Some Results on Degree of Vertices in Semitotal-Block Graph and Total-Block Graph

Bhavanari Satyanarayana[1], Devanaboina Srinivasulu[2]
[1,2]Department of MathematicsAcharyaNagarjuna University
Nagarjuna Nagar-522 510
Andhra Pradesh, India

Kuncham Syam Prasad
Department of Mathematics, MIT, Manipal University,
Manipal-576 104, India

Eswaraiah Setty S.
SGS College,
Jaggaiahpet, Krishna Dist.
Andhra Pradesh, India

ABSTRACT

We considersemitotal-block graph, total-block graph of a graph G (respectively, denoted as $T_b(G)$, $T_B(G)$). We prove that the number of edges in a semitotal-block graph of a given graph G is equal to $|E(G)| + |V(B_1)| + |V(B_2)| + \ldots + |V(B_m)|$, where $B_1, B_2, \ldots, B_m$ are the blocks of G. Further, we obtain that $T_B(G)$ is the ring sum of $T_b(G)$ and the block graph B(G). We introduce the concept "vertex-block graph (denoted by $B_v(G)$ of G)", andwe prove that$T_b(G)$ is the ring sum of G and $B_v(G)$. We also present some related fundamental results along with illustrations.

General Terms

Graph Theory, Characterization of different Block Graphs.

Keywords

Degree of vertex, Semitotal-block graph, Total-block graph.

1. INTRODUCTION

A *finitegraph* G = (V, E) consists of a finite nonempty set of objects, $V = \{v_1, v_2, \ldots\}$ called *vertices* and another finite set, $E = \{e_1, e_2, \ldots\}$ of elements called *edges* such that each edge e_k is identified with an unordered pair $\{v_i, v_j\}$ of vertices. An edge associated with a vertex pair $\{v_i, v_i\}$ is called a *self-loop*. The number of edges associated with the vertex is the degree of the vertex, and $\delta(v)$ denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called *parallel edges* (or) *multiple edges*. A graph that has does not have self-loop or parallel edges called a *simple graph*. Two vertices are said to be *adjacent* if they are the end vertices of the same edge. A finite alternating sequence of vertices and edges (no repetition of edge allowed) beginning and ending with vertices such that each edge is incident with the vertices preceding and following it, is called a *walk* and an open walk in which no vertex appears more than once, is called a *path*. A graph said to be *connected* if there is at least one path between every pair of vertices in G, otherwise it is called *disconnected*. In a connected graph, a vertex whose removal disconnects the graph is called a *cut-vertex*. The authors in [9] studied the cut vertices and a special type of symmetry in graphs. The *n*-cube defined as, for a set X with |X| = n and $\wp(X)$ be its power set. Then a graph having $\wp(X)$ as its vertex set; and there is an edge between two vertices A, B if and only if $|A \Delta B| = 1$ where $A \Delta B = (A \setminus B) \cup (B \setminus A)$, is called the *n-cube*. The graph n-cube is characterized and obtained an isomorphism theorem in [6, 7, and 10].

For any two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we define their *union* as the graph G = (V, E) where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, their *intersection*when $V_1 \cap V_2 \neq \phi$ is defined as the graph G = (V, E) where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$, and the *ring sum*$G_1 \oplus G_2$ of two graphs G_1 and G_2 is defined as the graph G = (V, E) where $V = V_1 \cup V_2$ and $E = (E_1 \cup E_2) \setminus (E_1 \cap E_2)$.

In this paper, we consider only finite simple graphs.

For the remaining fundamental definitions and results which are used in the paper, we refer [1, 2, 5, and 7].

2. SEMITOTAL-BLOCK GRAPHS

In this section, we define semitotal-block graphs and provide some examples. We prove that the number of edges in a semitotal-block graph is equal to the sum of the edges in a graph and the number of vertices in all the blocks of a graph.

1.1Definition [2]: A connected non–trivial graph having no cut point is a block. A block of a graph is a sub-graph that is a block and is maximal with respect to this property.

1.2 Note: The set of all Blocks of G is denoted by SB(G)

1.3 Example: (i). Consider the graph G in figure 1.

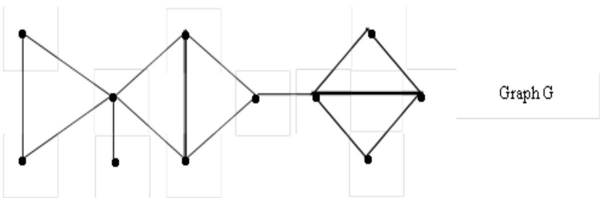

Figure 1

There are five blocks in G. They are $B_1(G)$, $B_2(G)$, $B_3(G)$, $B_4(G)$, $B_5(G)$. These blocks are shown in figure 2.

