to be complete if every Cauchy sequence in $X$ converges in $X$.

**Remark** 2.1 Since $*$ is continuous, it follows from $(FM-4)$ that limit of sequence is uniquely determined.

**Lemma** 2.1 [5] Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M,*)$ with the condition $(FM-6)$. If there exists a number $k \in (0,1)$ such that

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots$

then $\{y_n\}$ is a Cauchy sequence in $X$.

**Lemma** 2.2 [19] If for all $x, y \in X$, $t > 0$ and for a number $k \in (0,1)$

$$M(x, y, kt) \geq M(x, y, t) \text{ then } x = y.$$

**Definition** 2.7 [19] Let A and B be mappings from a fuzzy metric space $(X, M,*)$ into itself. The mappings A and B are said to be compatible if

$$\lim_{n\to\infty} M(ABx_n, BAx_n , t) = 1 , \text{ for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

**Definition** 2.8[14] A pair $A$ and $S$ is called weakly compatible pair in fuzzy metric space if they commute at a coincidence points.

**Example** 2.1 Let $X = [0, 2]$ with the metric $d$ defined by $d(x, y) = |x - y|$

for each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \qquad x, y \in X,$$

and define

$$M(x, y, 0) = 0, \qquad x, y \in X.$$

Clearly $(X, M, *)$ is a fuzzy metric space on $X$ where $*$ is defined by

$$a * b = a\,b \;\; or \;\; a * b = min\{a, b\}.$$

Define $A, B : X \longrightarrow X$ by

$$Ax = \begin{cases} x, \; if \; x \in [\,0, \frac{1}{4}) \\ \frac{1}{4}, \; if \; x \geq \frac{1}{4} \end{cases} \quad \text{and} \quad Bx = \frac{x}{1+x},$$

for all $x \in [0, 2]$.Consider the sequence $\{x_n = \frac{1}{3} + \frac{1}{n} : n \geq 1\}$

in $X$. Then

$$\lim_{n\to\infty} Ax_n = \frac{1}{4}, \lim_{n\to\infty} Bx_n = \frac{1}{4}$$

But

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = \frac{t}{t + \left|\frac{1}{4} - \frac{1}{5}\right|} \neq 1.$$

Thus $A$ and $B$ are non-compatible. But $A$ and $B$ are commuting at their coincidence point $x = 0$, that is weakly compatible at $x = 0$.

Thus, weakly compatible maps need not be compatible.

**Definition** 2.9 [20] Two mappings f and g of a fuzzy metric space $(X, M, *)$ into itself are said to be weakly commuting if

$$M(fgx, gfx, t) \geq M(fx, gx, t), for \; every \; x \in X.$$

**Definition** 2.10 [30] The mappings f and g of a fuzzy metric space (X, M, ∗) into itself are R-weakly commuting provided there exists some positive real number R such that

$$M(fx, gfx, t) \geq M(fx, gx, \tfrac{t}{R}), for \text{ all } x \in X.$$

Weak commutativity implies R-weak commutativity and the converse is true for $R \leq 1$.

## 3. MAIN RESULTS

**Theorem** 3.1 Let $S$ and $T$ be two self mappings of a fuzzy metric space $(X, M, *)$. Let $A$ be a self mapping of $X$ satisfying

(3.1) $A(X) \subseteq S(X) and \; A(X) \subseteq T(X)$

(3.2) $M(Ax, Ay, t) \geq r\left[\min\begin{Bmatrix} M(Sx, Ty, t), M(Sx, Ax, t), \\ M(Sx, Ay, t), M(Ty, Ay, t) \end{Bmatrix}\right].$

for all $x, y \in X$ where $r : [0, 1] \longrightarrow [0, 1]$ is a continuous function such that

$$r(t) > t \text{ for each } t < 1 \text{ and } r(t) = 1 \text{ for } t = 1.$$

(3.3) If one of $A(X), S(X), T(X)$ is a complete subspace of $X$,

then

(i) $A$ and $S$ have a coincidence point, and

(ii) $A$ and $T$ have a coincidence point.

