H_∞ control of discrete-time uncertain periodic systems with delays

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ABSTRACT

This paper deals with the problem of H_∞ control for a class of linear discrete-time periodic system with delays. The obtained results are then extended for the time-delay periodic system with Linear Fractional Representation (LFR) uncertainty. Furthermore, linear matrix inequality (LMI)-based sufficient conditions for H_∞ control are established. Two numerical examples are given to illustrate the applicability of the proposed approach.

Keywords:

Discrete systems, Periodic systems, Time-delay, State feedback stabilization, Linear Fractional Representation, H_∞ control, Asymptotic stabilization, robustness.

1. INTRODUCTION

The H_∞ norm as a measure of system performances has been thoroughly embedded in control theory [6].
During the last decade, the H_∞ control theory has attracted a lot of attention and made significant progress [5]. Recently, a great number of research papers focused on the problem of robust H_∞ control for linear systems with parameter uncertainty. The objective is to design a control law to stabilize an uncertain system while satisfying in the same time a $H_\infty-$norm bound constraint with the aim to insure disturbance attenuation in the presence of admissible uncertainties. The robust H_∞ control design for linear uncertain systems has been extensively studied. This has impulsed a large number of research results obtained on the analysis and synthesis for discrete and continuous-time uncertain system; see, e.g. [10, 12, 3, 8, 2, 7, 9]. Many researchers have extended the obtained results on the robust H_∞ control for delayed systems.
Time delays are frequently encountered in various engineering systems, such as chemical processes and long transmission lines in pneumatic systems [11]. Therefore, much attention has focused on the problems of asymptotic stability and stabilization with an H_∞ norm bound of time-delay systems with parameter uncertainties since time delays and parameter uncertainties are often the causes for instability and/or perfomance degradation of the considered system. Specifically, robust H_∞ control problems for time-delayed discrete-time systems have been considered and sufficient conditions has been derived in terms of Linear Matrix Inequalities in [4] for instance or in terms of modified Riccati inequalities in [5].
In [4], the problem of robust H_∞ state feedback control in which both robust stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainty and unknown time-delay. In terms of modified Riccati inequalities for discrete-time linear systems, a sufficient condition is presented and a state feedback control law is also given in [5].
The present paper investigates the H_∞ control problem of discrete-time uncertain periodic system with delays. The parameter uncertainties are assumed to comply to a Linear Fractional Representation. It is well known that linear matrix inequalities (LMIs) techniques have become essential tools for analysis and synthesis of control systems, and more specifically in the area of robust control [7]. therefore, in the terms of matrix inequalities, sufficient conditions for asymptotic stability as well as asymptotic stabilization with an H_∞ norm bound, using periodic state feedback controller are obtained for the considered class of systems.

Notation: We denote by $X^\top$ the transpose of matrix X, by the Hermitian expression $\mathsf{Sym}\{\cdot\}$: $\mathsf{Sym}\{X\} = X + X^\top$.
Matrix inequalities are considered in the sense of Löwner *i.e.* "< 0" ("≤ 0") means negative (semi–)definite and "> 0" ("≥ 0") positive (semi-)definite. I and 0 are, respectively, the identity the null matrix of suitable dimension.
We define the set $\mathbb{I}_p$ for $p \in \mathbb{N}$ as

$$\mathbb{I}_p = \{k \in \mathbb{N},\ 0 \le k \le p-1\}$$

2. PRELIMINARIES

Consider the following linear uncertain p-periodic system:

$$x(k+1) = (A(k) + \Delta A(k))\, x(k) \tag{2}$$

with

$$\begin{aligned} \Delta A(k) &= E(k)\Delta(k)N(k) \\ \Delta(k) &= F(k)\,(I - M(k)F(k))^{-1} \end{aligned} \tag{3}$$

where $A(k)$, $E(k)$, $M(k)$ and $N(k)$ are real p-periodic matrices with appropriate dimensions and $F(k)$ is an unknown p-periodic matrix that satisfies

$$F^\top(k)F(k) \le I, \quad \forall k \in \mathbb{I}_p. \tag{4}$$

Furthermore, we assume that

$$\mathcal{M}(k) = \begin{bmatrix} I & -M(k) \\ -M(k)^\top & I \end{bmatrix} > 0, \quad \forall K \in \mathbb{I}_p. \tag{5}$$

In this part, we will study the problem of asymptotic stability of uncertain periodic system such that in equations (2)-(4). Before proceeding too far, let us answer a question that imposes itself and reads as *Can we consider that the uncertainty matrix* $\Delta(k)$ *is periodic?*. To answer this question, let us consider that each matrix $A(k)$, for $k \in I_p$ is affected by a simple additive uncertainty, denoted $\Delta A(k)$. It comes then that from a periodic point of view, these uncertainties can be considered as periodic as is the case for matrix $A(k) = A(k+p)$, $k \in I_p$.
At this step, we present some definition related to stability and robust stability.

DEFINITION 1. *The linear p-periodic system* (2) *is robustly stable* `if and only if`,

$$\forall\Delta(k) \in \nabla(k),\ \exists\, X(k,\Delta(k)) = X^\top(k)(\Delta(k)) > 0,$$
$$\begin{bmatrix} I \\ A^\top(k,\Delta(k)) \end{bmatrix}^\top \begin{bmatrix} -X(k+1,\Delta(k+1)) & 0 \\ 0 & X(k,\Delta(k)) \end{bmatrix} \begin{bmatrix} I \\ A^\top(k,\Delta(k)) \end{bmatrix} < 0$$

for all $k = 0, \ldots, p-1$.