In this example, SB(G) = {$B_1(G)$, $B_2(G)$, $B_3(G)$, $B_4(G)$, $B_5(G)$}.

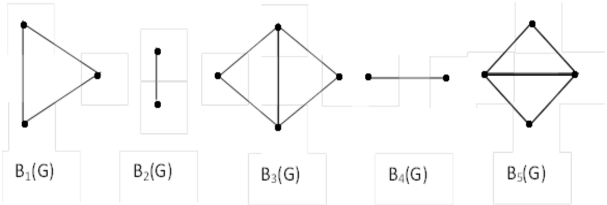

Figure 2

(ii) Observe that n-cube contains no cut point; and so it contains no other blocks except itself. That is, n-cube has just one block.

1.4 ***Definition*** [4]: The *semitotal–block graph* (denoted by $T_b(G)$) of a given graph G is defined as the graph having point set $V(G) \cup B(G)$, with two points adjacent if they correspond to two adjacent points of G or one corresponds to a block Bof G and other to a point v of G and v in B

1.5 ***Example***: Consider the graph G given in Figure 3.

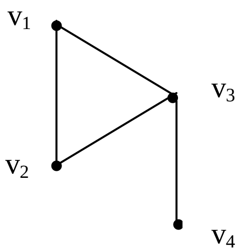

Figure 3

There are two blocks B_1 and B_2 of G are given in figure 4.

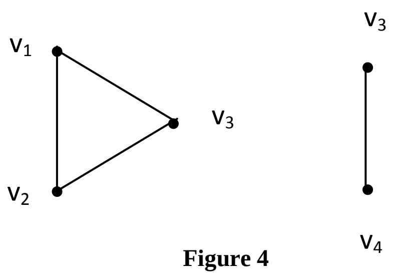

Figure 4

Then B(G) ={ B_1, B_2}. We construct the semi total-block graph $T_b(G)$.Now $V(T_b(G))$ = The vertex set of $T_b(G) = V(G) \cup B(G) = \{v_1, v_2, v_3, v_4, B_1, B_2\}$. Following the definition, we draw the semitotal-block graph $T_b(G)$ of G is given in figure 5.

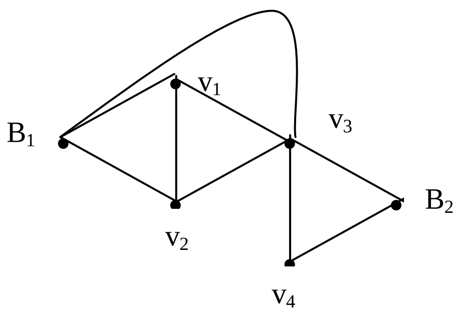

Figure 5

1.6 ***Note***: Let G be a graph. By the definition of $T_b(G)$, it follows that every edge in G is also an edge in T_b. Therefore $E(G) \subseteq E(T_b(G))$. Thus G is a subgraph of $T_b(G)$.

1.7 ***Definition***: Let G be a graph and B a block in G. An edge e in $T_b(G)$ is said to be an *edge related to block* B if one of the end points of e is B.

1.8 ***Lemma***: Let B be a block in the given graph G. Then

(i). The degree of the vertex B (of the semitotal-block graph) is equal to the number of vertices in the block B (of the given graph G).

(ii). The number of edges in $T_b(G)$ related to block B is $|V(B)|$.

Proof: (i) Suppose that the block B consists of 'k' vertices and $V(B) = \{v_1, v_2, \dots v_k\}$.

Since each v_i is in the block B, by the definition of semitotal-block graph, we get an edge between the vertex v_i and B (in the semitotal-block graph $T_b(G)$). This is true for all i with $1 \leq i \leq k$. Thus there are 'k' edges with end point B in $T_b(G)$. Any edge with an end point B in $T_b(G)$ is obtained in this way. Hence we get (i).

(ii) From (i), we get that degree of B (in $T_b(G)$) is k.

1.9 ***Theorem***: Let G be a connected graph. Then

$|E(T_b(G))| = |E(G)| + \sum_{i=1}^{m} |V(B_i)|$,where $B_1, B_2, \dots B_m$are the blocks of G.

Proof: Suppose $SB(G) = \{B_1, B_2, \dots, B_m\}$. Let e be an edge in $T_b(G)$.

If e is an edge between two vertices in V(G), then $e \in E(G)$. Otherwise, the edge formed in between a vertex and a block B_i (of G). Let B be a block in G with k vertices.

