Coupled Fixed Point Theorem for Weakly Compatible Mappings in Intuitionistic Fuzzy Metric Spaces

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ABSTRACT
The aim of present paper is to introduce the notion of t-conorm of H-type analogous to t-norm of H-type introduced by Hadzic [9] and using this notion we prove coupled fixed point theorems for weakly compatible mappings in intuitionistic fuzzy metric spaces.

Key Words
Coupled fixed point, Compatible maps, Fixed points and Intuitionistic Fuzzy Metric Spaces.

1. INTRODUCTION
Recently, Bhaskar and Lakshmikantham [10] introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later on these results were extended and generalized by Fang [4]and Xin-Qi Hu [12] etc .

As a generalization of fuzzy sets, Atanassove [1] introduced and studied the concept of intuitionistic fuzzy metric sets. Intuitionistic fuzzy sets deal with both degree of nearness and non-nearness. Motivated by the idea of intuitionistic fuzzy metric sets Park [8] introduced the concept of intuitionistic fuzzy metric spaces using continuous t-norms and continuous t-conorms. Later on , many authors[2-6] have studied fixed points results in intuitionistic fuzzy metric spaces.Fang [7] defined ϕ-contractive conditions and proved some important fixed point theorems under ϕ-contractions for compatible and weakly compatible maps in Menger PM-spaces using t-norm of H-type introduced by Hadzic [9].

In this paper , first we introduce the notion of t-conorm of H-type analogous to t-norm of H-type introduced by Hadžíc [9] and using this notion we prove a common coupled fixed point theorem for compatible mappings in intuitionistic fuzzy metric spaces.

2. DEFINITIONS AND PRELIMINARIES
For basic definitions and structure on intuitionistic fuzzy metric spaces we refer to [1-6].However we give some definitions in the sequel.

Definition 2.1 A binary operation * : [0,1] × [0,1] → [0,1] is continuous t-norm if $([0,1], *)$ is a topological abelian monoid with unit 1 such that t a * b ≤ c * d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0,1].

Definition 2.2. A binary operation ◊ : [0,1] × [0,1] → [0,1] is continuous t-conorm if $([0,1], \Diamond)$ is a topological abelian monoid with unit 1 such that) a ◊ b ≤ c ◊ d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0,1].

Definition 2.3[3]. A 5-tuple (X, M, N, *, ◊) is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, ◊ is a continuous t-conorm and M, N are fuzzy sets on $X^2\times(0,\infty)$ satisfying the following conditions:

(i) M(x, y, t) + N(x, y, t) ≤ 1 for all x, y ∈ X and t > 0,

(ii) M(x, y, 0) = 0 for all x, y ∈ X,

(iii) M(x, y, t) = 1 for all x, y ∈ X and t > 0 if and only if x = y,

(iv) M(x, y, t) = M(y, x, t) for all x, y ∈ X and t > 0,

(v) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s) for all x, y, z ∈ X and t, s > 0,

(vi) for all x, y ∈ X, M(x, y, .) : [0, ∞) → [0,1] is continuous,

(vii) $\lim_{t\to\infty} M(x,y,t) = 1$, for all x, y in X,

(viii) N(x, y, 0) = 1 for all x, y ∈ X,

(ix) N(x, y, t) = 0 for all x, y ∈ X and t > 0 if and only if x = y,

(x) N(x, y, t) = N(y, x, t) for all x, y ∈ X and t > 0,

(xi) N(x, y, t) ◊ N(y, z, s) ≥ N(x, z, t + s) for all x, y, z ∈ X and t, s > 0,

(xii) for all x, y ∈ X, N(x, y, .) : [0, ∞) → [0,1] is continuous,

(xiii) $\lim_{t\to\infty} N(x,y,t) = 0$, for all x, y in X.

Then (M, N) is called an intuitionistic fuzzy metric on X. the functions M(x, y, t) and N(x, y, t) denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Definition 2.4[3]. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. Then

(i) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if for all t > 0 and p ≥ 1,

$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$.

(ii) a sequence $\{x_n\}$ in X is said to be convergent to a point

x ∈ X, if for all t > 0,

$\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$.

Definition 2.5[9]. Let $\sup_{0<t<1} \Delta(t, t) = 1$. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at t = 1, where

$\Delta^1(t) = t$, $\Delta^{m+1}(t) = t\,\Delta\,(\Delta^m(t))$, m= 1, 2…… t ∈ [0, 1].