Further if

(3.4) $\{A, S\}$ and $\{A, T\}$ are $R$ −weakly commuting mappings, then

(iii) $A, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Let a sequence $\{y_n\}$ in $X$ be such that

$$y_{2n} = Ax_{2n} = Sx_{2n+1}$$

$$y_{2n+1} = Ax_{2n+1} = Tx_{2n+2}$$

and $T(X)$ be complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $T(X)$ and has a limit in $T(X)$, call it $z$.

Let $w \in T^{-1}(z)$, then $Tw = z$.

We shall use the fact that subsequences $\{y_{2n}\}$, $\{y_{2n+2}\}$ also converges to $z$.

By putting $x = x_{2n+1}, y = w$ in (3.2), we get

$$M(Ax_{2n+1}, Aw, t) = M(y_{2n+1}, Aw, t)$$
$$\geq r\left[min\begin{Bmatrix} M(y_{2n}, Tw, t), M(y_{2n}, y_{2n+1}, t), \\ M(y_{2n}, Aw, t), M(Tw, Aw, t) \end{Bmatrix}\right]$$

Taking limit as $n \to \infty$, we get

$$M(z, Aw, t) \geq rM(z, Aw, t) > M(z, Aw, t)$$

which is a contradiction. Therefore $Aw = Tw = z$ i.e. $w$ is a coincidence point of $A$ and $T$.

Since $A(X) \subset S(X), Tw = z$ implies that $z \in S(X)$.

Let $v \in S^{-1}z$. Then $Sv = z$.

By putting $x = v$ and $y = x_{2n+2}$ in (3.2), we get

$$M(Av, Ax_{2n+2}, t) = M(Av, y_{2n+2}, t)$$
$$\geq r\left[min\begin{Bmatrix} M(Sv, y_{2n+1}, t), M(Sv, Av, t), \\ M(Sv, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t) \end{Bmatrix}\right]$$

Taking limit as $n \to \infty$, we get

$$M(Av, z, t) \geq rM(z, Av, t) > M(z, Av, t)$$

which is a contradiction. Therefore $Av = Sv = z$ i.e. $v$ is a coincidence point of $A$ and $S$.

If $A(X)$ is complete then by (3.1) $z \in A(X) \subset T(X) or$

$z \in A(X) \subset S(X)$.

Thus (i) and (ii) are completely established.

Since the pair $\{A, T\}$ is R-weakly commuting, therefore we have

$$M(ATw, TAw, t) \geq M(Aw, Tw, \tfrac{t}{R}), \text{ for all } x \in X$$

which gives $ATw = TAw$, i.e. $Az = Tz$.

Similarly the R-weak commutativity of pair $\{A, S\}$ gives $Az = Sz$.

By putting $x = x_{2n+1}, y = z$ in 3.2 we get

$$M(Ax_{2n+1}, Az, t) = M(y_{2n+1}, Az, t)$$

$$\geq r\left[min\begin{Bmatrix} M(Sx_{2n+1}, Tz, t), M(Sx_{2n+1}, Ax_{2n+1}, t), \\ M(Sx_{2n+1}, Az, t), M(Tz, Az, t) \end{Bmatrix}\right].$$

Taking limit as $n \to \infty$, we get

$$M(z, Az, t) \geq rM(z, Az, t) > M(z, Az, t),$$

which is a contradiction. Thus $Az = z = Sz = Tz$ i.e. $z$ is common fixed point of $A, S$ and $T$.

**Theorem** 3.2 Let $S$ and $T$ be two self mappings of a fuzzy metric space $(X, M, *)$. Let $A, B, S$ and $T$ be self mappings of $X$ satisfying

(3.5) $A(X) \subseteq S(X)$ *and* $B(X) \subseteq T(X)$.