By Lemma 1.8, the number of edges related to block B is $|V(B)| = k$. Therefore the number of distinct edges (related to block B) that exist in $T_b(G)$ is $|V(B)|$.

This is true for each block B in G. Thus there are $\sum_{i=1}^{m} |V(B_i)|$ edges in $T_b(G)$ that are related to different blocks of G. Hence the number of edges in $T_b(G)$ is $|E(G)|$ + $\sum_{i=1}^{m} |V(B_i)|$. The proof is complete.

1.10. ***Theorem***: Let v be a vertex in a given graph G, then the degree of v in $T_b(G) = \delta(v) + |\{B \mid B$ is a block in G such that v lies in B$\}|$.

1.11 ***Corollary***: Let G be a graph and v a vertex in G. Then

(i) degree of v in $T_b(G) \geq$ degree of v in G

(ii) degree of v in $T_b(G)$ = degree of v in G

$\Leftrightarrow$v is not contained in any block of G.

Observe that every vertex lies in a block. So the degree of v in $T_b(G) \neq$ degree of v in G. Hence degree of v in $T_b(G)$ > degree of v in G.

1.12 ***Definition***: Let G be a graph. The *vertex-block graph* (denoted by $B_v(G)$) of G is defined as follows:

(i) $V(B_v(G)) = V(G) \cup SB(G)$

(ii)$E(B_v(G)) = \{ \overline{xy} \mid x \in V(G)$ and $y \in SB(G)$ such that x is a vertex of the block y}.

1.13 ***Note***: (i) $B_v(G)$ is a spanning subgraph of $T_b(G)$.

(ii) G is not a spanning subgraph of $T_b(G)$ if G contains a block.

1.14 ***Theorem***: $T_b(G) = G \oplus B_v(G)$.

Proof: Let G be a graph. Since G and $B_v(G)$ are subgraphs of $T_b(G)$, it follows that $G \cup B_v(G) \subseteq T_b(G)$.

By the definition of $T_b(G)$, every edge in $T_b(G)$ is either in G or in $B_v(G)$. So $T_b(G) \subseteq G \cup B_v(G)$. Hence $T_b(G) = G \cup B_v(G)$. By the definition of $B_v(G)$, no edge of G is in $B_v(G)$. Therefore Hence $E(G \oplus B_v(G)) = (E(G) \setminus E(B_v(G))) \cup (E(B_v(G)) \setminus E(G)) = E(G) \cup E(B_v(G))$. Hence $G \oplus B_v(G) = G \cup B_v(G)$, which shows that $T_b(G) = G \oplus B_v(G)$.

3. TOTAL BLOCK GRAPH

In this section, we study 'total-block graph' and related results with illustrations.

2.1 ***Definition***: The *total–block graph* (denoted as $T_B(G)$) of a given graph G is defined as the graph having the point set $V(G) \cup B(G)$, with two points adjacent if they corresponds to either two adjacent points of G or two blocks of G which have a common cut point or one corresponds to a block B_i of G and the other to a point v_j of G with v_j is in B_i.

2.2 ***Example***: A graph G and its total-block graph $T_B(G)$ is given in figure 6 and figure 7 respectively.

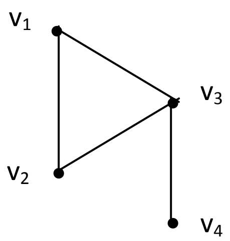

Figure 6

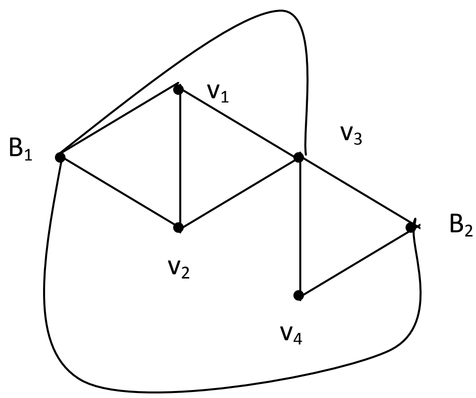

Figure 7

2.3 ***Note***: Let G be a graph. By the definition of $T_B(G)$, it follows that every edge of G is in $T_B(G)$. So $E(G) \subseteq E(T_B(G))$. Thus G is a subgraph of $T_B(G)$.

2.4***Definition***: Two blocks B_1 and B_2 in a given graph G are said to be *adjacent*blocks if they have a common cut vertex.

2.5 ***Theorem***: (i) The number of edges in $T_B(G)$ related to block B is equal to |V(B)| + (the number of adjacent blocks to B).

(ii) The degree of B in $T_B(G)$ = |V(B)| + (the number of adjacent blocks to B).

