Dominator Chromatic Number of Middle and Total Graphs

K. Kavitha
Research Scholar
Department of Mathematics, Madras Christian College, Chennai – 600 059

N.G. David
Associate Professor
Department of Mathematics, Madras Christian College, Chennai – 600 059

ABSTRACT
Dominator chromatic number of middle and total graphs of various graph families is found in this paper. Also these parameters are compared with dominator chromatic number of their respective graph families.

Key words: Middle graph, total graph and dominator coloring.
AMS Subject Classification: 05C15, 05C69

1. PRELIMINARIES
The notion of middle graph, total graph and dominator coloring are reviewed in this section [1, 2, 3, 4].

Definition 1.1
The *middle graph* M(G) of a graph G is defined as follows. The vertex set of M(G) is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of M(G) are adjacent in M(G) if either (i) x, y are in E(G) and x, y are adjacent in G or (ii) x is in V(G), y is in E(G) and x, y are incident in G. In other words, M(G) is obtained by subdividing each edge of G exactly once and joining all these newly added middle vertices of adjacent edges of G.

Definition 1.2
The *total graph* T(G) of a graph G is defined as a graph with vertex set $V(G) \cup E(G)$ and two vertices x, y of T(G) are adjacent in T(G) if either (i) x, y are in V(G) and x is adjacent to y in G or (ii) x, y are in E(G) and x, y are adjacent in G or (iii) x is in V(G), y is in E(G) and x, y are incident in G.

Definition 1.3
A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$, is the minimum number of colors required for a proper coloring of G. A *Color class* is the set of all vertices, having the same color. The color class corresponding to the color i is denoted by V_i.

Definition 1.4
A *dominator coloring* of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The convention is that if {v} is a color class, then v dominates the color class {v}. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G.

2.DOMINATOR CHROMATIC NUMBER OF MIDDLE GRAPHS
Dominator chromatic number of middle graphs of various classes of graphs is obtained in this section.

Theorem 2.1
For cycle graph C_n of order $n \geq 3$, $\chi_d[M(C_n)] = \lceil n/2 \rceil + 2$.

Proof

Let C_n be the cycle graph of order $n \geq 3$ and let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$, where $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_n v_1$. By the definition of middle graph, $M(C_n)$ is obtained by subdividing each edge $v_i v_{i+1}$, $1 \leq i \leq n-1$ of C_n exactly once by the vertex c_i, and subdividing $v_n v_1$ by c_n in $M(C_n)$ and joining (c_i and c_{i+1}), $1 \leq i \leq n-1$ and (c_n and c_1). Let $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{c_1, c_2, \ldots, c_n\}$. Then $V(M(C_n)) = V_1 \cup V_2$.

The following procedure gives a dominator coloring of $M(C_n)$. Color the vertices v_i, $1 \leq i \leq n$ by color 1. When n is odd, $c_2, c_4, c_6, \ldots, c_{n-3}$ and c_n are colored by color 2 and $c_1, c_3, c_5, \ldots, c_{n-2}$ and c_{n-1} by colors 3, 4, 5, …, $\lceil n/2 \rceil + 2$. When n is even, $c_2, c_4, c_6, \ldots, c_n$ are colored by color 2 and $c_1, c_3, c_5, \ldots, c_{n-1}$ by colors 3, 4, 5, …, $\lceil n/2 \rceil + 2$.

The vertices v_i and v_{i+1} dominate the color class of c_i, i = 1, 3, 5, …. The center vertices $c_1, c_3, c_5, \ldots$ dominate themselves and c_2 and c_n dominate the color class 3, as they are adjacent to c_1. The vertex c_{2i}, i = 2, 3, …, $\lfloor n/2 \rfloor$ dominates the color class of c_{2i+1}. Hence $\chi_d[M(C_n)] = \lceil n/2 \rceil + 2$.

The following example illustrates the procedure discussed in the above result.