The t-norm Δ_M = min. is an example of t-norm of H-type.

Remark 2.1. Δ is a H-type t-norm iff for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\Delta^m(t) > (1-\lambda)$ for all m ∈ N, when $t > (1-\delta)$.

We now define notion of t-conorm of H-type analogous to t-norm of H-type as follows.

Definition 2.6. Let $\inf_{0<t<1} \lozenge(t, t) = 0$. A t-conorm ◊ is said to be of H-type if the family of functions $\{\lozenge^m(t)\}_{m=1}^{\infty}$ is equicontinuous at t = 0, where $\lozenge^1(t) = t$, $\lozenge^{m+1}(t) = t \lozenge (\lozenge^m(t))$, m= 1, 2,… t ∈ [0, 1].

The t- conorm $\lozenge_m$ = max. is an example of t-conorm of H-type.

Remark 2.2.◊ is a H-type t-norm iff for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\lozenge^m(t) < \lambda$ for all m ∈ N, when

$t < \delta$.

Definition 2.7[11]. An element (x, y) ∈ X × X is called a coupled fixed point of the mapping f: X × X → X if
f(x, y) = x , f(y, x) = y.

Definition 2.8[11]. An element (x, y) ∈ X × X is called a coupled coincidence point of the mappings f: X × X → X and g: X → X if
f(x, y) = g(x), f(y, x) = g(y).

Definition 2.9[11]. An element (x, y) ∈ X × X is called

(i) a common coupled fixed point of the mappings f: X × X → X and g: X → X if
x = f(x, y) = g(x), y = f(y, x) = g(y).

(ii) a common fixed point of the mappings f: X × X → X and g: X → X if
x = f(x, x) = g(x).

Definition 2.10 [5]. The mappings f: X × X → X and g: X → X are said to be compatible if

$\lim_{n\to\infty} M(gf(x_n, y_n), f(g(x_n), g(y_n)), t) = 1$,

$\lim_{n\to\infty} M(gf(y_n, x_n), f(g(y_n), g(x_n)), t) = 1$

And ,$\lim_{n\to\infty} N(gf(x_n, y_n), f(g(x_n), g(y_n)), t) = 0$,

$\lim_{n\to\infty} N(gf(y_n, x_n), f(g(y_n), g(x_n)), t) = 0$,

for all t > 0 whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$,

$\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$,

for some x, y in X.

Lemma 2.1[2]. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space and for all x, y in X, t > 0, if there exists a number k ∈ (0,1) such that

M(x, y, kt) ≥ M(x, y, t) and N(x, y, kt) ≤ N(x, y, t) then

x = y.

3. MAIN RESULTS

Theorem 3.1. Let (X, M, N, *, ◊) be a Complete Intuitionistic Fuzzy Metric Space, * being continuous t–norm of H-type and ◊ being continuous t–conorm of H-type. Let A , B : X × X → X and S , T : X → X be four mappings satisfying the following conditions:

(3.1) A(X × X) ⊆ T(X), B(X × X) ⊆ S(X) and the pairs

(A, S) and (B, T) are weakly compatible

(3.2) S and T are continuous,

(3.3) $M(A(x,y), B(u,v), kt) \geq \phi[Min\{M(Sx,Tu,t), M(A(x,y),Sx,t), M(B(u,v),Tu,t)\}]$

(3.4) $N(A(x,y), B(u,v), kt) \leq \psi[Max\{N(Sx,Tu,t), N(A(x,y),Sx,t),$

$N(B(u,v),Tu,t)\}]$

$\forall x, y, u, v \in X, k \in (0,1)$ where

$\phi, \psi: 0,1 \rightarrow 0,1$ are continuous functions such that

$\phi(t) > t, \phi(1) = 1$ and $\psi(t) < t, \psi(0) = 0$ for

$0 < t < 1$. Then A,B,S and T have a unique common fixed point in X , i.e there exists unique x in X such that
A(x, x) = T(x) = B(x, x) = S(x) = x.