DEFINITION 2. *[1]The linear p-periodic system* (2) *is quadratically stable* `if and only if`,

$$\exists\ X(k) = X^\top(k) > 0, \quad \forall\Delta(k) \in \nabla(k)$$
$$\begin{bmatrix} I \\ A^\top(k,\Delta(k)) \end{bmatrix}^\top \begin{bmatrix} -X(k+1) & 0 \\ 0 & X(k) \end{bmatrix} \begin{bmatrix} I \\ A^\top(k,\Delta(k)) \end{bmatrix} < 0 \qquad (6)$$

for all $k = 0, \ldots, p-1$.

DEFINITION 3. *The linear p-periodic system* (2) *is asymptotically stable* `if` *condition* (6) *holds.*

In the three definitions $A(k,\Delta(k))$ is defined by:

$$A(k,\Delta(k)) = A(k) + \Delta A(k) \qquad (7)$$

where the uncertainty $\Delta A(k)$ is described by (3).

LEMMA 4. *Assume* $\Delta A(k) = 0$, $k \in \mathbb{N}$ *and suppose that there exits a symmetric positive definite p-periodic matrix* $P(k)$ *such that the condition*

$$A^\top(k)P(k+1)A(k) - P(k) < 0 \qquad (8)$$

holds for all $k = 0, 1, \ldots, p-1$, *then, the p-periodic system* (2) *is asymptotically stable .*

3. H_∞ STABILITY OF DELAYED PERIODIC SYSTEM

Consider a class of discrete time-delay p-periodic system described by

$$x(k+1) = A(k)x(k) + A_d(k)x(k-d) + B_w(k)w(k)$$
$$z(k) = C(k)x(k) \qquad (9)$$

The following lemma gives the stability for system (9).

LEMMA 5. *The linear time-delay p-periodic system* (9) *is asymptotically stable when* $w(k) = 0$, *if there exist a Lyapunov fonction* $V(k)$, *a positive scalar* α *and a positive* $p-$*periodic scalar* $\beta(k) \neq 0$ *satisfying* $\beta(k+1) \geq \beta(k)$ *for all* $k \in \mathbb{N}$, *such that*

(*1*)

$$0 \leq V(x(k)) \leq \alpha \max_{-d\leq i\leq 0} \|x(i+k)\|^2$$

(*2*)

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) \leq -\beta(k)\,\|x(k)\|^2$$

Proof:

It can be noted that

$$V(x(k+1)) = \Delta V(x(k)) + \Delta V(x(k-1)) + \cdots + \Delta V(x(1))$$
$$+\Delta V(x(0)) + V(x(0)) \leq -\sum_{i=0}^{k} \beta(i)\,\|x(i)\|^2 + V(x(0))$$

and thus

$$\beta(k)\,\|x(k)\|^2 \leq \sum_{i=0}^{k} \beta(i)\,\|x(i)\|^2 \leq V(x(0)) - V(x(k+1)) \leq V(x(0))$$
$$\leq \alpha \max_{-d\leq i\leq 0} \|x(i)\|^2$$

Hence

$$\|x(k)\|^2 \leq \frac{\alpha}{\beta(k)} \max_{-d\leq i\leq 0} \|x(i)\|^2 \qquad (11)$$

which means that $\|x(k)\|^2$ is small enough for a small enough $\max_{-d\leq i\leq 0} \|x(i)\|^2$. Moreover, $\sum_{i=0}^{k} \beta(i)\,\|x(i)\|^2$ is bounded for any k, and hence the series $\beta(i)\,\|x(i)\|^2$ converges to 0. Keeping in mind that $\beta(i)$ is an increasing function, it comes that $x(i)$ converges to 0, which implies the asymptotic stability of the considered p-periodic system.

THEOREM 6. *The p-periodic system with delays* (9) *is said to be asymptotically stable with an* H_∞ *norm bound* γ *if there exist symmetric p-periodic matrices* $X(k) > 0$ *and* $G(k) > 0$ *such that the following matrix inequality holds for all* $k \in \mathbb{I}_p$:

$$\begin{bmatrix} \Psi_{11}(k) & \Psi_{12}^\top(k) \\ \Psi_{12}(k) & \Psi_2(k) \end{bmatrix} < 0 \qquad (12)$$

with

$$\Psi_{11}(k) = \begin{bmatrix} -X(k) & 0 & 0 \\ 0 & -G(k-d) & 0 \\ 0 & 0 & -\gamma I \end{bmatrix},$$
$$\Psi_{12}(k) = \begin{bmatrix} C(k)X(k) & 0 & 0 \\ A(k)X(k) & A_d(k)G(k-d) & B_w(k) \\ X(k) & 0 & 0 \end{bmatrix},$$
$$\Psi_{22}(k) = \begin{bmatrix} -\gamma I & 0 & 0 \\ 0 & -X(k+1) & 0 \\ 0 & 0 & -G(k) \end{bmatrix}.$$

Proof:

Now, define the following Lyapunov function

$$V(k) = x^\top(k)P(k)x(k) + \sum_{j=1}^{d} x^\top(k-j)Q(k-j)x(k-j) \qquad (13)$$

where $P(k)$ and $Q(k)$ are symmetric positive definite $p-$periodic matrices with appropriate dimensions.
By some algebraic calculations, we get

$$\begin{aligned}\Delta V(k) \;=\;& x^\top(k)\left(Q(k)-P(k)+A^\top(k)P(k+1)A(k)\right)x(k)+\\ & x^\top(k)A^\top(k)P(k+1)A_d(k)x(k-d)+\\ & x^\top(k)A^\top(k)P(k+1)B_w(k)w(k)+\\ & x^\top(k-d){A_d}^\top(k)P(k+1)A(k)x(k)+\\ & x^\top(k-d)\left(A_d^\top(k)P(k+1)A_d(k)-Q(k-d)\right)x(k-d)+\\ & x^\top(k-d)A_d^\top(k)P(k+1)B_w(k)w(k)+\\ & w^\top(k)B_w^\top(k)P(k+1)A(k)x(k)+\\ & w^\top(k)B_w^\top(k)P(k+1)A_d(k)x(k-d)+\\ & w^\top(k)B_w^\top(k)P(k+1)B_w(k)w(k) \qquad (14)\end{aligned}$$

Condition (14) can be written as

$$\Delta V(k) = \xi(k)\Psi(k)\xi(k) \qquad (15)$$

with

$$\xi(k) = \begin{bmatrix} x^\top(k) & x^\top(k-d) & w^\top(k)\end{bmatrix}^\top \qquad (16)$$

and

$$\Psi(k) = \begin{bmatrix} -P(k) & 0 & 0\\ 0 & -Q(k-d) & 0\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix} A^\top(k)\\ {A_d}^\top(k)\\ B_w^\top(k)\end{bmatrix} P(k+1)\begin{bmatrix} A(k) & A_d(k) & B_w(k)\end{bmatrix} \qquad (17)$$

In the case of null initial conditions, the H_∞ performance of system (9) is defined as

$$\sum_{k=0}^{\infty}\left(z^\top(k)z(k)-\gamma^2 w^\top(k)w(k)\right) < 0, \qquad (18)$$

for any nonzero $w(k) \in l_2[0,\infty)$ which means that $\|z(\cdot)\|_2 < \gamma\|w(\cdot)\|_2$ or in other words, that the system is stable with a H_∞ bound γ.

REMARK 3.1. *In the case of non null initial conditions, the condition above becomes*

$$\sum_{k=0}^{\infty}\left(z^\top(k)z(k)-\gamma^2 w^\top(k)w(k)\right) - V(x(0)) < 0.$$

In order to set a condition that implies the inequality (18), asume that the initial conditions satisfy $V(x(0)) = 0$ and consider the expression

$$\sum_{k=0}^{\infty}\left(z^\top(k)z(k)-\gamma^2 w^\top(k)w(k)+\Delta V(x(k))\right) = \sum_{k=0}^{\infty}\xi^\top(k)\Theta(k)\xi(k)$$

where

$$\Theta(k) = \begin{bmatrix} -P(k) & 0 & 0\\ 0 & -Q(k-d) & 0\\ 0 & 0 & -\gamma^2 I\end{bmatrix} + \begin{bmatrix} A(k) & A_d(k) & B_w(k)\\ I & 0 & 0\\ C(k) & 0 & 0\end{bmatrix}^\top \times \begin{bmatrix} P(k+1) & 0 & 0\\ 0 & Q(k) & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix} A(k) & A_d(k) & B_w(k)\\ I & 0 & 0\\ C(k) & 0 & 0\end{bmatrix} \qquad (19)$$

Hence, if

$$\Theta(k) < 0 \qquad (20)$$

then,

$$\sum_{k=0}^{\infty}\left(z^\top(k)z(k)-\gamma^2 w^\top(k)w(k)+\Delta V(x(k))\right) = \sum_{k=0}^{\infty}\left(z^\top(k)z(k)-\gamma^2 w^\top(k)w(k)\right) + V(x(\infty)) - V(x(0)) < 0$$

and since $V(x(0)) = 0$ and $V(x(\infty)) \geq 0$, then we get

$$\|z\|_2^2 \leq \gamma^2\|w\|_2^2 \qquad (21)$$

for any nonzero $w(k) \in l_2[0,\infty)$ which means that we have an H_∞ norm bound less than γ.
Multiplying (20) on the left and on the right by

$$\begin{bmatrix} \gamma^{\frac{1}{2}}P^{-1}(k) & 0 & 0\\ 0 & \gamma^{\frac{1}{2}}Q^{-1}(k-d) & 0\\ 0 & 0 & \gamma^{-\frac{1}{2}}I\end{bmatrix},$$

we get:

$$\begin{bmatrix} -\gamma P^{-1}(k) & 0 & 0\\ 0 & -\gamma Q^{-1}(k-d) & 0\\ 0 & 0 & -\gamma^{-1}I\end{bmatrix} + \mathcal{A}(k)^\top\begin{bmatrix} \gamma^{-1}P(k+1) & 0 & 0\\ 0 & \gamma^{-1}Q(k) & 0\\ 0 & 0 & \gamma I\end{bmatrix}\mathcal{A}(k) < 0, \qquad (22)$$

with

$$\mathcal{A}(k) = \begin{bmatrix} \gamma A(k)P^{-1}(k) & \gamma A_d(k)Q^{-1}(k-d) & B_w(k)\\ \gamma P^{-1}(k) & 0 & 0\\ \gamma C(k)P^{-1}(k) & 0 & 0\end{bmatrix}.$$

Replacing, respectively, $\gamma^{-1}P(k)$ and $\gamma^{-1}Q(k-d)$ by $X^{-1}(k)$ and $G^{-1}(k-d)$ and applying a Schur complement technique, condition (22) can be written as (12).

