Dominator Coloring of Central Graphs

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ABSTRACT

Dominator chromatic number of central graph of various graph families such as cycles, paths, wheel graphs, complete graphs and complete bipartite graphs are found in this paper. Also these parameters are compared with dominator chromatic number of their respective graph families.

Key words: Central graph, dominator coloring.

AMS Subject Classification: 05C15, 05C69

1. PRELIMINARIES

The notion of central graph and dominator coloring are reviewed in this section [1, 2, 3, 4].

Definition 1.1

Let G be a simple and undirected graph and let its vertex set and edge set be denoted by V(G) and E(G). The *central graph* of G, denoted by C(G) is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in C(G).

Definition 1.2

A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$, is the minimum number of colors required for a proper coloring of G. A *Color class* is the set of all vertices, having the same color. The color class corresponding to the color i is denoted by V_i.

Definition 1.3

A *dominator coloring* of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The convention is that if {v} is a color class, then v dominates the color class {v}. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G.

2. DOMINATOR CHROMATIC NUMBER OF CENTRAL GRAPHS

Dominator chromatic number of Central graph of various classes of graphs is obtained in this section.

Theorem 2.1

For cycle graph C_n of order $n \geq 3$,

$$\chi_d[C(C_n)] = \begin{cases} \lceil 2n/3 \rceil + 2 & \text{when } n=3 \\ \lceil 2n/3 \rceil + 1 & \text{otherwise} \end{cases}$$

Proof

Let C_n be the cycle graph of order $n \geq 3$ and let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$, where $e_i = v_iv_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_nv_1$. By the definition of central graph, $C(C_n)$ is obtained by subdividing each edge v_iv_{i+1}, $1 \leq i \leq n-1$ of C_n exactly once by newly added vertex c_i, and subdividing v_nv_1 by c_n and joining v_i with v_j, $1 \leq i, j \leq n$, $i \neq j$ and $v_iv_j \notin E(C_n)$. Let $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{c_1, c_2, \ldots, c_n\}$. Then $V(C(C_n)) = V_1 \cup V_2$.

The following procedure gives a dominator coloring of $C(C_n)$. Since all newly added vertices c_i, $1 \leq i \leq n$ in $C(C_n)$ form an independent set, color these vertices by color 1. When $n = 3k$, $k \geq 2$, the vertices v_i and v_{i+1}, $i = 1, 4, 7, \ldots, (n-2)$ are colored by color $2\lceil i/3 \rceil$ and the remaining vertices v_j, $j = 3, 6, 9, \ldots, n$ are colored by $1+(2j/3)$. When $n = 3k-1$, $k \geq 2$, the vertices v_i and v_{i+1}, $i = 1, 4, 7, \ldots, (n-4)$ are colored by color $2\lceil i/3 \rceil$ and the remaining vertices v_j, $j = 3, 6, 9, \ldots, (n-2)$ are colored by color $1+(2j/3)$ and v_{n-1} and v_n are colored by $\lceil 2n/3 \rceil$ and $\lceil 2n/3 \rceil + 1$. When $n = 3k+1$, $k \geq 2$, the vertices v_i and v_{i+1}, $i = 1, 4, 7, \ldots, (n-3)$ are colored by color $2\lceil i/3 \rceil$ and the remaining vertices v_j, $j = 3, 6, 9, \ldots, n-1$ are colored by $1+(2j/3)$ for and v_n is colored by $\lceil 2n/3 \rceil+1$. When $n = 3$, the vertex v_i is colored by color $i+1$, $1 \leq i \leq 3$. When $n = 4$, the vertices v_i are colored by the color sequence (2, 2, 3, 4).

The vertices v_i and v_{i+1}, $i = 1, 4, 7, \ldots$ dominate one of the color classes of v_j, $j = 3, 6, 9, \ldots$. The vertex v_j, $j = 3, 6, 9, \ldots$ dominates itself. The center vertices c_i, $i = 1, 4, 7, \ldots$ dominate the color class of v_i. The center vertices c_i and c_{i+1}, $i = 2, 5, 8, \ldots$ dominate the color class of v_{i+1}, as they are adjacent to v_{i+1}. When $n = 3$, it is easy to see that $\chi_d[C(C_3)] = \lceil 2n/3 \rceil + 2$ and when $n = 4$, it is seen that $\chi_d[C(C_4)] = \lceil 2n/3 \rceil + 1$.