Proof: Let x_0, y_0 be two arbitrary points in X. Since A(X × X) ⊆ T(X), we can choose x_1, y_1 in X such that $T(x_1) = A(x_0, y_0)$, $T(y_1) = A(y_0, x_0)$. Again, Since B(X × X) ⊆ S(X), we can choose x_2, y_2 in X such that $S(x_2) = B(x_1, y_1)$ and

$S(y_2) = B(y_1, x_1)$. Continuing in this way, we can construct two sequences $\{z_n\}$ and $\{z'_n\}$ in X such that

$z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1})$,

$z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2})$

And ,$z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1})$,

$z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2})$, for all $n \geq 0$.

Step 1: We first show that $\{z_n\}$ and $\{z'_n\}$ are Cauchy sequences.Using (3.3) and (3.4) ,

$$M(z_{2n+1}, z_{2n+2}, kt) = M(A(x_{2n}, y_{2n}), B(x_{2n+1}, y_{2n+1}), kt)$$

$$\geq \phi\left[Min\begin{Bmatrix} M(Sx_{2n}, Tx_{2n+1}, t), \\ M(A(x_{2n}, y_{2n}), Sx_{2n}, t), \\ M(B(x_{2n+1}, y_{2n+1}), Tx_{2n+1}, t) \end{Bmatrix}\right]$$

$$= \phi\left[Min\begin{Bmatrix} M(A(x_{2n-1}, y_{2n-1}), B(x_{2n}, y_{2n}), t), \\ M(A(x_{2n}, y_{2n}), B(x_{2n-1}, y_{2n-1}), t), \\ M(B(x_{2n+1}, y_{2n+1}), A(x_{2n}, y_{2n}), t) \end{Bmatrix}\right]$$

$$= \phi[Min\{M(z_{2n}, z_{2n+1}, t), M(z_{2n}, z_{2n+1}, t), M(z_{2n+2}, z_{2n+1}, t)\}]$$

If $M(z_{2n+2}, z_{2n+1}, t) \leq M(z_{2n+1}, z_{2n}, t)$, a contradiction as $M(x, y, t)$ is increasing , therefore ,

$$M(z_{2n+2}, z_{2n+1}, kt) \geq \phi[M(z_{2n+1}, z_{2n}, t)]$$

$$\geq M(z_{2n+1}, z_{2n}, t)$$

And ,

$$N(z_{2n+1}, z_{2n+2}, kt) = N(A(x_{2n}, y_{2n}), B(x_{2n+1}, y_{2n+1}), kt)$$

$$\leq \psi\left[Max\begin{Bmatrix} N(Sx_{2n}, Tx_{2n+1}, t), \\ N(A(x_{2n}, y_{2n}), Sx_{2n}, t), \\ N(B(x_{2n+1}, y_{2n+1}), Tx_{2n+1}, t) \end{Bmatrix}\right]$$

$$= \psi\left[Max\begin{Bmatrix} N(A(x_{2n-1}, y_{2n-1}), B(x_{2n}, y_{2n}), t), \\ N(A(x_{2n}, y_{2n}), B(x_{2n-1}, y_{2n-1}), t), \\ N(B(x_{2n+1}, y_{2n+1}), A(x_{2n}, y_{2n}), t)\} \end{Bmatrix}\right]$$

$$= \psi[Max\{N(z_{2n}, z_{2n+1}, t), N(z_{2n}, z_{2n+1}, t), N(z_{2n+2}, z_{2n+1}, t)\}]$$

If $N(z_{2n+2}, z_{2n+1}, t) \geq N(z_{2n+1}, z_{2n}, t)$, a contradiction as $N(x, y, t)$ is decreasing , therefore ,

$$N(z_{2n+2}, z_{2n+1}, kt) \leq \psi[N(z_{2n+1}, z_{2n}, t)] \leq N(z_{2n+1}, z_{2n}, t)$$

Similarly , we can show that

$M(z_{2n+3}, z_{2n+2}, kt) \geq M(z_{2n+2}, z_{2n+1}, t)$ and
$N(z_{2n+3}, z_{2n+2}, kt) \leq N(z_{2n+2}, z_{2n+1}, t)$

In general, $M(z_n, z_{n+1}, kt) \geq M(z_{n-1}, z_n, t)$ and $N(z_n, z_{n+1}, kt) \leq N(z_{n-1}, z_n, t)$. Thus by lemma [2.1] z_n is a Cauchy sequence . Similarly we can show $\{z'_n\}$ is Cauchy sequence.