4. H_∞ CONTROL OF DELAYED PERIODIC SYSTEM

Consider the following p-periodic system with delays

$$\begin{aligned} x(k+1) &= A(k)x(k) + A_d(k)x(k-d) + B(k)u(k) + B_w(k)w(k),\\ z(k) &= C(k)x(k) + D(k)u(k). \qquad (23)\end{aligned}$$

In the remainder of this part, we will establish a stabilization condition with an H_∞ norm bound for the closed loop p-periodic system with delays (23). Our aim is to design a p-periodic state feedback controller

$$u(k) = K(k)x(k);\; K(k+p) = K(k) \text{ for all } k \in \mathbb{N} \qquad (24)$$

such that for a given scalar $\gamma > 0$, for all nonzero $w(k) \in l_2[0,+\infty)$ and null initial conditions, we have

$$\|z\|_2 \leq \gamma\|w\|_2. \qquad (25)$$

In this situation, the p-periodic system (23) with the controller (24) is said to achieve an H_∞ norm bound less than γ.

THEOREM 7. *The p-periodic system with delays* (23) *with the controller* (24) *is said to be asymptotically stabilizable with an H_∞ norm bound γ if there exist symmetric p-periodic matrices $X(k) > 0$, $G(k) > 0$ and $Y(k)$ such that the following matrix inequality holds for all $k = 0, 1, \ldots, p-1$:*

$$\begin{bmatrix} -X(k) & 0 & 0 & \Gamma_{41}^{\top}(k) & \Gamma_{51}^{\top}(k) & X(k) \\ 0 & -G(k-d) & 0 & 0 & \Gamma_{52}^{\top}(k) & 0 \\ 0 & 0 & -\gamma I & 0 & B_w^{\top}(k) & 0 \\ \Gamma_{41}(k) & 0 & 0 & -\gamma I & 0 & 0 \\ \Gamma_{51}(k) & \Gamma_{52}(k) & B_w(k) & 0 & -X(k+1) & 0 \\ X(k) & 0 & 0 & 0 & 0 & -G(k) \end{bmatrix} < 0, \quad (26)$$

where

$$\Gamma_{41}(k) = (C(k)X(k) + D(k)Y(k)),$$

$$\Gamma_{51}(k) = (A(k)X(k) + B(k)Y(k)),$$

$$\Gamma_{52}(k) = A_d(k)G(k-d).$$

Furthermore, if the matrix inequality (30) *has a feasible solution* $X(k)$, $G(k)$ *and* $Y(k)$, *then the state feedback control law is given by:*

$$u(k) = Y(k)X^{-1}(k)x(k). \quad (27)$$

Proof: the proof is omitted because it can be carried out by following a similar line as in the proof of theorem 6.

5. ROBUST CONTROLLER DESIGN

Consider the following uncertain p-periodic system with delays

$$\begin{aligned} x(k+1) &= (A(k) + \Delta A(k))\, x(k) + (A_d(k) + \Delta A_d(k))\, x(k-d) + \\ &\quad (B(k) + \Delta B(k))\, u(k) + B_w(k)w(k), \\ z(k) &= C(k)x(k) + D(k)u(k), \end{aligned} \quad (28)$$

with

$$\begin{aligned} \begin{bmatrix} \Delta A(k) & \Delta A_d(k) & \Delta B(k) \end{bmatrix} &= E(k)\Delta(k) \begin{bmatrix} N(k) & N_{A_d}(k) & L(k) \end{bmatrix}, \\ \Delta(k) &= F(k)\,(I - M(k)F(k))^{-1}, \end{aligned} \quad (29)$$

where $A(k)$, $A_d(k)$, $E(k)$, $B(k)$, $M(k)$, $N(k)$, $N_{A_d}(k)$ and $L(k)$ are real p-periodic matrices with appropriate dimensions and F_k is an unknown matrix that satisfies:

$$F^{\top}(k)F(k) \leq I. \quad (4)$$

Furthermore, we assume that

$$\mathcal{M}(k) = \begin{bmatrix} I & -M(k) \\ -M(k)^{\top} & I \end{bmatrix} > 0. \quad (5)$$

In this section, we are concerned with the problem of robust p-periodic state feedback control for the uncertain p-periodic system (28)-(4) for all admissible uncertainties. Our aim is to design a p-periodic state feedback controller (24) such that for a given scalar $\gamma > 0$, for all nonzero $w(k) \in l_2\,[0, +\infty)$ and for all parameter uncertainties satisfying (29) and (4),

$$\|z\|_2 \leq \gamma\, \|w\|_2\,. \quad (30)$$

In this situation, the p-periodic system (28) with the controller (24) is said to achieve a robust H_∞ performance (30).

Definition 8. *Let a constant* γ *be given. The uncertain p-periodic system* (28)-(4) *is said to be stabilizable with an* H_∞ *norm bound* γ *if there exists a p-periodic state feedback control law* (24), *such that for any admissible parameter uncertainty* $\Delta A(k)$, $\Delta A_d(k)$ *and* $\Delta B(k)$ *the following conditions are satisfied.*

(*1*) *The closed-loop p-periodic delayed system is asymptotically stable when* $w(k) = 0$,

(2) *Subject to the assumption of the zero initial condition, the controlled output* $z(k)$ *satisfies* (30).

