Some Properties of Direct Product Graphs of Cayley Graphs with Arithmetic Graphs

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ABSTRACT

Nathanson was the pioneer in introducing the concepts of Number Theory, particularly, the "Theory of Congruences" in Graph Theory, thus paving way for the emergence of a new class of graphs, namely "Arithmetic Graphs". Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph.

In this paper, we present some results related to basic properties of direct product graphs of Euler totient Cayley graphs with Arithmetic V_n graph.

Keywords

Euler Totient Cayley Graph, Arithmetic V_n Graph , Direct Product Graph.

AMS(MOS) Subject Classification: 6905c

1. INTRODUCTION

EULER TOTIENT CAYLEY GRAPH $G(Z_n, \varphi)$ AND ITS PROPERTIES

Madhavi [3] introduced the concept of Euler totient Cayley graphs and studied some of its properties. She gave methods of enumeration of disjoint Hamilton cycles and triangles in these graphs.

For any positive integer n, let $Z_n = \{0,1,2, \dots .. n-1\}$. Then $(Z_n, \oplus)$, where, $\oplus$ is addition modulo n, is an abelian group of order n. The number of positive integers less than n and relatively prime to n is denoted by $\varphi(n)$ and is called Euler totient function. Let S denote the set of all positive integers less than n and relatively prime to n.

That is $S = \{r / 1 \leq r < n \text{ and } \mathrm{GCD}\,(r, n) = 1\}$. Then $|S| = \varphi(n)$.

Now we define Euler totient Cayley graph as follows.

For each positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all integers less than n and relatively prime to n. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set is given by $Z_n = \{0,1,2, \dots . n-1\}$ and the edge set is $E = \{(x, y) / x - y \in S \text{ or } y - x \in S\}$.

Clearly as proved by Madhavi [3], the Euler totient Cayley graph $G(Z_n, \varphi)$ is

1. a connected, simple and undirected graph,
2. $\varphi\,(n)$ - regular and has $\frac{n.\varphi(n)}{2}$ edges,
3. Hamiltonian,
4. Eulerian for $n \geq 3$,
5. bipartite if n is even and
6. complete graph if n is a prime.

ARITHMETIC V_n GRAPH

Vasumathi and Vangipuram [4] introduced the concept of Arithmetic V_n graphs and studied some of its properties

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . . p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if GCD $(u, v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence we consider Arithmetic graph V_n without vertex 1 as the contribution of this isolated vertex is nothing when the properties of these graphs and enumeration of some domination parameters are studied.

Clearly, V_n graph is a connected graph. Because if n is a prime, then V_n graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacency in V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n.

DIRECT PRODUCT GRAPHS

In the literature, the direct product is also called as the tensor product, categorical product, cardinal product, relational product, Kronecker product, weak direct product, or conjunction. As an operation on binary relations, the tensor product was introduced by Alfred North Whitehead and Bertrand Russell in their Principia Mathematica [6]. It is also equivalent to the Kronecker product of the adjacency matrices of the graphs given by Weichsel [5].

If a graph can be represented as a direct product, then there may be multiple different representations (direct products do not satisfy unique factorization) but each representation has the same number of irreducible factors. Wilfried Imrich [1] gives a polynomial time algorithm for recognizing tensor product graphs and finding a factorization of any such graph.

This product is commutative and associative in a natural way (refer [2] for a detailed description on product graphs).

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \ldots, u_l\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ respectively. Then the direct product of these two graphs denoted by $G_1 \times G_2$ is defined to be a graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 such that any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if u_1u_2 is an edge of G_1 and v_1v_2 is an edge of G_2.

The cross symbol $\times$, shows visually the two edges resulting from the direct product of two edges.

2. RESULTS

Let G_1 denote Euler totient Cayley graph and G_2 denote an Arithmetic V_n graph. Then G_1 and G_2 are simple graphs. Therefore by the definition of direct product, $G_1 \times G_2$ is a simple graph.