Step 2: Since X is complete, there exists point a, b in X such that

$\lim_{n\to\infty} z_n = a$ and $\lim_{n\to\infty} z'_n = b$,

that is,$\lim_{n\to\infty} z_{2n+1} = \lim_{n\to\infty} A(x_{2n}, y_{2n}) = \lim_{n\to\infty} T(x_{2n+1}) = a$,

$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n\to\infty} S(x_{2n+2}) = a$

and ,$\lim_{n\to\infty} z'_{2n+1} = \lim_{n\to\infty} A(y_{2n}, x_{2n}) = \lim_{n\to\infty} T(y_{2n+1}) = b$,

$\lim_{n\to\infty} z'_{2n} = \lim_{n\to\infty} B(y_{2n+1}, x_{2n+1}) = \lim_{n\to\infty} S(y_{2n+2}) = b$.

We first show that S(a) = T(a). As S and T are continuous , so $SSx_{2n} \to Sa, SSy_{2n} \to Sb, SA(x_{2n}, y_{2n}) \to Sa.$ and
$TTx_{2n} \to Ta, TTy_{2n} \to Tb, TB(x_{2n}, y_{2n}) \to Ta.$
But the pairs (A,S) and (B,T) are weakly compatible , so ,

$SA(x_{2n}, y_{2n}) = A(Sx_{2n}, Sy_{2n}) \to Sa$ and
$TB(x_{2n}, y_{2n}) = B(Tx_{2n}, Ty_{2n}) \to Ta$ .Using (3.3) and (3.4), we have ,

$$M(A(Sx_{2n}, Sy_{2n}), B(Tx_{2n}, Ty_{2n}), kt))$$

$$\geq \phi\left[Min\begin{Bmatrix} M(SSx_{2n}, TTx_{2n}, t), \\ M(A(Sx_{2n}, Sy_{2n}), SSx_{2n}, t), \\ M(B(Tx_{2n}, Ty_{2n}), TTx_{2n}, t) \end{Bmatrix}\right]$$

Taking $n \to \infty$, we get

$$M(Sa,Ta,kt) \geq \phi[Min\{M(Sa,Ta,t),1,1\}$$

$$\geq M(Sa,Ta,t)$$

And ,

$$N(A(Sx_{2n},Sy_{2n}),B(Tx_{2n},Ty_{2n}),kt))$$

$$\leq \psi\left[Max\begin{Bmatrix} N(SSx_{2n},TTx_{2n},t), N(A(Sx_{2n},Sy_{2n}),SSx_{2n},t), \\ N(B(Tx_{2n},Ty_{2n}),TTx_{2n},t) \end{Bmatrix}\right]$$

Taking $n \to \infty$, and using lemma (3.1) ,we get

$$N(Sa,Ta,kt) \leq \psi[Min\{N(Sa,Ta,t),0,0\}]$$

$$\leq N(Sa,Ta,t).$$

Which gives Sa = Ta .

Now , we prove that Sa = B(a,b) , again using (3.3) and (3.4) we have

$$M(A(Sx_{2n},Sy_{2n}),B(a,b),kt)$$

$$\geq \phi[Min\{M(SSx_{2n},Ta,t),M(A(Sx_{2n},Sy_{2n}),SSx_{2n},t),$$

$$M(B(a,b),Ta,t)\}]$$

And ,

$$N(A(Sx_{2n},Sy_{2n}),B(a,b),kt)$$

$$\leq \psi[Max\{N(SSx_{2n},Ta,t),N(A(Sx_{2n},Sy_{2n}),SSx_{2n},t),$$

$$N(B(a,b),Ta,t)\}]$$

Taking $n \to \infty$, we get Sa = B(a,b) = Ta . Now , we prove that B(a,b) = A(a,b)

$$M(A(a,b),B(a,b),kt)$$

$$\geq \phi[Min\begin{Bmatrix} M(Sa,Ta,t),M(A(a,b),Sa,t), \\ M(B(a,b),Ta,t) \end{Bmatrix}$$

$$= \phi[Min\ 1, M(A(a,b),B(a,b),t),1$$

$$> M(A(a,b),B(a,b),t)$$

And ,

$$N(A(a,b),B(a,b),kt)$$

$$\leq \psi\left[Max\begin{Bmatrix} N(Sa,Ta,t),N(A(a,b),Sa,t), \\ N(B(a,b),Ta,t) \end{Bmatrix}\right].$$

Thus , A(a,b) = B(a,b) = Sa = Ta . Similarly , we can show that A(b, a) = B(b,a) = Sb = Tb.