Hence $$\chi_d[C(C_n)] = \begin{cases} \lceil 2n/3 \rceil + 2 & \text{when } n=3 \\ \lceil 2n/3 \rceil + 1 & \text{otherwise} \end{cases}$$

The following example illustrates the procedure discussed in the above result.

Example 2.2

In figure 1, central graph of C_7 is depicted with a dominator coloring.

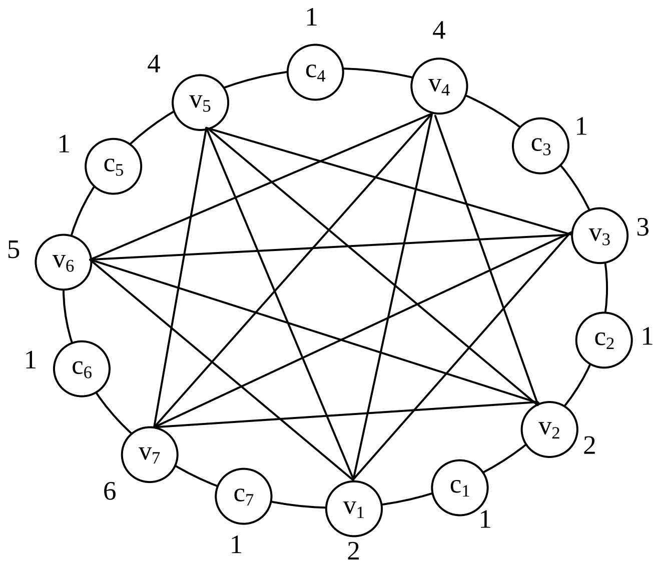

Figure 1

The color classes of $C(C_7)$ are, $V_1 = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$, $V_2 = \{v_1, v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4, v_5\}$, $V_5 = \{v_6\}$ and $V_6 = \{v_7\}$. The dominator chromatic number is, $\chi_d[C(C_7)] = 6$.

Theorem 2.3

For path graph P_n of order $n \geq 2$,

$$\chi_d[C(P_n)] = \begin{cases} \lceil n/2 \rceil + 1 & \text{when n is odd} \\ \lceil n/2 \rceil + 2 & \text{when n is even} \end{cases}$$

Proof

Let P_n be a path of order $n \geq 3$ and let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. The central graph $C(P_n)$ is obtained by subdividing each edge v_iv_{i+1}, $1 \leq i \leq n-1$ of P_n exactly once by adding a new vertex c_i in $C(P_n)$ and joining each vertex v_j, $1 \leq j \leq n-2$ with each vertex v_k, $j+2 \leq k \leq n$. Let $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{c_1, c_2, \ldots, c_{n-1}\}$. Then $V(C(P_n)) = V_1 \cup V_2$. Relabel the vertices of $C(P_n)$ by $u_1 = v_1$, $u_2 = c_1$, $u_3 = v_2$, …, u_{2n-1} consecutively.

The following procedure gives a dominator coloring of $C(P_n)$. When $n \geq 6$, the vertex u_1 is colored by color 1, u_2 is colored by color 2 and the vertex u_i, $i = 4, 6, 8, \ldots, 2n-2$ is colored by color 1. If n is odd, the vertex u_i, $i = 5, 9, 13, \ldots, 2n-1$ is colored by color 2 and the vertices u_j, $j = 3, 7, 11, \ldots, 2n-3$ are respectively colored by individual colors 3, 4, 5, …, $\lceil n/2 \rceil+1$. If n is even, the vertex u_i, $i = 5, 9, 13, \ldots, 2n-3$ is colored by color 2 and the vertices u_j, $j = 3, 7, 11, \ldots, 2n-1$ are colored respectively by individual colors 3, 4, 5, …, $\lceil n/2 \rceil+2$. When n = 3, 4 or 5, the vertices of $C(P_n)$ are colored by the color sequences (1, 2, 1, 3, 2), (1, 2, 3, 1, 2, 4, 2) or (1, 2, 3, 1, 2, 1, 4, 2, 1) in order to get a dominator coloring.