Theorem 2.1: The degree of a vertex in $G_1 \times G_2$ is given by

$$deg_{G_1 \times G_2}(u_i, v_j) = deg_{G_1}(u_i) \cdot deg_{G_2}(v_j) = \varphi(n) \cdot deg_{G_2}(v_j).$$

Proof: Let$deg_{G_1}(u_i) = l$ and $deg_{G_2}(v_j) = m$. Let u_i be adjacent to the vertices $u_1, u_2, \ldots, u_l$ in G_1 and v_j be adjacent to the vertices $v_1, v_2, \ldots, v_m$ in G_2. Then in $G_1 \times G_2$, the vertex (u_i, v_j) is adjacent to the following vertices.

$$\begin{array}{lll} (u_1, v_1), & (u_1, v_2), & \ldots\ldots\ldots (u_1, v_m), \\ (u_2, v_1), & (u_2, v_2), & \ldots\ldots\ldots (u_2, v_m), \\ \vdots & \vdots & \vdots \\ (u_l, v_1), & (u_l, v_2), & \ldots\ldots\ldots (u_l, v_m). \end{array}$$

Also if (u_r, v_s) is any other vertex in $G_1 \times G_2$ then it is not adjacent to vertex (u_i, v_j), for $r > l$ or $s > m$. This is because vertex u_i is not adjacent to vertex u_r if $r > l$ and vertex v_j is not adjacent to vertex v_s if $s > m$, since $deg_{G_1}(u_i) = l$ and $deg_{G_2}(v_j) = m$.

Hence $deg_{G_1 \times G_2}(u_i, v_j) = deg_{G_1}(u_i) \cdot deg_{G_2}(v_j)$.

Since the graph G_1 is $\varphi(n)$- regular,

we have $deg_{G_1}(u_i) = \varphi(n)$.

Thus $deg_{G_1 \times G_2}(u_i, v_j) = \varphi(n) \cdot deg_{G_2}(v_j)$. ■

In what follows, we show that the number of vertices of $G_1 \times G_2$ is the product of number of vertices of G_1 and number of vertices of G_2. Further the number of edges of $G_1 \times G_2$ is twice the product of number of edges of G_1 and number of edges of G_2.

Theorem 2.2: The number of vertices and edges in $G_1 \times G_2$ is given respectively by

1. $|V_{G_1 \times G_2}| = |V_{G_1}|\,|V_{G_2}|$.
2. $|E_{G_1 \times G_2}| = 2\,|E_{G_1}|\,|E_{G_2}|$.

Proof: Let n_1, n_2, n denote the number of vertices and m_1, m_2, m denote the number of edges of graphs G_1, G_2 and $G_1 \times G_2$ respectively.

Since $V = V_1 \times V_2$, it follows that $|V| = |V_1 \times V_2| = |V_1|\,|V_2|$

Hence $|V_{G_1 \times G_2}| = |V_{G_1}|\,|V_{G_2}|$.

We know that

$$|E_{G_1}| = m_1 = \frac{1}{2}\sum_{u_i \in V_1} \deg(u_i)$$

$$|E_{G_2}| = m_2 = \frac{1}{2}\sum_{v_j \in V_2} \deg(v_j)$$

Now $$|E_{G_1 \times G_2}| = m = \frac{1}{2}\sum_{i,j} \deg(u_i, v_j)$$

$$= \frac{1}{2}\left\{\sum_{i,j} \deg(u_i)\deg(v_j)\right\} \text{ (by Theorem 2.1)}$$

$$= \frac{1}{2}\left(\sum_{i \in V_{G_1}} \deg(u_i)\right)\left(\sum_{j \in V_{G_2}} \deg(v_j)\right)$$

$$= \frac{1}{2}(2m_1)(2m_2)$$

$$= 2\ m_1 m_2$$

$$= 2\,|E_{G_1}|\,|E_{G_2}|.$$ ■

Theorem 2.3: If $n = p^2$, then the graph $G_1 \times G_2$ is $\varphi(n)$ −regular.

Proof: We know that graph G_1 is $\varphi(n)$ −regular. Then $deg_{G_1}(u_i) = \varphi(n)$ for any i.

If $n = p^2$, then G_2 contains vertices p and p^2 and there is an edge between p, p^2 since GCD$(p, p^2) = p$. Hence $deg_{G_2}(v_j) = 1$, for any $v_j \in V_2$.

Now for any vertex (u_i, v_j) in $G_1 \times G_2$, we have

$$deg_{G_1 \times G_2}(u_i, v_j) = deg_{G_1}(u_i) \cdot deg_{G_2}(v_j) = \varphi(n) \cdot 1 = \varphi(n).$$

Thus every vertex in $G_1 \times G_2$ is of degree $\varphi(n)$.

Therefore $G_1 \times G_2$ is $\varphi(n)$ −regular. ■

Remark 1: If $n = p^2$, then $\varphi(n) = \varphi(p^2) = p^2 - p = p(p-1)$.

So $G_1 \times G_2$ is $p(p-1)$-regular, when $n = p^2$.

Theorem 2.4: If n is a prime, then $G_1 \times G_2$ is a completely disconnected graph on n vertices.