Definition 9. *Let a constant* γ *be given. The uncertain p-periodic delayed system* (28)-(29) *is said to be quadratically stabilizable with an* H_∞ *norm bound* γ *if there exists a linear p-periodic state feedback law* (24) *and real symmetric positive definite p-periodic matrices* $P(k)$ *and* $Q(k)$ *such that the inequality*

$$\begin{bmatrix} \Gamma_{11}(k) & \Gamma_{12}^{\top}(k) & \Gamma_{13}^{\top}(k) \\ \Gamma_{12}(k) & \Gamma_{22}(k) & \Gamma_{23}^{\top}(k) \\ \Gamma_{13}(k) & \Gamma_{23}(k) & \Gamma_{33}(k) \end{bmatrix} < 0 \quad (31)$$

with

$$\begin{aligned} \Gamma_{11}(k) &= A^{c\top}(k, \Delta_k)P(k+1)A^c(k, \Delta_k) - P(k) + Q(k) + C^{c\top}(k)C^c(k) \\ \Gamma_{12}(k) &= A_d^{\top}(k, \Delta(k))P(k+1)A^c(k, \Delta(k)) \\ \Gamma_{13}(k) &= B_w^{\top}(k)P(k+1)A^c(k, \Delta_k) \\ \Gamma_{22}(k) &= A_d^{\top}(k, \Delta(k))P(k+1)A_d(k, \Delta(k)) - Q(k-d) \\ \Gamma_{23}(k) &= B_w^{\top}(k)P(k+1)A_d(k, \Delta_k) \\ \Gamma_{33}(k) &= B_w^{\top}(k)P(k+1)B_w(k) - \gamma^2 I \end{aligned}$$

holds for any admissible uncertainty $\Delta A(k)$, $\Delta A_d(k)$ *and* $\Delta B(k)$, *where* $A^c(k, \Delta(k)) = A(k, \Delta(k)) + B(k, \Delta(k))K(k)$ *and* $C^c(k) = C(k) + D(k)K(k)$

Initially, we will show that quadratic stabilization with an H_∞ norm bound $\gamma > 0$ implies stabilization with the same H_∞ norm bound γ.

Lemma 10. *If the uncertain p-periodic system* (28)-(29) *with the uncertainty satisfying* (4) *and* (5) *is quadratically stabilizable with an* H_∞ *norm bound* $\gamma > 0$, *then it is also stabilizable with the same* H_∞ *norm bound* γ.

Proof:
By following similar arguments as in the proof of theorem 6, we obtain, for any admissible uncertainty $\Delta A(k)$, $\Delta A_d(k)$ and $\Delta B(k)$ that (31) holds
which implies that
$\|z\|_2 \leq \gamma\, \|w\|_2$ for any nonzero $w(k) \in l_2\,[0, \infty)$.

Theorem 11. *The uncertain p-periodic system* (28)-(29) *is said to be asymptotically stabilizable by the p-periodic state feedback controller* (24) *with an* H_∞ *norm bound* γ *if there exist symmetric p-periodic matrices* $X(k) > 0$ *and* $G(k) > 0$ *and a p-periodic scalar* $\epsilon(k) > 0$ *such that the following matrix inequality holds:*

$$\begin{bmatrix} \Phi_{11}(k) & \Phi_{12}(k) \\ \Phi_{12}^{\top}(k) & \Phi_{22}(k) \end{bmatrix} < 0, \quad (32)$$

with

$$\Phi_{11}(k) = \begin{bmatrix} -X(k) & 0 & 0 & \Lambda_1^{\top}(k) \\ 0 & -G(k-d) & 0 & 0 \\ 0 & 0 & -\gamma I & 0 \\ \Lambda_1(k) & 0 & 0 & -\gamma I \end{bmatrix},$$

$$\Phi_{12}(k) = \begin{bmatrix} \Lambda_2^{\top}(k) & X(k) & \Lambda_3^{\top}(k) & 0 \\ G(k-d)A_d^{\top}(k) & 0 & G(k-d)N_{A_d}^{\top}(k) & 0 \\ B_w^{\top}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Phi_{22}(k) = \begin{bmatrix} -X(k+1) & 0 & 0 & \epsilon(k)E(k) \\ 0 & -G(k) & 0 & 0 \\ 0 & 0 & -\epsilon(k)I & \epsilon(k)M(k) \\ \epsilon(k)E_k^{\top} & 0 & \epsilon(k)M^{\top}(k) & -\epsilon(k)I \end{bmatrix},$$

where

$$\begin{aligned}\Lambda_1(k) &= (C(k)X(k) + D(k)Y(k)),\\ \Lambda_2(k) &= (A(k)X(k) + B(k)Y(k)),\\ \Lambda_3(k) &= (N(k)X(k) + L(k)Y(k)).\end{aligned}$$