Example 2.2

In figure 1, middle graph of C_6 is depicted with a dominator coloring.

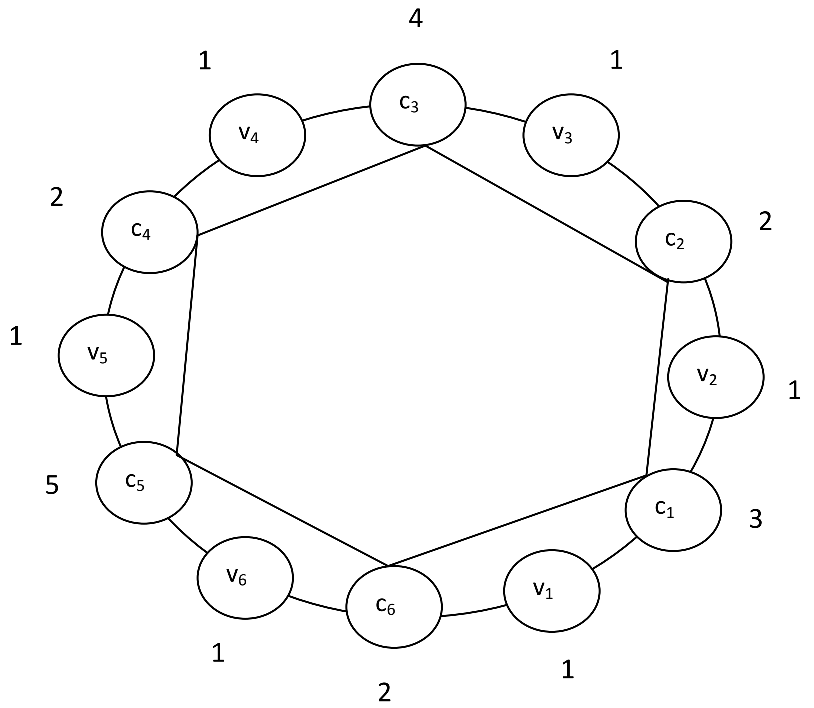

Figure 1

The color classes of $M(C_6)$ are $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V_2 = \{c_2, c_4, c_6\}$, $V_3 = \{c_1\}$, $V_4 = \{c_3\}$and $V_5= \{c_5\}$. The dominator chromatic number is, $\chi_d[M(C_6)] = 5$.

Theorem 2.3

For path graph P_n of order $n \geq 2$,

$$\chi_d[M(P_n)] = \begin{cases} n & \text{when } n = 3, 4 \\ 2 + \lceil n/2 \rceil & \text{otherwise.} \end{cases}$$

Proof

Let P_n be a path of order $n \geq 2$ with vertex labels $v_1, v_2, ..., v_n$. By the definition of middle graph , $M(P_n)$ has the vertex set $V(P_n) \cup E(P_n) = \{v_i \,/\, 1 \leq i \leq n\} \cup \{e_i \,/\, 1 \leq i \leq n\text{-}1\}$ in which each e_i is adjacent to v_i and $v_{i+1,}$ $1 \leq i \leq n\text{-}1$ and also adjacent to e_{i+1}, for $1 \leq i \leq n\text{-}2$.

A dominator coloring of $M(P_n)$ is obtained by coloring the vertices v_i, $1 \leq i \leq n$ by color 1, even subscripted middle vertices c_2, c_4, …, by color 2 and the remaining odd subscripted middle vertices c_1, c_3, c_5, … are colored respectively by colors 3, 4, 5, …, $\lceil n/2 \rceil + 2$.