When $n \geq 6$, the vertex u_1 dominates the color class of u_7. If n is odd, vertices u_i and u_{i+2}, $i = 2, 6, 10, 14, \ldots, 2n-3$ dominate the color class of u_{i+1}. The vertex u_j, for $j = 3, 7, 11, \ldots, 2n-3$ dominates itself. If n is even, vertices u_i and u_{i+2}, $i = 2, 6, 10, 14, \ldots, 2n-6$ dominate the color class of u_{i+1} and u_{2n-2} dominates the color class of u_{2n-1}. The vertex u_j, for $j = 3, 7, 11, \ldots, 2n-3$ dominates itself. The vertex u_j, for $j = 5, 9, 13, \ldots, 2n-1$ dominate either u_{j+6}, u_{j-6} or both.

Hence $$\chi_d[C(P_n)] = \begin{cases} \lceil n/2 \rceil + 1 & \text{when n is odd} \\ \lceil n/2 \rceil + 2 & \text{when n is even} \end{cases}$$

The following example illustrates the procedure discussed in the above result.

Example 2.4

In figure 2, central graph of P_6 is depicted with a dominator coloring.

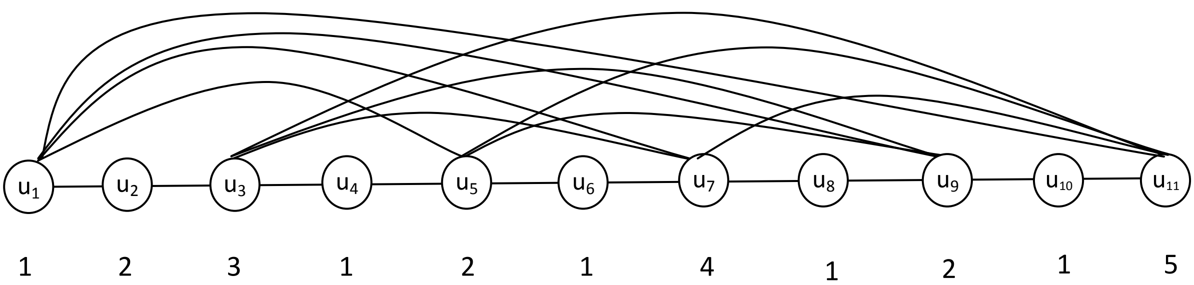

Figure 2

The color classes of $C(P_6)$ are, $V_1 = \{u_1, u_4, u_6, u_8, u_{10}\}$, $V_2 = \{u_2, u_5, u_9\}$, $V_3 = \{u_3\}$, $V_4 = \{u_7\}$, $V_5 = \{u_{11}\}$. The dominator chromatic number is, $\chi_d[C(P_6)] = 5$.

Theorem 2.5

For wheel graph $W_{1,n}$ of order $n \geq 3$,

$$\chi_d[C(W_{1,n})] = \begin{cases} \lceil 2n/3 \rceil + 3 & \text{when } n = 3 \\ \lceil 2n/3 \rceil + 2 & \text{otherwise} \end{cases}$$

Proof

Let $W_{1,n}$ be a wheel graph of order $n \geq 3$. Let the vertex at the centre be v_1 and the vertices on the rim be $v_2, v_3, \ldots, v_{n+1}$. Central graph $C(W_{1,n})$ is obtained by subdividing each edge of $W_{1,n}$ exactly once and join all the non-adjacent vertices of $W_{1,n}$. Let the middle vertices on the edges v_1v_i, $i = 2, \ldots, n+1$ of $W_{1,n}$ be c_{i-1} and the middle vertices on v_iv_{i+1}, i = 2, …, n of $W_{1,n}$ be c_{n+i-1} and the middle vertex on $v_{n+1}v_2$ be c_{2n}.

