Properties P and Q for Suzuki-type fixed point theorems in metric spaces

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ABSTRACT

The aim of this paper is to present several results for maps defined on a metric space involving contractive conditions of Suzuki-type which satisfy properties P and Q. An interesting fact about this study is that none of these maps has any nontrivial periodic points.

Keywords
Property P; Property Q; Metric space; Suzuki contraction.

1. INTRODUCTION

The Banach contraction principle [15] states that every contraction T on a complete metric space has a unique fixed point. Recently, Suzuki [20] introduced a new type of mapping and presented a generalization of the Banach contraction principle as follows:

Theorem 1.1.[20] Define a non-increasing function θ from [0, 1) onto $\left(\frac{1}{2},1\right]$ by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}(\sqrt{5}-1) \\ \frac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5}-1) \le r \le \frac{1}{\sqrt{2}} \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into itself. Assume that there exists $r \in [0, 1)$ such that for all $x, y \in X$

$\theta(r)\, d(x, Tx) \le d(x, y)$ implies $d(Tx, Ty) \le r\, d(x, y)$.

Then there exists a unique $z \in X$ such that $z \in Tz$.

The elegant technique employed to prove Theorem 1.1 attracted several authors to work along these lines and subsequently Theorem 1.1 was generalized and extended in various ways (see for instance, [1], [3], [4], [7-14], [16-19], [21], [22] and others).

We will denote the set all fixed points of a self mapping T from X into itself by F(T), i.e., F(T)= {z∈X:Tz=z}. It is obvious that if z is a fixed point of T then it is also a fixed point of T^n for each $n \in N$, *i. e.,* $F(T) \subset F(T^n)$ if $F(T) \ne \phi$. However converse is false. Indeed the mapping T: R→R defined by Tx= $\frac{1}{2} - x$ has a unique fixed point, i.e., F (T) = $\left\{\frac{1}{4}\right\}$, but every x∈R is a fixed point for T^2. If $F(T) = F(T^n)$, for each $n \in N$, then we say that T has no periodic points.

In 2005, Jeong and Rhoades [5] examine a number of situations in which the fixed point sets for maps and their iterates are the same.

They state that a map T has ***property P*** if $F(T) = F(T^n)$ for each $n \in N$. Also a pair of maps S and T have ***property Q*** if $F(S) \cap F(T) = F(S^n) \cap F(T^n)$ for each $n \in N$.

Several works has been done related to Property P and Q (see for instance [2] and [6]).

Now we continue this study for mappings satisfying Suzuki type contractive conditions in metric space. In section I, we discuss property P for a map which involve Suzuki contractive conditions. In section II, we prove property Q for pairs of maps involving above contractive conditions. An important of this study is that if a map satisfies property P then every periodic point is a fixed point. The same situation is true for maps satisfying property Q.

2. PROPERTY P

Theorem 2.1. Define a nonincreasing function φ from [0, 1) into (0, 1] by

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \le r \le \frac{1}{2} \\ 1-r, & \text{if } \frac{1}{2} \le r \le 1. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into itself. Assume that there exists $r \in [0, 1)$ such that for all $x, y \in X$

$\varphi(r)\, d(x, Tx) \le d(x, y)$ implies

$$d(Tx,Ty) \le r \max\left\{ d(x,y),\, d(x,Tx),\, d(y,Ty),\, \frac{d(x,Ty)+d(y,Tx)}{2} \right\} .$$

Then T has property P.

Proof: From corollary 2.3 of [3], T has a fixed point. In other words, $F(T) \ne \emptyset$. Therefore $F(T^m) \ne \emptyset$ for each positive integer m. Let n be a fixed positive integer greater than 1 and suppose that $z \in F(T^n)$. We claim that $z \in F(T)$, that is, z is a fixed point of T.