The odd subscripted middle vertices c_1, c_3, c_5, … dominate themselves and the vertices v_i and v_{i+1} dominate the color class of c_i, i = 1, 3, 5, …. The vertex c_i dominates the color class of c_{i+1}, i = 2, 4, …. Hence

$$\chi_d[M(P_n)] = \begin{cases} n & \text{when } n = 3, 4 \\ 2 + \lceil n/2 \rceil & \text{otherwise.} \end{cases}$$

Theorem 2.4

For wheel graph $W_{1,\, n}$ of order $n \geq 3$,

$$\chi_d[M(W_{1,n})] = n + 2.$$

Proof

Let $W_{1,\, n}$ be a wheel graph of order $n \geq 3$. Let the vertex at the centre be v_1 and the vertices on the rim be v_2, v_3, …, v_{n+1}. By the definition of middle graph, we subdivide each edge exactly once and join all the middle vertices of adjacent edges of $W_{1,\, n}$. Let the middle vertices on the edges v_1v_i, i = 2,…, n+1 of $W_{1,\, n}$ be c_{i-1} and the middle vertices on $v_iv_{i+1,}$ i = 2, …, n of $W_{1,n}$ be c_{n+i-1} and the middle vertex on $v_{n+1}v_2$ be $c_{2n.}$

A dominator coloring of $M(W_{1,\, n})$ is obtained by the following procedure. Color the vertex v_1 by color 1 and assign colors 2, 3, …, n+1 for the induced clique c_1, c_2, …, c_n. Color the vertices v_2, v_3 and c_{n+1} by colors 3, 2 and (n+2) respectively. The remaining 2n-3 vertices on the rim of $M(W_{1,\, n})$ are colored by using colors 1, 2 and 3 in a particular fashion.

The vertices c_3, c_4, …, c_n and c_{n+1} dominate themselves. The vertices c_1, c_2, v_2 and v_3 dominate the color class (n+2), as they are adjacent to the vertex c_{n+1}. The remaining 2n-3 vertices v_4, c_{n+3}, v_5, c_{n+4}, …, c_{2n} and v_n on the rim of $M(W_{1,\, n})$ dominate respective color classes from 4 to (n+1). The vertex v_1 dominates all color classes between 4 and (n+1). Hence $\chi_d[M(W_{1,n})] = n + 2$.

Example 2.5

In figure 2, middle graph of $W_{1,5}$ is depicted with a dominator coloring.