Proof: Suppose n is a prime. Then G_1 is a complete graph and G_2 is a single vertex graph . Therefore there are no adjacent vertices in G_2. Hence by the definition of the direct product, edges do not exist between the vertices of $G_1 \times G_2$. As the product contains $|V(G_1)|$ vertices, and $|V(G_1)| = n$, it implies that $G_1 \times G_2$ becomes a completely disconnected graph on n vertices. ■

Theorem 2.5: If n is not a prime, then $G_1 \times G_2$ is a finite graph without isolated vertices.

Proof: Suppose n is not a prime. Then G_1, G_2 are finite graphs without isolated vertices. Therefore $deg_{G_1}(u_i) \neq 0$ for any i and also $deg_{G_2}(v_j) \neq 0$ for any j.

Moreover by Theorem 2.1,

$$deg_{G_1 \times G_2}(u_i, v_j) = deg_{G_1}(u_i) \cdot deg_{G_2}(v_j)$$

Thus $deg_{G_1 \times G_2}(u_i, v_j) \neq 0$ for any i and j.

So $G_1 \times G_2$ does not have any isolated vertices. ∎

We now examine the connectivity property of the direct product graph of Euler totient Cayley graph with Arithmetic V_n graph.

Let us recall the following Theorem proved by Weishel [5], which characterizes connectedness in direct product of two graphs.

Weishel Theorem: Suppose G and H are connected non-trivial graphs in a set of finite simple graphs. If at least one of G or H has an odd cycle then $G \times H$ is connected. If both G and H are bipartite, then $G \times H$ has exactly two components.

A similar result for connectedness is given by Imrich and Klavžar [2]. They obtained a necessary and sufficient condition for the connectedness of direct product of two graphs [[2], Theorem 5.29] which is given in the following.

Theorem: Let G and H be graphs with atleast one edge. Then $G \times H$ is connected if and only if both G and H are connected and at least one of them is non-bipartite. Furthermore, if both G and H are connected and bipartite, then $G \times H$ has exactly two components.

If n is not a prime, then G_1 and G_2 both are connected graphs with at least one edge. So, $G_1 \times G_2$ is connected if either G_1 or G_2 is non - bipartite. That is either G_1 or G_2 contains an odd cycle.

Hence we make the following results related to the connectedness of $G_1 \times G_2$.

Theorem 2.6: Let n be an odd number which is not a prime. Then $G_1 \times G_2$ is a connected graph.

Proof: Suppose n is an odd number which is not a prime. Then G_1, G_2 are connected graphs. Further G_1 contains a Hamilton cycle. Since n is odd, this cycle is an odd cycle.

Hence by Weishel Theorem, $G_1 \times G_2$ is a connected graph. ∎

Theorem 2.7: Let n be an even number such that $n > 2$, $n \neq 2^k$ and $n \neq 2p$, where p is a prime. Then the graph $G_1 \times G_2$ is connected. Otherwise it is disconnected.

Proof: Suppose n is an even number such that $n > 2$, $n \neq 2^k$ and $n \neq 2p$ where p is a prime. Then n is not a prime number. Hence both G_1, G_2 are connected graphs. Furthermore n being an even number, G_1 is a bipartite graph. Hence there exists no odd cycles in G_1.

We now show that G_2 contains an odd cycle. Since the even number n is not in the form 2^k and $2p$, it can be written as $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots\ldots, p_k$ are odd primes and $\alpha_i \geq 1$. Then G_2 contains three distinct vertices 2, $2p_i$, $2p_j$ with GCD $(2, 2p_i) = 2$, GCD $(2, 2p_j) = 2$, and GCD $(2p_i, 2p_j) = 2$. This implies that these vertices are connected. Hence G_2 contains an odd cycle.

Thus $G_1 \times G_2$ is connected.

Suppose $n = 2^k$ or $n = 2p$. Then we show that in these cases $G_1 \times G_2$ is disconnected.

Suppose $n = 2^k$. Then G_2 contains vertices $2, 2^2, 2^3, \ldots, 2^k$. Since GCD$(2^i, 2^j) = 2^i$ if $1 < i < j$ or GCD $(2^i, 2^j) = 2^j$ if $1 < j < i$, which is not a prime. So there is no edge between any two powers of 2. The only edges are in between 2 and its powers. Hence we cannot find an odd cycle in G_2.

Since $n = 2^k$ is an even number it follows that G_1 is a bipartite graph. Hence it contains no odd cycles.

Therefore $G_1 \times G_2$ is disconnected.