Suppose that $z \ne Tz$. Then

$d(z, Tz) = d(T^n z, T(T^n z)) = d(T^n z, T^{n+1} z)$,

which is of the form $d(Tx,Ty)$, here

$y = T^n z,\ x = T^{n-1} z.$

Now $\varphi(r) d(x,Tx) = \varphi(r)\, d(T^{n-1}z, T(T^{n-1}z))$

$= \varphi(r) d(T^{n-1}z, T^n z) \le d(x, y)$,

that is, $\varphi(r)\, d(x, Tx) \le d(x, y)$ implies

$$d(Tx, Ty) \le r \max\left\{ d(x,y),\, d(x,Tx),\, d(y,Ty),\, \frac{d(x,Ty)+d(y,Tx)}{2} \right\}$$

that is, $d(T(T^{n-1}z), T(T^n z))$

$$\le r \max\left\{ d(T^{n-1}z,T^n z),\, d(T^{n-1}z,T^n z),\, d(T^n z,T^{n+1}z),\, \frac{d(T^{n-1}z,T^{n+1}z)+d(T^n z, T^n z)}{2} \right\},$$

$$= r \max\left\{ d(T^{n-1}z,T^n z),\, d(T^n z,T^{n+1}z),\, \frac{d(T^{n-1}z,T^{n+1}z)}{2} \right\},$$

thatis, $d(T^n z, T^{n+1}z) \le r \max$

$\le r \max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\}$.

Then $d(T^n z, T^{n+1}z) \le r\, d(T^{n-1}z, T^n z)$.

Continuing like this, we have
$d(T^n z, T^{n+1}z) \le r\, d(T^{n-1}z, T^n z) \le r^2 d(T^{n-2}z, T^{n-1}z) \le \ldots\ldots \le r^n d(z, Tz)$,
that is, $d(z, Tz) \le r^n d(z, Tz) < d(z, Tz)$,
that is, $d(z, Tz) < d(z, Tz)$,
which is a contradiction.
So our supposition that $z \ne Tz$ is wrong.Thus, $z = Tz$ and so $z \in F(T)$.
Therefore $F(T^n) \subseteq F(T)$.Also $F(T) \subseteq F(T^n)$.
Thus, $F(T) = F(T^n)$.Hence T satisfies property P.

Special cases of Theorem 2.1 are contractive conditions appearing in Theorem 3.3 of [19], Theorem 2.2 and Theorem 3.1 of [8], Theorem 2 of [20], Corollary 3.4 of [17] and Corollary 4.4 of [18].

Theorem 2.2. Let (X, d) be a compact metric space and let T be a mapping on X. Assume that

$\frac{1}{2} d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$

for$x, y \in X$. Then T has property P.
Proof: From Theorem 3 of [21], T has a unique fixed point.In other words, $F(T) \ne \emptyset$. Therefore $F(T^m) \ne \emptyset$ for each positive integer m.Let n be a fixed positive integer greater than 1 and suppose that $z \in F(T^n)$.We claim that $z \in F(T)$, that is, z is a fixed point of T.
Suppose that $z \ne Tz$. Then
$d(z, Tz) = d(T^n z, T(T^n z)) = d(T^n z, T^{n+1}z)$,
which is of the form $d(Tx, Ty)$, here $y = T^n z$, $x = T^{n-1}z$.

Now $\frac{1}{2} d(x, Tx) = \frac{1}{2} d(T^{n-1}z, T(T^{n-1}z)) = \frac{1}{2} d(T^{n-1}z, T^n z) < d(T^{n-1}z, T^n z)$,

that is, $\frac{1}{2} d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$,

that is, $d(T(T^{n-1}z), T(T^n z)) < d(T^{n-1}z, T^n z)$,

that is, $d(T^n z, T^{n+1}z) < d(T^{n-1}z, T^n z)$.

Continuing like this, we have
$d(T^n z, T^{n+1}z) < d(z, Tz)$,
That is, $d(z, Tz) < d(z, Tz)$, which is a contradiction.
So our supposition that $z \ne Tz$ is wrong.Thus, $z = Tz$ and so $z \in F(T)$.
Therefore $F(T^n) \subseteq F(T)$.Also $F(T) \subseteq F(T^n)$.
Thus, $F(T) = F(T^n)$.Hence T satisfies property P.