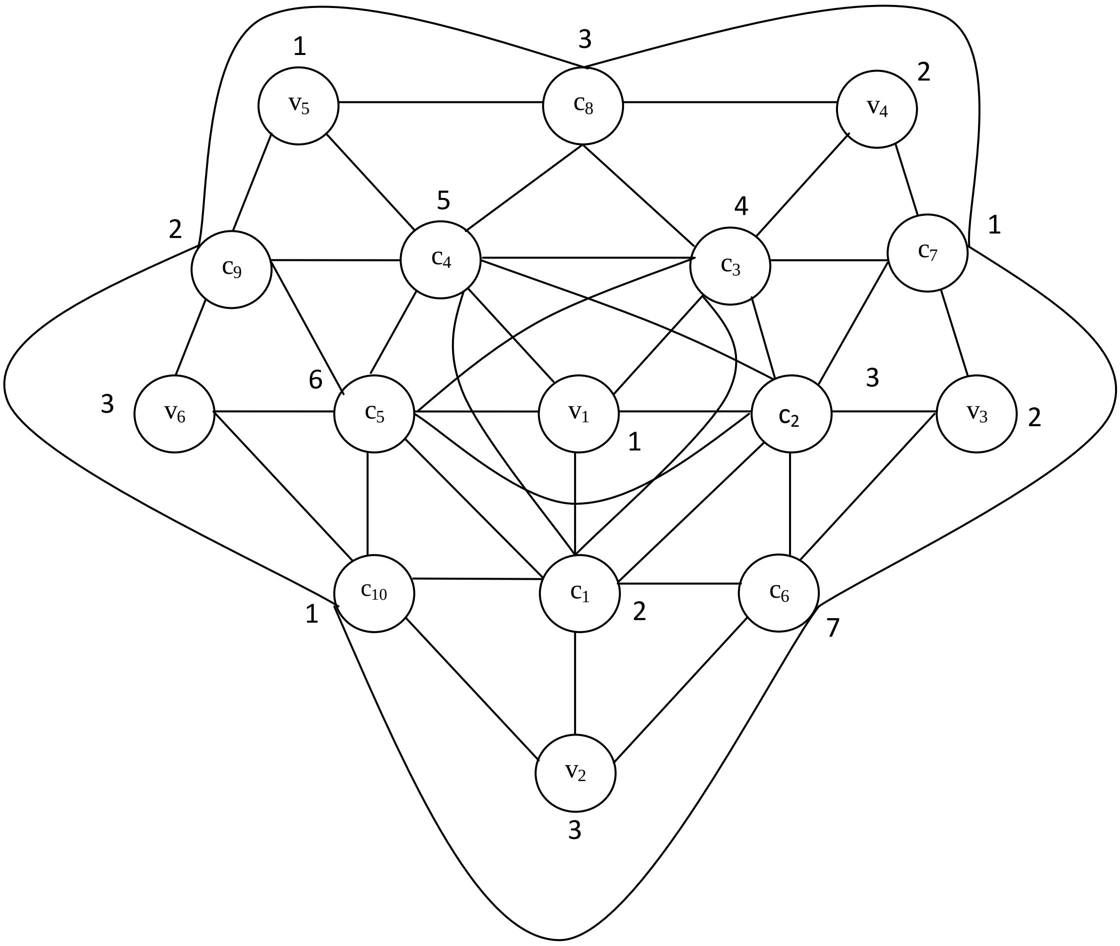

Figure 2

The color classes of $M(W_{1,5})$ are $V_1 = \{v_1, c_7, v_5, c_{10}\}$, $V_2 = \{c_1, v_3, v_4, c_9\}$, $V_3 = \{c_2, v_2, c_8, v_6\}$, $V_4 = \{c_3\}$, $V_5 = \{c_4\}$, $V_6 = \{c_5\}$ and $V_7 = \{c_6\}$. The dominator chromatic number is, $\chi_d[M(W_{1,5})] = 7$.

3. DOMINATOR CHROMATIC NUMBER OF TOTAL GRAPHS

In this section, dominator chromatic number of total graphs of various classes of graphs is obtained

Theorem 3.1

For cycle graph C_n of order $n \geq 3$,

$$\chi_d[T(C_n)] = \begin{cases} 2 + \lceil 2n/5 \rceil & \text{when } n = 3 \\ 3 + \lceil 2n/5 \rceil & \text{otherwise.} \end{cases}$$

Proof

Let C_n be the cycle graph of order $n \geq 3$ and let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$, where $e_i = v_iv_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_nv_1$. By the definition of total graph, $T(C_n)$ can be obtained by subdividing each edge v_iv_{i+1}, $1 \leq i \leq n-1$ of C_n exactly once in $T(C_n)$ and joining all these middle vertices of adjacent edges of C_n and also joining the adjacent vertices of C_n. Let the vertices of $T(C_n)$ be labeled by $u_1 = v_1, u_2, u_3, \ldots, u_{2n}$.

A dominator coloring of $T(C_n)$, $n \geq 4$ is obtained by the following procedure. The vertex u_{1+5j} is colored by $4+j$, $0 \leq j \leq \lfloor (2n-1)/5 \rfloor$. The vertices u_{2+5j}, u_{3+5j}, u_{4+5j}, u_{5+5j}, which lie between u_{1+5j}, $u_{1+5(j+1)}$, $0 \leq j \leq \lfloor (2n-1)/5 \rfloor$, are colored by (1, 2, 3, 1) or (2, 3, 1, 2) depending on whether j is even or odd.