Proof:
Using the same approach as in the proof of theorem 6, the uncertain p-periodic system is asymptotically stabilizable with an H_∞ norm bound γ if there exist symmetric p-periodic matrices $X(k) > 0$ and $G(k) > 0$ such that

$$\begin{bmatrix}\Gamma_{11}(k) & \Gamma_{12}^\top(k)\\ \Gamma_{12}(k) & \Gamma_{22}(k)\end{bmatrix} < 0 \quad (33)$$

with

$$\Gamma_{11}(k) = \begin{bmatrix}-X(k) & 0 & 0\\ 0 & -G(k-d) & 0\\ 0 & 0 & -\gamma I\end{bmatrix},$$

$$\Gamma_{12}(k) = \begin{bmatrix}C^c(k)X(k) & 0 & 0\\ A^c(k,\Delta(k))X(k) & A_d(k,\Delta(k))G(k-d) & B_w(k)\\ X(k) & 0 & 0\end{bmatrix},$$

$$\Gamma_{22}(k) = \begin{bmatrix}-\gamma I & 0 & 0\\ 0 & -X(k+1) & 0\\ 0 & 0 & -G(k)\end{bmatrix}.$$

which is in fact condition (12) rewritten for the uncertain closed-loop system, *i.e.*, $A(k)$ is replaced by

$$\begin{aligned}A^c(k,\Delta_k) &= A(k) + E(k)\Delta(k)N(k) + (B(k) + E(k)\Delta(k)L(k))\,K(k)\\ &= \underbrace{(A(k) + B(k)K(k))}_{A^c(k)} + E(k)\Delta(k)\underbrace{(N(k) + L(k)K(k))}_{N^c(k)}\\ &= A^c(k) + E(k)\Delta(k)N^c(k). \quad (34)\end{aligned}$$

where $\Delta(k) = F(k)\,(I - M(k)F(k))^{-1}$.

Therefore, condition (33) can be displayed as follows

$$\begin{bmatrix}\Phi_{11}(k) & \Phi_{12}^\top(k)\\ \Phi_{12}(k) & \Phi_{22}(k)\end{bmatrix} + \text{Sym}\left\{\begin{bmatrix}0\\0\\0\\0\\E_k\\0\end{bmatrix} F(k)\,(I - M(k)F(k))^{-1}\begin{bmatrix}X(k)N^c(k)^\top\\ G(k-d)N_{A_d}^\top(k)\\ 0\\0\\0\\0\end{bmatrix}^\top\right\} < 0. \quad (35)$$

with

$$\begin{aligned}\Phi_{11}(k) &= \begin{bmatrix}-X(k) & 0 & 0\\ 0 & -G(k-d) & 0\\ 0 & 0 & -\gamma I\end{bmatrix},\\ \Phi_{12}(k) &= \begin{bmatrix}C^c(k)X(k) & 0 & 0\\ A^c(k)X(k) & A_d(k)G(k-d) & B_w(k)\\ X(k) & 0 & 0\end{bmatrix},\\ \Phi_{22}(k) &= \begin{bmatrix}-\gamma I & 0 & 0\\ 0 & -X(k+1) & 0\\ 0 & 0 & -G(k)\end{bmatrix}. \quad (36)\end{aligned}$$

Using [8, Lemma 2.6], we can state that there exists an $\epsilon > 0$ such that condition (35) above is equivalent for all admissible uncertainties $F(k)$ to

$$\begin{bmatrix}\Phi_{11}(k) & \Phi_{12}^\top(k)\\ \Phi_{12}(k) & \Phi_{22}(k)\end{bmatrix} + \mathcal{Z}(k)\mathcal{M}_k^{-1}\mathcal{Z}^\top(k) < 0, \quad (37)$$

with

$$\mathcal{Z}(k) = \begin{bmatrix}\epsilon^{-\frac{1}{2}}(k)X(k)N_k^{c\top} & 0\\ \epsilon^{-\frac{1}{2}}(k)G(k-d)N_{A_d}^\top(k) & 0\\ 0 & 0\\ 0 & 0\\ 0 & \epsilon^{\frac{1}{2}}(k)E_k\\ 0 & 0\end{bmatrix},$$

and $\mathcal{M}_k$ given by (5).
At this step, we notice that performing a Schur complement operation shows that condition (37) is in fact equivalent to

$$\begin{bmatrix}\Gamma_{11}(k) & \Gamma_{12}(k)\\ \Gamma_{12}^\top(k) & \Gamma_{22}(k)\end{bmatrix} < 0, \quad (38)$$

with

$$\Gamma_{11}(k) = \begin{bmatrix}-X(k) & 0 & 0 & X(k)C^{c\top}(k)\\ 0 & -G(k-d) & 0 & 0\\ 0 & 0 & -\gamma I & 0\\ C^c(k)X(k) & 0 & 0 & -\gamma I\end{bmatrix},$$

$$\Gamma_{12}(k) = \begin{bmatrix}X(k)A^{c\top}(k) & X(k) & X(k)N_k^{c\top} & 0\\ G(k-d)A_d^\top(k) & 0 & X(k-d)N_{A_d}^\top(k) & 0\\ B_w^\top(k) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Gamma_{22}(k) = \begin{bmatrix}-X(k+1) & 0 & 0 & \epsilon(k)E_k\\ 0 & -G(k) & 0 & 0\\ 0 & 0 & -\epsilon(k)I & \epsilon(k)M(k)\\ \epsilon(k)E_k^\top & 0 & \epsilon(k)M^\top(k) & -\epsilon(k)I\end{bmatrix},$$

where, in addition, we have multiplied both sides of the obtained condition by the diagonal matrix $\text{diag}\{I, I, I, I, I, I, \epsilon^{\frac{1}{2}}(k)I, \epsilon^{\frac{1}{2}}(k)I\}$. Hence, condition (32) is then recovered easily from above.