Theorem 2.3: Define a function η from [0,1) into (1/2, 1] by

$$\eta(r) = \begin{cases} 1 & if \quad 0 \le r < 1/2 \\ (1+r)^{-1} & if \quad 1/2 \le r < 1 \end{cases}.$$

Let (X, d) be a complete metric space and let T be a mappings form X into CB(X). Assume that there exists $r \in [0, 1)$ such that for all $x, y \in X$
$\eta(r)\, d(x, Tx) \le d(x, y)$ implies $\delta(Tx, Ty) \le r d(x, y)$. (2.3.1)
Then T has property P.

Proof: From theorem 4 of [10], T has a unique fixed point z and $Tz = \{z\}$. Therefore, $F(T^n) \ne \phi$ for each positive integer n. Let n be a fixed positive integer greater than 1 and suppose that $u \in F(T^n)$. We claim that $u \in F(T)$, that is, u is a fixed point of T.
Let $u \in F(T^n)$Then for any positive integer i, j satisfying $0 \le i, j \le n$, we obtain
$\eta(r)\ d(T^{i-1}u, T(T^{j-1}u)) = \eta(r) d(T^{i-1}u, T^j u) \le d(T^{i-1}u, T^j u)$.
Then contractive condition (2.3.1) implies that
$\delta(T(T^{i-1}u), T(T^j u)) \le r d(T^{i-1}u, T^j u)$. (2.3.2)
Define $\delta = \max_{0 \le i, j \le n} \delta(T^i u, T^j u)$.

Since, if $j = n$, then $T^{j+1}u = Tu$.
Assuming $\delta > 0$, it then follows from (2.3.2) that $\delta \le r\,\delta$, which is a contradiction.
Therefore $\delta = 0$. Thus $\delta(Tu, u) = 0$ implies $\{u\} = Tu$. Hence $u \in F(T)$.
Hence T satisfies property P.

3. PROPERTY Q

Theorem 3.1. Define a strictly decreasing function η from [0, 1) onto $\left(\frac{1}{2}, 1\right]$ by $\eta(r) = \frac{1}{1+r}$.

Let (X, d) be a complete metric space and let T and S be mappings from X into itself. Assume that there exists $r \in [0, 1)$ such that for all $x, y \in X$
$\eta(r) \min\{d(x, Tx), d(y, Sy)\} \le d(x, y)$ implies $d(Tx, Sy) \le r M(x, y)$ (3.1.1)

where$M(x, y) = \max\left\{ d(x,y), \frac{d(x,Tx)+d(y,Sy)}{2}, \frac{d(x,Sy)+d(y,Tx)}{2} \right\}$

Then S and T have property Q.
Proof: From corollary 2.3 of [12], S and T have a unique common fixed point. In other words, $F(S) \cap F(T) \ne \emptyset$.
Therefore, $F(S^m) \cap F(T^m) \ne \emptyset$ for each positive integer m.
Let n be a fixed positive integer greater then 1and suppose that $z \in F(S^n) \cap F(T^n)$.
We claim that $z \in F(S) \cap F(T)$. To prove this,it is sufficient to show that z is a fixed point of T.
Suppose that $z \ne Tz$. Then
$d(z, Tz) = d(S^n z, T(T^n z)) = d(T(T^n z), S(S^{n-1}z))$,
which is of the form $d(Tx, Sy)$, here $x = T^n z$, $y = S^{n-1}z$.
Now $\eta(r)\, d(x, Tx) =$
$\eta(r)\ d(T^n z, T(T^n z)) = \eta(r)\, d(T^n z, T^{n+1}z)$
$\le d(T^n z, T^{n+1}z)$. (3.1.2)
Case-I If$d(x, Tx) \le d(y, Sy)$.
Then $d(T^n z, T^{n+1}z) \le d(S^{n-1}z, S^n z) = d(S^{n-1}z, z)$
$= d(S^{n-1}z, T^n z) = d(T^n z, S^{n-1}z)$. (3.1.3)
Combining (3.1.2) and (3.1.3) we have
$\eta(r)\, d(x, Tx) \le d(x, y)$.
Then by contractive condition (3.1.1), we have

$$d(Tx, Sy) \le r \max\left\{ d(x,y), \frac{d(x,Tx)+d(y,Sy)}{2}, \frac{d(x,Sy)+d(y,Tx)}{2} \right\}.$$