Suppose $n = 2p$. Then G_2 contains the vertices $2, p$ and $2p$. Then by the definition of edges in G_2, there are edges between 2 and $2p$ since GCD $(2, 2p.) = 2$; and p and $2p$ since GCD $(p, 2p.) = p$. Since p is an odd prime, we get GCD $(2, p) = 1$. This implies that there is no edge between the vertices 2 and p in G_2. Hence there exists no odd cycle in G_2, if $n = 2p$.

Again as n is even, G_1 becomes bipartite and contains no odd cycle.

Hence $G_1 \times G_2$ is disconnected. ∎

We use the following result given in [2], to prove our subsequent results.

Result: If either G_1 or G_2 is a bipartite graph, then $G_1 \times G_2$ is a bipartite graph.

We now find out for **what values of n, $G_1 \times G_2$ is a bipartite graph?**

Theorem 2.8: Suppose $n > 2$ is an even number. Then the graph $G_1 \times G_2$ is a bipartite graph.

Proof: If n is an even number then the graph G_1 is a bipartite graph [3]. Hence $G_1 \times G_2$ is a bipartite graph. (By using the above Result). ∎

Theorem 2.9: The graph $G_1 \times G_2$ is a bipartite graph if n is an odd number such that

$n = p^{\alpha}, \alpha > 1$ or $n = p_i p_j$ where p_i, p_j are distinct odd primes.

Proof: Suppose n is an odd number. Then G_1 is not a bipartite graph because it contains an odd cycle. Hence $G_1 \times G_2$ is a bipartite graph if G_2 is a bipartite graph.

Suppose $n = p^{\alpha}, \alpha > 1$ or $n = p_i p_j$.

We show that G_2 is a bipartite graph.

If $n = p^{\alpha}$, $\alpha > 1$ then among the vertices p, $p^2, \ldots\ldots p^{\alpha}$ of G_2, no two prime powers of p are connected by an edge as GCD $(p^r, p^s) = 1$, for $r, s > 1$. So G_2 contains no odd cycle.

In a similar way, if $n = p_i p_j$ then also G_2 contains no odd cycle, because among the vertices p_i, p_j and $p_i p_j$ of G_2, there is no edge between the vertices p_i and p_j as

GCD (p_i, p_j)= 1, for $i \neq j$.

Hence in either case, G_2 becomes a bipartite graph.

Thus $G_1 \times G_2$ is a bipartite graph. ■

Theorem 2.10: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_k^{\alpha_k}$, where $p_1, p_2 \dots, p_k$ are odd primes such that either $k > 2$ or $k = 2$ with at least one $\alpha_i > 1$, then $G_1 \times G_2$ is not a bipartite graph.

Proof: We know that G_1 is Hamiltonian, and hence it contains a cycle which is odd since n is odd. Therefore G_1 is not bipartite. So $G_1 \times G_2$ is bipartite if G_2 is a bipartite graph. Since $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_k^{\alpha_k}$, where either $k > 2$ or $k = 2$ with at least one $\alpha_i > 1$, it follows that the graph G_2 contains at least three vertices namely

$$p_i,\ p_i p_j,\ p_i p_k \quad \text{or} \quad p_i,\ p_i^2,\ p_i p_j.$$

Case 1: Suppose G_2 contains the vertices $p_i, p_i p_j, p_i p_k$. Then these three vertices are joined by three edges because GCD$(p_i, p_i p_j)$ = GCD$(p_i p_j, p_i p_k)$ = GCD $(p_i, p_i p_k) = p_i$.

This gives the existence of odd cycles in G_2. Hence G_2 is also not a bipartite graph. This implies that $G_1 \times G_2$ is not a bipartite graph.

Case 2: Suppose G_2 contains the vertices $p_i, p_i^2, p_i p_j$. Then these vertices are joined by three edges because GCD (p_i, p_i^2) = GCD $(p_i^2, p_i p_j)$ = GCD $(p_i, p_i p_j) = p_i$.

This implies that G_2 contains an odd cycle so that G_2 is not a bipartite graph. Therefore $G_1 \times G_2$ is not a bipartite graph.

Thus, in either case we have proved that $G_1 \times G_2$ is not a bipartite graph. ■

ILLUSTRATIONS

Let n = 4.