Example1:
Consider the problem of H_∞ control of the discrete-time 2-periodic system (23) with

$$\begin{bmatrix}A(0) & A_d(0) & C(0) & B(0)\\ A(1) & A_d(1) & C(1) & B(1)\end{bmatrix} = \left[\begin{array}{cc|cc|cc|c}1.3 & -0.6 & 0.2 & -0.2 & 0.3 & 0.25 & 1.2\\ 0.2 & 0.2 & 0.4 & 0.3 & 0.12 & 0.2 & 0.3\\ 1.2 & -0.2 & 0.3 & -0.15 & 0.4 & 0.15 & 1.1\\ 0.2 & 0.3 & 0.1 & 0.3 & 0.1 & 0.3 & 0.4\end{array}\right];$$

$$\begin{bmatrix}B_w(0) & D(0) & B_w(1) & D(1)\end{bmatrix} = \left[\begin{array}{cc|c|cc|c}0.3 & 0.2 & 0.2 & 0.1 & 0.2 & 0.12\\ 0.15 & 0.2 & 0.25 & 0.22 & 0.3 & 0.3\end{array}\right];$$

Using theorem 7, we obtain the following matrices:

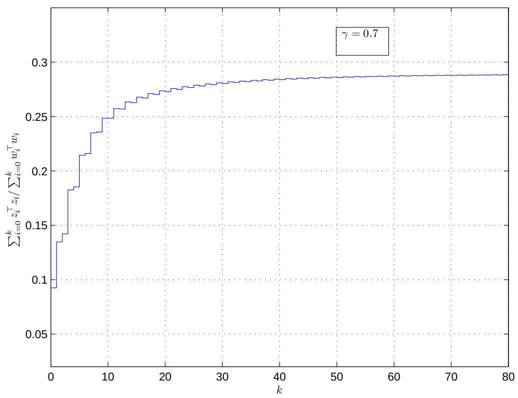

Fig. 1. Variation of the ratio $\dfrac{\sum_{i=0}^{k} z_i^\top z_i}{\sum_{i=0}^{k} w_i^\top w_i}$.

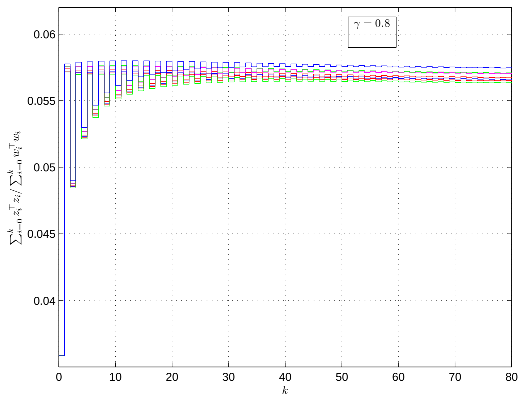

Fig. 2. The variation of $\dfrac{\sum_{i=0}^{k} z_i^\top z_i}{\sum_{i=0}^{k} w_i^\top w_i}$ for some values of Δ_k satisfying (4).

$$\begin{bmatrix} X(0) & G(0) \\ X(1) & G(1) \end{bmatrix} = \left[\begin{array}{cc|cc} 0.4102 & 0.0570 & 0.9995 & -0.1869 \\ 0.0570 & 0.3204 & -0.1869 & 0.8498 \\ 0.2666 & 0.0634 & 0.3453 & -0.0720 \\ 0.0634 & 0.2893 & -0.0720 & 0.9370 \end{array}\right]$$

Then, the state feedback gain matrices are given by:

$$\begin{bmatrix} Y(0) & K(0) \\ Y(1) & K(1) \end{bmatrix} = \left[\begin{array}{cc|cc} -0.4053 & 0.0619 & -1.0407 & 0.3786 \\ -0.2475 & -0.0884 & -0.9026 & -0.1078 \end{array}\right];$$

Hence, we check that the statements of definition 8 are verified with a norm bound $\gamma = 0.7$.
In order illustrate the obtained result, we will take the disturbance signal $w(k) = w_0 r^{-\alpha k}$ with $r > 1$.
Figure 1 shows that the 2-periodic system (23) with the controller (24) achieves a H_∞ norm bound less than $\gamma = 0.7$ complying with the performance index (30).

Example2:

Consider the problem of robust H_∞ control of the discrete-time 2-periodic system (28)-(4) with:

$$\begin{bmatrix} A(0) & B(0) & C(0) \\ A(1) & B(1) & C(1) \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 1.4 & -0.4 & 1.4 & 0.22 & -0.2 & 0.1 \\ 0.4 & 0.6 & 0.35 & 0.12 & 0.2 & 0.1 \\ 1.2 & -0.5 & 1.6 & 1 & -0.1 & 0.2 \\ 0.5 & 0.6 & 0.8 & 0.21 & 0.25 & 0.2 \end{array}\right];$$

$$\begin{bmatrix} D(0) & E(0) & M(0) \\ D(1) & E(1) & M(1) \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0.15 & 0.2 & 0.02 & 0.015 & 0.05 & 0.02 \\ 0.12 & 0.2 & 0.02 & 0.02 & 0.03 & 0.04 \\ 0.16 & 0.15 & 0.02 & 0.03 & 0.02 & 0.03 \\ 0.1 & 0.18 & 0.02 & 0.03 & 0.02 & 0.03 \end{array}\right];$$