This implies that $d(T(T^n z), S(S^{n-1}z)) \le$

$$r\max\left\{ d(T^n z, S^{n-1}z),\ \frac{d(T^n z, T^{n+1}z) + d(S^{n-1}z, S^n z)}{2},\ \frac{d(T^n z, S^n z) + d(S^{n-1}z, T^{n+1}z)}{2} \right\},$$

That is, $d(T^{n+1}z, S^n z) \le$

$$r\max\left\{ d(T^n z, S^{n-1}z),\ \frac{d(T^n z, T^{n+1}z) + d(S^{n-1}z, S^n z)}{2},\ \frac{d(T^n z, S^n z) + d(S^{n-1}z, T^{n+1}z)}{2} \right\}.$$

Case-II If $d(y, Sy) \le d(x, Tx)$,

i.e.$d(S^{n-1}z, S(S^{n-1}z) \le d(T^n z, T(T^n z))$,

i.e. $d(S^{n-1}z, S^n z) \le d(T^n z, T^{n+1}z)$,

Now $\eta(r)\ d(y, Sy) = \eta(r)\ d(S^{n-1}z, S^n z) \le d(S^{n-1}z, S^n z) = d(S^{n-1}z, z)$

$= d(S^{n-1}z, T^n z) = d(x, y)$.

As $d(y, Sy) \le d(x, Tx)$ and $\eta(r)\ d(y, Sy) \le d(x, y)$.

So by contractive condition (3.1.1), we have

$$d(Tx, Sy) \le r \max r\max\left\{ d(x,y),\ \frac{d(x,Tx) + d(y,Sy)}{2},\ \frac{d(x,Sy) + d(y,Tx)}{2} \right\}.$$

This implies that $d(T(T^n z), S(S^{n-1}z)) \le$

$$r\max\left\{ d(T^n z, S^{n-1}z),\ \frac{d(T^n z, T^{n+1}z) + d(S^{n-1}z, S^n z)}{2},\ \frac{d(T^n z, S^n z) + d(S^{n-1}z, T^{n+1}z)}{2} \right\},$$

that is, $d(T^{n+1}z, S^n z) \le$

$$r\max\left\{ d(T^n z, S^{n-1}z),\ \frac{d(T^n z, T^{n+1}z) + d(S^{n-1}z, S^n z)}{2},\ \frac{d(T^n z, S^n z) + d(S^{n-1}z, T^{n+1}z)}{2} \right\}.$$

Thus from bothcase-I and case-II, we obtain

$d(T^{n+1}z, S^n z) \le$

$$r\max\left\{ d(T^n z, S^{n-1}z),\ \frac{d(T^n z, T^{n+1}z) + d(S^{n-1}z, S^n z)}{2},\ \frac{d(T^n z, S^n z) + d(S^{n-1}z, T^{n+1}z)}{2} \right\},$$

that is, $d(z, Tz) \le$

$$r\max\left\{ d(z, S^{n-1}z),\ \frac{d(z,Tz) + d(S^{n-1}z, z)}{2},\ \frac{d(z,z) + d(S^{n-1}z, Tz)}{2} \right\},$$

that is, $d(z, Tz) \le r$ max

$$r\max\left\{ d(z, S^{n-1}z),\ \frac{d(z,Tz) + d(S^{n-1}z, z)}{2},\ \frac{d(S^{n-1}z, Tz)}{2} \right\}. \tag{3.1.4}$$

Since, $\dfrac{d(S^{n-1}z, Tz)}{2} \le \dfrac{d(z,Tz) + d(S^{n-1}z, z)}{2}$.

So (3.1.4) takes the form

$$d(z, Tz) \le r\max\left\{ d(z, S^{n-1}z),\ \frac{d(z,Tz) + d(S^{n-1}z, z)}{2} \right\}.$$

Case-(a) If $d(z, Tz) \le r\ d(z, S^{n-1}z)$.