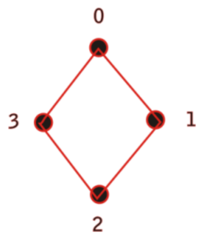

Fig. 1
$G_1 = G(Z_4, \varphi)$

Fig. 2
$G_2 = G(V_4)$

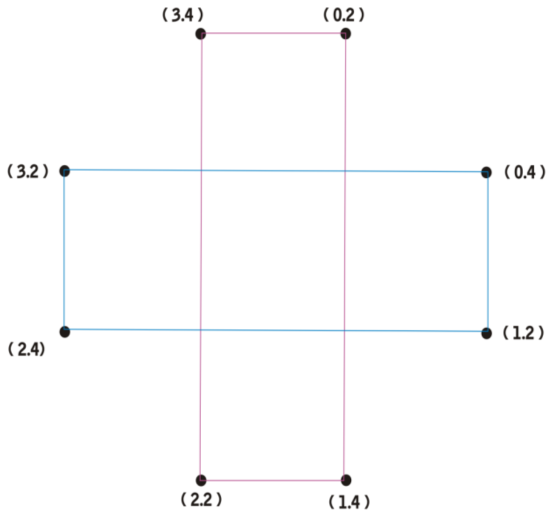

Fig. 3: $G_1 \times G_2$ is a bipartite graph

Let n = 8.

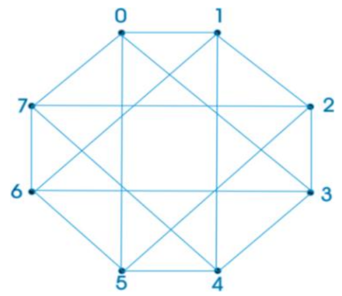

Fig. 4
$G_1 = G(Z_8, \varphi)$

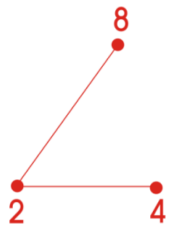

Fig. 5
$G_2 = G(V_8)$

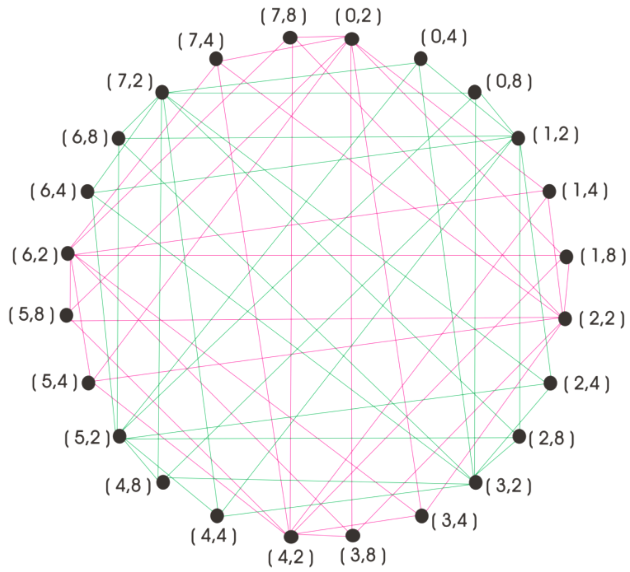

Fig. 6
$G_1 \times G_2$ is a disconnected graph

Let n = 11.

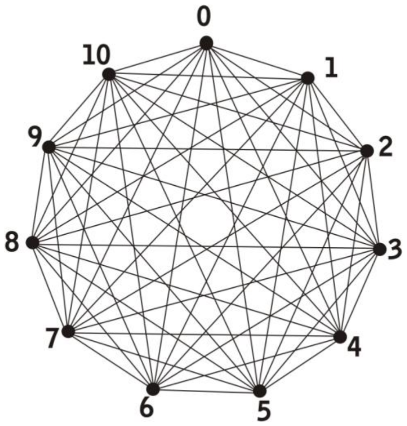

Fig. 7
$\boldsymbol{G_1 = G(Z_{11}, \varphi)}$

Fig. 8
$\boldsymbol{G_2 = G(V_{11})}$

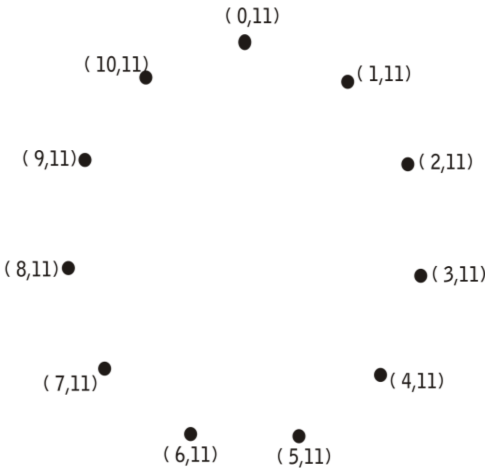

Fig. 9
$\boldsymbol{G_1 \times G_2}$ is a null graph

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