(3.6) $aM(Tx, Sy, t) + bM(Tx, Ax, t) + cM(Sy, By, t) + max\{M(Ax, Sy, t), M(By, Tx, t)\} \leq qM(Ax, By, t)$,

for all $x, y \in X$ where $a, b, c \geq 0, q > 0$ with $q < a + b + c + 1$.

(3.7) If one $A(X), B(X), S(X), T(X)$ is complete subspace of $X$ then

(i) $A$ and $T$ have a coincidence point, and

(ii) $B$ and $S$ have a coincidence point,

Further if

(3.8) $\{A, T\}$ and $\{B, S\}$ are $R$ −weakly commuting pairs, then

(iii) $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Suppose that $T(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $T(X)$ and has a limit in $T(X)$, call it $z$.

Let $w \in T^{-1}(z)$, then $Tw = z$.

We shall use the fact that subsequence $\{y_{2n}\}$ also converges to z.

By putting $x = w$ and $y = x_{2n+1}$ in (3.6), we get

$$aM(Tw, y_{2n}, t) + bM(Tw, Aw, t) + cM(y_{2n}, y_{2n+1}, t)$$
$$+ max\begin{Bmatrix} M(Aw, y_{2n}, t), \\ M(y_{2n+1}, Tw, t) \end{Bmatrix} \leq qM(Aw, y_{2n+1}, t).$$

As $n \to \infty$, we get

$$M(Aw, z, t) \geq \frac{a + c}{q - b - 1} > 1,$$

which is a contradiction.

Thus, $Aw = z = Tw$, i.e. $w$ is a coincidence point of $A$ and $T$.

Since $A(X) \subseteq S(X)$ , $Aw = z$ implies that $z \in S(X)$.

Let $v \in S^{-1}z$. Then $Sv = z$.

By putting $x = x_{2n+2}, y = v$ in (3.6), we get

$$aM(y_{2n+1}, Sv, t) + bM(y_{2n+1}, y_{2n+2}, t) + cM(Sv, Bv, t)$$
$$+ max\begin{Bmatrix} M(y_{2n+2}, Sv, t), \\ M(Bv, y_{2n+1}, t) \end{Bmatrix} \leq qM(y_{2n+2}, Bv, t).$$

As $n \to \infty$, we have

$$M(Bv, z, t) \geq \frac{a+b}{q-c-1} > 1,$$

a contradiction. Therefore $Bv = z = Sv$, i.e. $v$ is a coincidence point of $B$ and $S$.

If $A(X)$ *or* $B(X)$ is complete then by (3.5)

$$z \in A(X) \subseteq S(X) \text{ or } z \in B(X) \subseteq T(X).$$

Thus (i) and (ii) are completely established.

Since the pair $\{A, T\}$ is $R$ −weakly commuting therefore we have

$$M(ATw, TAw, t) \geq M(Aw, Tw, t).$$

which gives $ATw = TAv$ *i.e.* $Az = Tz$.

Similarly $Bz = Sz$.

By putting $x = z, y = x_{2n+1}$ in (3.6)

$$aM(Tz, y_{2n}, t) + bM(Tz, Az, t) + cM(y_{2n}, y_{2n+1}, t) + max\begin{Bmatrix} M(Az, y_{2n}, t), \\ M(y_{2n+1}, Tz, t) \end{Bmatrix} \leq qM(Az, y_{2n+1}, t),$$

As $n \to \infty$, we have

$$M(Az, z, t) \geq \frac{b-c}{q-a-1} < 1,$$

which is a contradiction. Thus $Az = z = Bz = Sz = Tz$.

## 4. CONCLUSION

The theorems in this paper are the improved ,extended and generalized form of some earlier results on common fixed point theorems in fuzzy metric spaces given by Pant[20], vasuki [30], and som [28],[29]. The proven results in fuzzy metric spaces for R-weak commutative mappings without taking any mapping continuous shows that for existence of fixed point in fuzzy metric space , continuity of any mapping is not needed.

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