A dominator coloring of $C(W_{1,n})$ is obtained by the following procedure. Color the vertex v_1 by color 1 and assign color 2 to the center vertices c_i, $1 \leq i \leq 2n$, as they form an independent set. When n = 3k, k ≥ 2, the vertices v_i and v_{i+1}, for i = 2, 5, 8, …, (n–1) are colored by color $1+\lceil 2i/3 \rceil$ and the remaining vertices v_j, for j = 4, 7, 10, …, (n+1) are colored by color $2+\lfloor 2j/3 \rfloor$. When n = 3k–1, k ≥ 2, the vertices v_i and v_{i+1}, for i = 2, 5, 8, …, (n–3) are colored by color $1+\lceil 2i/3 \rceil$ and the remaining vertices v_j, for j = 4, 7, 10, …, (n–1) are colored by color $2+\lfloor 2j/3 \rfloor$ and the vertices v_n and v_{n+1} are colored respectively by colors $\lceil 2n/3 \rceil + 1$ and $\lceil 2n/3 \rceil + 2$. When n = 3k+1, k ≥ 2, the vertices v_i and v_{i+1}, for i = 2, 5, 8, …, (n–2) are colored by color $1+\lceil 2i/3 \rceil$ and the remaining vertices v_j, for j = 4, 7, 10, …, n are colored by color $2+\lfloor 2j/3 \rfloor$ and the vertex v_{n+1} is colored by color $\lceil 2n/3 \rceil + 2$. When n = 3, v_i is colored by color i+1, $2 \leq i \leq 4$. When n = 4, v_i and v_{i+1} is colored by color i+1, i = 2 and remaining vertices v_i are colored by color i, i = 4 and 5.

The vertex v_1 and vertices v_i, i = 4, 7, 10, … of $C(W_{1,n})$, dominate themselves. The vertices v_i and v_{i+1}, i = 2, 5, 8, … dominate the color class of v_{i+2}. The center vertices c_i, $1 \leq i \leq n$ dominate the color class of v_1. The center vertices on the rim c_i and c_{i+1}, i = n+2, n+5, n+8, … dominate the color class of v_{i-n+2} and the center vertices on the rim c_i, i = n+1, n+4, n+7, … dominate the color class of v_{i-n+1}. When n = 3, it is easy to see that $\chi_d[C(W_{1,3})] = \lceil 2n/3 \rceil + 3$ and when n = 4, it is seen that $\chi_d[C(W_{1,4})] = \lceil 2n/3 \rceil + 2$.

Hence $\chi_d[C(W_{1,n})] = \begin{cases} \lceil 2n/3 \rceil + 3 & \text{when } n = 3 \\ \lceil 2n/3 \rceil + 2 & \text{otherwise} \end{cases}$

The following example illustrates the procedure discussed in the above result.

Example 2.6

In figure 3, central graph of $w_{1,5}$ is depicted with a dominator coloring.

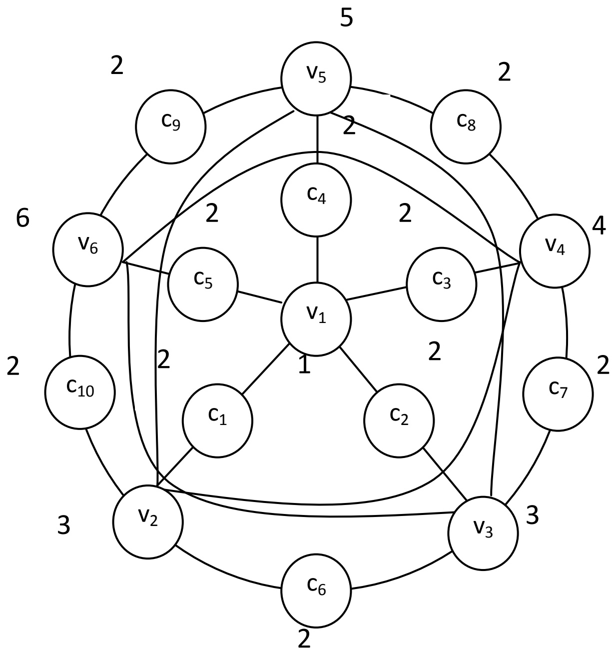

Figure 3

The color classes of $C(W_{1,5})$ are, $V_1 = \{v_1\}$, $V_2 = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}\}$, $V_3 = \{v_2, v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_5\}$ and $V_6 = \{v_6\}$. The dominator chromatic number is, $\chi_d[C(W_{1,5})] = 6$.