Proof: Let B be a block and e be an edge in $T_B(G)$ related to block B

Then either e $=\overline{vB}$for some $v \in B$ or e $=\overline{BB_1}$, for some adjacent block B_1 of B.

The number of edges of the form $\overline{vB}$is |V(B)|. The number of edges of the form$\overline{BB_i}$, is equal to the number of distinct blocks B_i, which are adjacent to B.

Hence the number of edges in $T_B(G)$ related to block B is equal to |V(B)|+(the number of adjacent blocks to B). Thus we get (i).

(ii). F v2 fron v4 • B2

2.6 ***L*** . ***n*** (: G . The *block graph* (denoted by B(G)) is defined as follows:

V(B(G)) = SB(G), the set of all blocks of G; and E(B(G)) = $\{\overline{B_1B_2} \mid B_1, B_2 \in V(B(G))$ and B_1 and B_2 have a common cut vertex}.

2.7 ***Remark***: Let G be a graph and B be a block. Then degree of B in B(G), (the block graph) is equal to the number of distinct adjacent blocks to B in G.

The following theorem states a relation between the graphs: total-block graph; semitotal-block graph; vertex-block graph; and block graph.

2.8 ***Theorem***: For a connected graph G, (i) $T_B(G) = T_b(G) \oplus B(G)$, and

(ii) $T_B(G) = G \oplus B_v(G) \oplus B(G)$.

Proof: By the definition of $T_B(G)$, it follows that $T_b(G)$ and B(G) are subgraphs of $T_B(G)$.

$V(T_B(G)) = V(G) \cup SB(G) = (V(G) \cup SB(G)) \cup SB(G)$ (by idempotent and associative laws of sets) $= V(T_b(G)) \cup V(B(G))$.

Let s be an edge in $T_B(G)$. Then $s \in E(G)$ or $s = \overline{vB}$ for some $v \in V(G)$, $B \in SB(G)$ with $v \in B$ or $s = \overline{B_1B_2}$ for some B_1, $B_2 \in SB(G)$ with B_1, B_2 are adjacent blocks in G.

Now $s \in E(G)$ or $s \in E(B_v(G))$ or $s \in E(B(G)) \Rightarrow s \in E(G) + E(B_v(G)) = E(T_b(G))$ or $s \in E(B(G))$. Therefore $E(T_B(G)) \subseteq E(T_b(G)) \cup E(B(G))$.

Since $T_b(G)$ and B(G) are subgraphs of $T_B(G)$ we have $E(T_b(G)) \cup E(B(G)) \subseteq E(T_B(G))$.

Hence $E(T_B(G)) = E(T_b(G)) \cup E(B(G))$. This shows that $T_B(G) = T_b(G) \cup B(G)$, the union of the graph $T_b(G)$ & B(G). Since $T_B(G)$ and B(G) have no edge in common, we conclude that $T_B(G) = T_b(G) \oplus B(G)$, the ring sum of the graphs $T_b(G)$ & B(G).

(ii). By Theorem 1.15, we have that $T_b(G) = G \oplus B_v(G)$.

Now $T_B(G) = T_b(G) \oplus B(G)$ (by (i)) $= G \oplus B_V(G) \oplus B(G)$ (by the Theorem 1.15)

The proof is complete.

The following Corollary answers the question "How many edges are there in total-block graph".

2.9 ***Corollary***: $|E(T_B(G))| = |E(G)| + |V(B_1)| + \ldots + |V(B_m)| + |E(B(G))|$, where $B_1, B_2, \ldots, B_m$ are the blocks of G.

Proof: By Theorem 2.8, we have that $T_B(G) = T_b(G) \oplus B(G)$. Therefore $|E(T_B(G)) = |E(T_b(G))| + |E(B(G))| = |E(G)| + |V(B_1)| + \ldots + |V(B_m)| + |E(B(G))| =$ (by the Theorem 1.9). The proof is complete.

A straightforward observation leads to the following.

2.10 ***Corollary***: Let v be a vertex in a given graph G. Then (i) Degree of v in $T_B(G)$ = degree of v in $T_b(G)$, (ii) Degree of v in $T_B(G) = \delta(v) + |\{B \,/\, B$ is a block in G such that v lies in $B\}|$.

2.11 ***Corollary***: Let G be a graph and v a vertex in G.

(i) degree of v in $T_B(G) \geq$ degree of v in G

(ii) degree of v in $T_B(G)$ = degree of v in G $\Leftrightarrow$ degree of v in $T_b(G)$ = degree of v in G $\Leftrightarrow$ v is not contained in any block of G.

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