A Note on a Theorem for Two pairs of Weakly Commuting Maps

T. Phaneendra
Applied Analysis Division,
School of Advanced Sciences,
VIT University-632014,
Tamil Nadu State, India

Swatmaram,
Department of Mathematics,
Chaitanya Bharati Institute of Technology,
Hyderabad-500075
Andhra Pradesh State, India

ABSTRACT

Major corrections are suggested to a theorem of Joshi and Mehta (2010) for two pairs of weakly commuting self-maps on a complete metric space.

Keywords

Complete metric space, implicit relation, Weakly Commuting self-maps, and common fixed point.

1. INTRODUCTION

Let X be a metric space with metric d. If $x\in X$ and f is a self-map on X, the f-image of $x$and the range of f are denoted by fx and $f(X)$ respectively. Also Sf denotes the composition of self-maps S and f on X.

Self-maps S and f on X are commuting $fS = Sf$. Sessa [1] introduced a weaker condition of commutativity as given below:

Definition 1.1 Self-maps S and f on X are *weakly commuting* if $d(Sfx, fSx) \le d(Sx, fx)$ for all $x\in X$.

Obviously every commuting pair is weakly commuting. But the converse is not true as seen in [1] and [2].

This was further generalized by Jungck [3].

Definition 1.2 Self-maps S and f on X are *compatible* if $\lim_{n\to\infty} d(Sfx_n, fSx_n) = 0$ whenever there is a $\langle x_n\rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \text{ for some } t \in X. \qquad \ldots \quad (1)$$

A compatible pair need not be weakly commuting one [3]. One can refer to [4], [5], [6] etc. for nice works related to compatibility.

Remark 1.1 Non-vacuously compatible self-maps which ensure the existence of sequence $\langle x_n\rangle_{n=1}^{\infty}$ in X with choice (3) commute at their coincidence points.

Self-maps which commute at their coincidence points are called weakly compatible maps [7]. Further study of weakly compatible maps can be found in [8],[9], [10], [11], [12] etc.

The following was mentioned as Proposition 2.1 in [13]:

Lemma: *Let (S, f) be compatible pair self-maps on a metric space X where f is continuous. Then the pair is weakly compatible*.

The continuity of f is not necessary indeed to obtain the conclusion of the lemma. In fact, writing $x_n = x$ for all n in Definition 1.2, it follows that $Sfx = fSx$ whenever $fx = Sx$. In other words S and f are weakly compatible.

It may also be noted that the lemma plays no role in proving their main theorem.

Let F^* be the class of real valued non-negative functions $F:[0,\infty)^5 \to [0,\infty)$ with the following choice:

(F1) F is non-decreasing in its fourth and fifth coordinate variables

(F2) there exists $h \in [0,1)$ such that for every $u \ge 0$, $v \ge 0$

$$u \le \max\{F(v,v,u,u+v,0), F(v,u,v,u+v,0)\} \Rightarrow u \le hv.$$

(F3) If $u \le \max\{F(u,0,0,u,u), F(0,u,0,0,u), F(0,0,u,u,0)\}$, then $u = 0$.

The result of Joshi and Mehta [13] is

Theorem 1. *Let S, T, f and g be self-maps on a complete metric space X such that*

$$S(X) \subset g(X) \quad \text{and} \quad T(X) \subset f(X) \qquad \ldots \quad (2)$$

and satisfy the inequality:

$$d(Sx,Ty) \le F(d(fx,gy), d(fx,Sx), d(gy,Ty), d(fx,Ty), d(Sx,gy)) \quad \text{for all } x, y \in X, \qquad \ldots \quad (3)$$

Then S, T, f and g have a unique common fixed point z in X. Further z is a unique common fixed point of the pairs (S, f) and (T, g).

The statement of Theorem 1 is incomplete in its present form and requires some additional conditions. Based on some meticulous observations, in the next section major corrections are suggested to Theorem 1 and is restated appropriately.