Case-(b) If $d(z, Tz) \le r\,\dfrac{d(z,Tz) + d(S^{n-1}z, z)}{2}$.

that is, $2d(z, Tz) \le rd(z, Tz) + r\, d(S^{n-1}z, z)$,

$(2-r)\, d(z, Tz) \le r\, d(S^{n-1}z, z)$,

$d(z, Tz) \le \left(\dfrac{r}{2-r}\right) d(S^{n-1}z, z)$,

that is, $d(z, Tz) \le r_1\ d(S^{n-1}z, z)$, where $r_1 < 1$.

Thus from case-(a) and case-(b), we have

$d(z, Tz) \le \beta\, d(S^{n-1}z, z)$, where $\beta < 1$,

that is, $d(Tz, z) \le \beta\, d(z, S^{n-1}z)$,

that is, $d(T(T^n z), S^n z) \le \beta\ d(T^n z, S^{n-1}z)$.

Thus we get $d(T^{n+1}z, S^n z) \le \beta\ d(T^n z, S^{n-1}z)$

Continuing like this, we have

$d(T^{n+1}z, S^n z) \le \beta^n\, d(Tz, z) < d(Tz, z)$,

that is, $d(Tz, z) < d(Tz, z)$, which is a contradiction.

So, oursupposition that $z \ne Tz$ is wrong.Thus $z = Tz$.

Analogously $z = Sz$.Therefore $z \in F(T) \cap F(S)$.

So $F(S^n) \cap F(T^n) \subseteq F(T) \cap F(S)$.

Also $F(T) \cap F(S) \subseteq F(S^n) \cap F(T^n)$.

Thus, $F(T) \cap F(S) = F(S^n) \cap F(T^n)$.

Hence S and T satisfy property Q.

Theorem 3.2. Define a non-increasing function θ as in Theorem 1.1 and let X be a complete metric space, f, T: X → X satisfying the following:

(i) f is continuous.

(ii) $T(X) \subset f(X)$.

(iii) f and T commute.

Assume there exists $r \in [0, 1)$ such that for each $x, y \in X$,

$\theta(r)\, d(fx, Tx) \le d(fx, fy)$ implies

$$d(Tx, Ty) \le r\max\left\{ d(fx, fy),\ d(fx, Tx),\ d(fy, Ty),\ \frac{d(fx, Ty) + d(fy, Tx)}{2} \right\}. \tag{3.2.1}$$

Then f and T have property Q.

Proof: From theorem 2.1 of [11], f and T have a unique common fixed point.

In other words, $F(f) \cap F(T) \ne \emptyset$. Therefore $F(f^n) \cap F(T^n) \ne \emptyset$ for each positive integer n. Let n be a fixed positive integer greater than 1 and suppose that $u \in F(f^n) \cap F(T^n)$. We claim that

$u \in F(f) \cap F(T)$.

Since $u \in F(f^n) \cap F(T^n)$. Then for any positive integer i, r satisfying $0 \le i, r \le n$, we obtain

$\theta(r)\, d(f(T^{i-1}f^{r-1}u), T(T^{i-1}f^{r-1}u))$

$\le d(T^{i-1}f^r u, T^i f^{r-1}u)$

$= d(f(T^{i-1}f^{r-1}u),\ f(T^i f^{r-2}u))$.

Then contractive condition (3.2.1) implies that

$d(T^i f^{r-1}u, T^{i+1}f^{r-2}u) \le$

$$r\max\left\{ \begin{array}{l} d(T^{i-1}f^r u, T^i f^{r-1}u), d(T^{i-1}f^r u, T^i f^{r-1}u), \\ d(T^i f^{r-1}u, T^{i+1}f^{r-2}u), \\ \dfrac{d(T^{i-1}f^r u, T^{i+1}f^{r-2}u) + d(T^i f^{r-1}u, T^i f^{r-1}u)}{2} \end{array} \right\} \tag{3.2.2}$$

Define $\delta = \max_{0 \le i,r,l,t \le n} d(T^i f^r u, T^l f^t u)$

Since, if $i = n$, then $T^{i+1}u = Tu$.