Theorem 2.7

For complete graph K_n of order n ≥ 3, $\chi_d[C(K_n)] = n + 1$.

Proof

Let K_n be the complete graph of order n ≥ 3 and let $V(K_n) = \{v_1, v_2, …, v_n\}$. The central graph is obtained by subdividing each edge v_iv_j exactly once by the center vertex c_{ij}, $1 \leq i, j \leq n$, $i \neq j$. Let $V_1 = \{v_1, v_2, …, v_n\}$ and $V_2 = \{ c_{ij} / 1 \leq i, j \leq n, i \neq j\}$. Then $V(C(K_n)) = V_1 \cup V_2$.

A dominator coloring of $C(K_n)$ is obtained by the following procedure. Since the center vertices form an independent set, color c_{ij}, $1 \leq i, j \leq n$, $i \neq j$, by color 1. The vertex v_i is colored by the color (i+1), $1 \leq i \leq n$.

The center vertex c_i, $1 \leq i \leq n$ dominates at least one of the color classes of v_i, $1 \leq i \leq n$. The vertices $v_1, v_2, …, v_n$ dominate themselves. Hence $\chi_d[C(K_n)] = n + 1$.

Theorem 2.8

For complete bipartite graph $K_{m, n}$ of order m, $n \geq 1$, $\chi_d[C(K_{m,n})] = m + n.$

Proof

Let $\{v_i: 1 \leq i \leq m\}$ and $\{u_j: 1 \leq j \leq n\}$ be the vertices of $K_{m, n}$ and by the definition of complete bipartite graph , every vertex in $\{v_i: 1 \leq i \leq m\}$ is adjacent to every vertex in $\{u_j : 1 \leq j \leq n\}$. Let $\{e_{ij} : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ be the set of edges of $K_{m, n}$. By the definition of central graph, each edge e_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$ is subdivided by a vertex c_{ij} in $C(K_{m, n})$ and let $V = \{v_1, v_2, \ldots, v_m\}$, $V' = \{u_1, u_2, \ldots, u_n\}$ and $V'' = \{c_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$. Clearly $V(C(K_{m, n})) = V \cup V' \cup V''$. Note that in $C(K_{m, n})$, the induced subgraphs $\langle V = \{v_1, v_2, \ldots, v_m\}\rangle$ and $\langle V' = \{u_1, u_2, \ldots, u_n\}\rangle$ are complete.

The following procedure gives a dominator coloring of $C(K_{m, n})$. The set of vertices $\{c_{ij} / 1 \leq i \leq m; 1 \leq j \leq n\}$ is independent and hence assign the color 1 to these vertices. The vertices v_1 and u_1 are non–adjacent in $C(K_{m, n})$ and therefore they are colored by color 2. As the induced subgraphs $\langle V \rangle$ and $\langle V' \rangle$ are complete, the vertex v_i is colored by the color (i+1), $2 \leq i \leq m$ and the vertex u_j is colored by the color (m+j), $2 \leq j \leq n$.

The vertex v_1 dominates the color classes of v_i, $2 \leq i \leq m$ and the vertex u_1 dominates the color classes of u_j, $2 \leq j \leq n$. The vertices v_i and u_j, $2 \leq i \leq m$; u_j: $2 \leq j \leq n$ dominate themselves. The center vertex c_{ij} dominates one of the color classes of v_i and u_j, $2 \leq i \leq m$; u_j: $2 \leq j \leq n$. Hence $\chi_d[C(K_{m,n})] = m + n.$

By combining the observations of [2] and above theorems, we have the following result.

Result 2.9

(i) $\chi_d[C(C_n)] > \chi_d(C_n)$

(ii) $\chi_d[C(P_n)] > \chi_d(P_n)$

(iii) $\chi_d[C(W_{1, n})] > \chi_d(W_{1, n})$

(iv) $\chi_d[C(K_n)] > \chi_d(K_n)$

3. CONCLUSION

In this paper, we obtained the dominator chromatic number of central graph of various graph families are obtained and compared with dominator chromatic number of their corresponding graph families. This paper can further be extended by identifying graph families of graphs for which these chromatic numbers are equal to other kinds of chromatic numbers.

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