Let A(a,b) = B(a,b) = Sa = Ta = x and A(b, a) = B(b,a) = Sb = Tb = y. Since (A,S) and (B,T) are weakly compatible , so ,

$Sx = SA(a,b) = A(Sa,Sb) = A(x,y)$ and
$Sy = SA(b,a) = A(Sb,Sa) = A(y,x)$.

$Tx = TB(a,b) = B(Ta,Tb) = B(x,y)$ and
$Ty = TB(b,a) = B(Tb,Ta) = B(y,x)$.

Step 3: We next show that x = y . From (3.3) and (3.4) ,

$$M(x,y,kt) = M(A(a,b),B(a,b),kt)$$

$$\geq \phi\left[Min\begin{Bmatrix} M(Sa,Ta,t),M(A(a,b),Sa,t) \\ M(B(a,b),Ta,t) \end{Bmatrix}\right] \geq 1$$

And ,

$$N(x,y,kt) = N(A(a,b),B(a,b),kt)$$

$$\leq \psi\left[Max\begin{Bmatrix} N(Sa,Ta,t),N(A(a,b),Sa,t) \\ N(B(a,b),Ta,t) \end{Bmatrix}\right] \leq 0$$

Therefore , x = y .

Step 4: Now , we prove that Sx = Tx, again using (3.3) and (3.4)

$$M(Sx,Tx,kt) = M(A(x,y),B(x,y),kt)$$

$$\geq \phi\left[Min\begin{Bmatrix} M(Sx,Tx,t),M(A(x,y),Sx,t), \\ M(B(x,y),Tx,t) \end{Bmatrix}\right]$$

$$\geq M(Sx,Tx,t)$$

$$N(Sx,Tx,kt) = N(A(x,y),B(x,y),kt)$$

$$\leq \psi\left[Max\begin{Bmatrix} N(Sx,Tx,t),N(A(x,y),Sx,t), \\ N(B(x,y),Tx,t) \end{Bmatrix}\right]$$

$$\leq N(Sx,Tx,t),$$

Thus we get Sx = Tx .

Step 5: Lastly , we prove that Sx = x , From (3.3) and (3.4) , we have

$$M(Sx,x,kt) = M(A(x,y),B(x,y),kt)$$

$$\geq \phi\left[Min\begin{Bmatrix} M(Sx,Tx,t),M(A(x,y),Sx,t), \\ M(B(x,y),Tx,t) \end{Bmatrix}\right] \geq 1$$

$$N(Sx,x,kt) = N(A(x,y),B(x,y),kt)$$

$$\leq \psi\left[Max\begin{Bmatrix} N(Sx,Tx,t),N(A(x,y),Sx,t), \\ N(B(x,y),Tx,t) \end{Bmatrix}\right] \leq 0$$

Thus , we get Sx = x . Hence $x = Sx = Tx = A(x,x) = B(x,x)$. This shows that A, B, S, T have a common fixed point and uniqueness of x follows easily from (3.3) and (3.4).

Next we give an example in support of theorem 3.1

Example 3.1. Let X = R and d be the usual metric on X. Denote a * b = ab and a ◊ b = min{1, a + b} for all a, b in [0,1] and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$M_d(x, y, t) = \frac{t}{t+|x-y|}$, $N_d(x, y, t) = \frac{|x-y|}{t+|x-y|}$.

Then (X, M, N, *, ◊) is an intuitionistic fuzzy metric space. Define the mappings

$A, B : X \times X \to X$ and $S, T : X \to X$ as follows

$$A(x,y) = \begin{cases} x+y, x \in 0,2 \ , y \in X \\ 1, otherwise \end{cases}$$ and

$$B(x,y) = \begin{cases} x-y, x \in 0,2 \ , y \in X \\ 2, otherwise \end{cases}$$

$S(x) = x$ and $T(x) = 2x$, then the pairs (A,S) and (B,T) are weakly compatible and we see A(X × X) ⊆ T(X), B(X × X) ⊆ S(X), So, all the conditions of our theorem are satisfied . Thus A,B,S and T have a unique common coupled fixed point in X. Indeed, x = 0 is the unique common fixed point.

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