Assuming $\delta > 0$, it then follows from (3.2.2) that
$\delta \le r\max\{\delta, \delta, \delta, \delta\}$,
that is, $\delta < \delta$ which is a contradiction.Therefore $\delta = 0$.
Thus $d(fu, u) = d(Tu, u) = 0$ implies $u = fu = Tu$.
Hence $u \in F(f) \cap F(T)$.
So $F(f^n) \cap F(T^n) \subseteq F(f) \cap F(T)$.
Also $F(f) \cap F(T) \subseteq F(f^n) \cap F(T^n)$.
Thus,$F(f) \cap F(T) = F(f^n) \cap F(T^n)$.
Hence f and T satisfy property Q.

Special case of Theorem 3.2 is contractive condition appearing in Theorem 3 of [7].

Theorem 3.3. Let (X, d) be a complete metric space. Let f and T be mappings on X satisfying
(i)-(iii) in Theorem 3.2. Assume that

$\frac{1}{2} d(fx, Tx) < d(fx, fy)$ implies

$$d(Tx, Ty < \max\left\{ d(fx, fy), \frac{d(fx, Tx) + d(fy, Ty)}{2} \right\} \quad (3.3.1)$$

for all $x, y \in X$, and that for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x, y \in X$

$\frac{1}{2} d(fx, Tx) < d(fx, fy)$ and max

$\max\left\{ d(fx, fy), \frac{d(fx, Tx) + d(fy, Ty)}{2} \right\} < \epsilon + \delta(\epsilon)$

implies $d(Tx, Ty) \le \epsilon$.
Then f and T have property Q.
Proof: From theorem 3.1 of [11], f and T have a unique common fixed point.
In other words, $F(f) \cap F(T) \neq \emptyset$. Therefore $F(f^n) \cap F(T^n) \neq \emptyset$ for each positive integer n. Let n be a fixed positive integer greater than 1 and suppose that $u \in F(f^n) \cap F(T^n)$. We claim that $u \in F(f) \cap F(T)$.
Since $u \in F(f^n) \cap F(T^n)$. Then for any positive integer i, r satisfying $0 \le i, r \le n$, we obtain

$$\frac{1}{2} d(f(T^{i-1} f^{r-1} u), T(T^{i-1} f^{r-1} u)) < d(T^{i-1} f^r u, T^i f^{r-1} u)$$
$$= d(f(T^{i-1} f^{r-1} u), f(T^i f^{r-2} u)).$$

Then contractive condition $(3.3.1)$ implies that

$$d(T^i f^{r-1} u, T^{i+1} f^{r-2} u) < \max\left\{ \begin{array}{l} d(T^{i-1} f^r u, T^i f^{r-1} u), \\ \frac{d(T^{i-1} f^r u, T^i f^{r-1} u) + d(T^i f^{r-1} u, T^{i+1} f^{r-2} u)}{2} \end{array} \right\} \quad (3.3.2)$$

Define $\delta = \max_{0 \le i,r,l,t \le n} d(T^i f^r u, T^l f^t u)$

Since, if $i = n$, then $T^{i+1}u = Tu$.

Assuming $\delta > 0$, it then follows from (3.3.2) that
$\delta < \max\{\delta, \delta\}$,
that is, $\delta < \delta$ which is a contradiction. Therefore $\delta = 0$.
Thus $d(fu, u) = d(Tu, u) = 0$ implies $u = fu = Tu$.
Hence $u \in F(f) \cap F(T)$.

So $F(f^n) \cap F(T^n) \subseteq F(f) \cap F(T)$.
Also $F(f) \cap F(T) \subseteq F(f^n) \cap F(T^n)$.
Thus, $F(f) \cap F(T) = F(f^n) \cap F(T^n)$.Hence f and T satisfy property Q.

Special case of Theorem 3.3 is Meir-Keeler contractive condition appearing in Theorem 4 of [7].

4. CONCLUSION

In this paper, we have studied a number of Suzuki-type contractive conditions defined on a metric space for which fixed point sets for maps and their iterates are same. An important fact about this study is that if maps satisfy property P or Q then every periodic point is a fixed point.

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