2. OBSERVATIONS AND DISCUSSION

The following are some observations from the proof of Theorem 1:

Observation 2.1 The proof uses the continuity of any one of the four maps S, T, f and g.

Observation 2.2 The authors of Theorem 1 used only the weak commutativity of the self-maps.

Observation 2.3 If S and f are weakly commuting and g is continuous, then they have a common fixed point, say z which will also be a common fixed point of the other weakly commuting pair (T, g). It follows from (3) that z will be a common fixed point of all the four maps.

In view of the above remarks, Theorem 1 can be restated with significant corrections as follows:

Theorem 1. *Let S, T, f and g be self-maps on a complete metric space X satisfying inclusions* (2) *and the inequality* (3). *Suppose that any one of the four maps S, T, f and g is continuous. Then*

(a) *S and f will have a common fixed point z, provided they are weakly commuting. Further if* (T, g) *is also weakly commuting, then z will be a common fixed point of* (T, g) *as well.*

(b) (T, g) *has a common fixed point z, provided they are weakly commuting. Further if* (S, f) *is also weakly commuting, then z will be a common fixed point of* (S, f) *as well.*

If (S, f) *and* (T, g) *are weakly commuting, then all the four maps S, T, f and g have a common fixed point in X. Indeed the common fixed point is unique.*

Since neither compatibility nor weak compatibility is utilized in Theorem 1, it is appropriate to change the title of [13] as *Common fixed point for two pairs of weakly commuting self-maps on complete metric spaces*.

Moreover, the references of Jungck et al., Srinivas and U.M. Rao, Brown, Hua and Gao, Popa, and Renu Chug and Sanjay Kumar are *redundantly* mentioned in [9].

Acknowledgement. The authors are highly thankful to the referee for the valuable suggestions in further improving the paper.

3. REFERENCES

[1] Sessa, S. 1982. On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd, 32 (46), 146-153

[2] Rhoades, B.E. and Sessa, S. 1986. Common fixed point theorems under weak commutativity condition, Indian J. pure App. Math., 17 (1), 47-57

[3] Gerald Jungck. 1986. Compatible maps and common fixed points, Int. J. Math. & Math. Sci., 9, 771-779

[4] '—' 1988. Common Fixed Points for Commuting and Compatible Maps on Compacta, Proc. Amer. Math. Soc. 103 (3), 977-983

[5] Rhoades B. E. et al. *A common Fixed Point Theorem for compatible mappings*, Indian J. Pure Appl. Math. 26 (5) (1995), 403 -409

[6] Pathak, H. K. and Khan M.S. 1997. A comparison of various types of compatible maps and common fixed points, Indian J. Pure Appl. Math. 28 (5), 611-615

[7] Gerald Jungck and Rhoades B. E. 1998. Fixed point for set valued functions without continuity, Indian J. pure appl. Math., 29 (3): 227-238

[8] Singh, S. L. and Anita Tomar 2003. Weaker forms of commuting maps and existence of fixed points, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 10(3), 145-161

[9] Phaneendra T. 2004. Coincidence points of two weakly compatible self-maps and common fixed point theorem through orbits, Indian J. Math. 46 (2-3), 173-180

[10] Phaneendra, T. 2004. Asymptotic Regularity and Common Fixed Point, Pure and Appl. Math. Sci., 59 (1-2), 45 – 49

[11] Sastry, K. P. R., Ismail S. and Murthy, I. S. R. K. 2004. Common fixed points for two self-maps under strongly partially commuting condition, Bull. Cal. Math. Soc. 96 (1), 1-10

[12] Imdad, M., Ali, J. 2008. Jungck′s common fixed point theorem and E.A. property, Acta Mathematica Sinica, 24 (1), 87-94

[13] Joshi, M. L. and Mehta, J. G. 2010. Common fixed point for weakly compatible maps in complete metric spaces, Inter. Jour. Comp. Appl., 11 (4), 32-34