$$\begin{bmatrix} N(0) & L(0) & B_w(0) \\ N(1) & L(1) & B_w(1) \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0.3 & 0.4 & 0.2 & 0.3 & 0.03 & 0.02 \\ 0.2 & 0.45 & 0.12 & 0.22 & 0.01 & 0.03 \\ 0.3 & 0.5 & 0.1 & 0.2 & 0.02 & 0.01 \\ 0.22 & 0.3 & 0.2 & 0.2 & 0.02 & 0.02 \end{array}\right];$$

$$\begin{bmatrix} A_d(0) & N_{A_d}(0) \\ A_d(1) & N_{A_d}(1) \end{bmatrix} = \left[\begin{array}{cc|cc} 0.4 & -0.5 & 0.04 & 0.02 \\ 0.3 & -0.4 & 0.03 & 0.02 \\ 0.5 & -0.4 & 0.03 & 0.04 \\ 0.2 & -0.3 & 0.02 & 0.03 \end{array}\right];$$

Let $\gamma = 0.8$.
Using theorem 11, we obtain the following matrices:

$$\begin{bmatrix} X(0) & G(0) & Y(0) \\ X(1) & G(1) & Y(1) \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0.0085 & 0.0070 & 0.0424 & 0.0428 & 0.0058 & 0.0129 \\ 0.0070 & 0.0100 & 0.0428 & 0.0523 & -0.0738 & -0.1024 \\ 0.0985 & 0.0639 & 6.0283 & 4.4847 & -0.1500 & -0.1122 \\ 0.0639 & 0.0512 & 4.4847 & 3.3742 & 0.1532 & 0.1280 \end{array}\right]$$

and the scalars $\epsilon(0) = 6.9667$ and $\epsilon(1) = 0.4428$.
Then, the state feedback gain matrices are given by:

$$\begin{bmatrix} K(0) & K(1) \end{bmatrix} = \left[\begin{array}{cc|cc} -0.8899 & 1.9106 & -0.5339 & -1.5254 \\ -0.6507 & -9.7866 & -0.3469 & 2.9338 \end{array}\right];$$

Hence, we noticed that the statements of definition 8 are verified for any admissible parameter uncertainty $\Delta A(k)$, $\Delta A_d(k)$ and $\Delta B(k)$.
For the simulation purposes, we'll take the disturbance signal in the form $w_k = w_0 r^k$ with $r < 1$.

Figure 2 shows the evolution of $\frac{\sum_{i=0}^{k} z_i^\top z_i}{\sum_{i=0}^{k} w_i^\top w_i}$ for some values of Δ_k satisfying (4). The figure shows explicitly that the closed-loop uncertain 2-periodic system exhibits a H_∞ norm bound less than $\gamma = 0.8$.

6. CONCLUSION

In this paper, a sufficient condition for H_∞ control problem of discrete-time periodic system with delays has been presented in terms of Linear Matrix Inequality (LMI). The obtained results are then extended for the uncertain time-delay periodic system. The uncertainties are of the form of Linear Fractional Representation type which includes the case of norm bounded uncertainties as a special case. numerical examples are given to illustrate the proposed results.

7. REFERENCES

[1] B. R. Barmish. Necessary and sufficient conditions for qudratic stabilizability of an uncertain system. *Journal of Optimization Theory and Applications*, 46(4):399–408, 1985.

[2] S. W. Kau, Y. S. Liu, C. H. Lee, L. Hong, and C. H. Fang. An LMI approach to robust H_∞ control for uncertain continuous-time systems. *Asian Journal of Control*, 7(2):182–186, 2005.

[3] P. P. Khargonekar, I. Petersen, and K. Zhou. Robust stabilization of uncertain linear systems: quadratic stabilizability and H^∞ control theory. *IEE Transaction on Automatic Control*, 35(3):356–361, 1990.

[4] S. P.Shue P. Shi, E. K. Boukas and R. K. Agarwa. H_∞ control of discrete-time linear uncertain systems with delayed-state. In 37^{th}*Conference on Decision and Control*, pages 4551–4552, Tampa, Florida USA, 1998.

[5] S. H. Song and J. K. Kim. H_∞ control of discrete-time linear systems with norm-bounded uncertainties and time delay in state. *Automatica*, 34(1):137–139, 1998.

[6] A. Stoorvogel. *The H_∞ control problem*. Systems and Control Engineering. Prentice Hall International (UK) Ltd, Eindhoven university of technology, 1992.

[7] M. Sun, Y. Jia, J. Du, and F. Yu. Robust h_2/h_∞ control for time-delay systems with polytopic uncertainty. *Asian Journal of Control*, 11(1):11–20, 2009.

[8] L. Xie. Output feedback H_∞ control of systems with parameter uncertainty. *International Journal of Control*, 63(4):741–750, 1996.

[9] L. Xie and C. E. De Souza. Robust H_∞ control for linear time-invariant systems with norm-bounded uncertainty in the input matrix. *Systems and Control Letters*, 14:389–396, 1990.

[10] L. Xie and C. E. De Souza. Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty. *IEEE Transactions on Automatic Control*, 37(8):1188–1191, 1992.

[11] S. Xu, J. Lam, and Y. Zou. Improved conditions for delay-dependent robust stability and stabilization of uncertain discrete time-delay systems. *Asian Journal of Control*, 7(3):344–348, 2005.

[12] L. Yuan, L. E. K. Achenie, and W. Jiang. Robust H_∞ control for linear discrete-time systems with norm-bounded time-varying uncertainty. *Systems and Control Letters*, 27:199–208, 1996.