Vertex u_{1+5j} dominates itself and u_{1+5j-2}, u_{1+5j-1}, u_{1+5j+1}, u_{1+5j+2} dominate the color class (4+j), as they are adjacent to u_{1+5j}, $0 < j \le \lfloor (2n-1)/5 \rfloor$ and when j = 0, u_2, u_3, u_{2n}, u_{2n-1} dominate the color class 4. When n = 3, it is easy to see that $\chi_d[T(C_3)] = 2 + \lceil 2n/5 \rceil$. Hence

$$\chi_d[T(C_n)] = \begin{cases} 2 + \lceil 2n/5 \rceil & \text{when } n = 3 \\ 3 + \lceil 2n/5 \rceil & \text{otherwise.} \end{cases}$$

Theorem 3.2

For path graph P_n of order n ≥ 3,

$$\chi_d[T(P_n)] = \begin{cases} \lceil 2n/3 \rceil + 2 & \text{when } n = 3 \\ \lceil 2n/3 \rceil + 1 & \text{otherwise.} \end{cases}$$

Proof

Let P_n be a path of order n ≥ 3 with vertex labels $v_1, v_2, ..., v_n$. By the definition of total graph, $T(P_n)$ can be obtained by subdividing each edge of P_n exactly once in $T(P_n)$ and joining all these middle vertices of adjacent edges of P_n and also joining the adjacent vertices of P_n. The vertex set of $T(P_n)$ is $V(P_n) \cup E(P_n) = \{v_i / 1 \le i \le n\} \cup \{e_i / 1 \le i \le n-1\}$. Relabel the vertices of $T(P_n)$ by $u_1 = v_1$, $u_2 = c_1$, $u_3 = v_2$, …, u_{2n-1} consecutively.

A dominator coloring of $T(P_n)$, n ≥ 4 is obtained by coloring the vertex u_{1+3i} by color 1, u_{2+3i} by color 2 and u_{3+3i} by color (3+i), $0 \le i \le 1 + \lceil 2n/3 \rceil$. Vertex u_{3+3i} dominates itself and u_{3+3i-2}, u_{3+3i-1}, u_{3+3i+1}, u_{3+3i+2} dominate the color class (3+i), as they are adjacent to u_{3+3i}, $0 \le i \le 1 + \lceil 2n/3 \rceil$. When n = 3, it is easy to see that $\chi_d[T(P_3)] = \lceil 2n/3 \rceil + 2$. Hence

$$\chi_d[T(P_n)] = \begin{cases} \lceil 2n/3 \rceil + 2 & \text{when } n = 3 \\ \lceil 2n/3 \rceil + 1 & \text{otherwise.} \end{cases}$$

The following example illustrates the procedure discussed in the above result.

Example 3.3

In figure 3, total graph of P_5 is depicted with a dominator coloring.

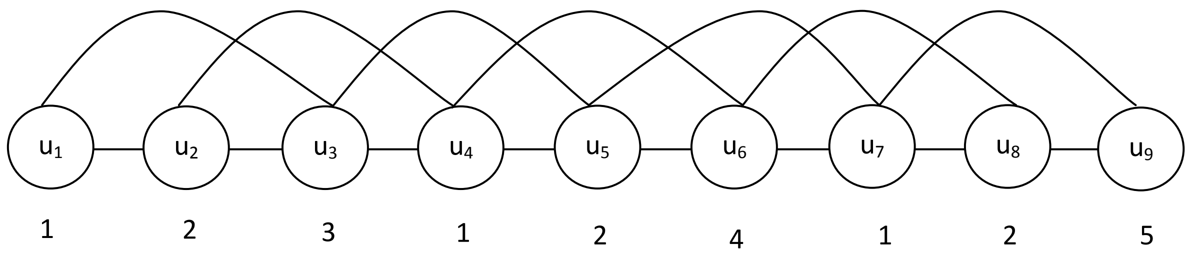

Figure 3

The color classes of $T(P_5)$ are $V_1 = \{u_1, u_4, u_7\}$, $V_2 = \{u_2, u_5, u_8\}$, $V_3 = \{u_3\}$, $V_4 = \{u_6\}$ and $V_5 = \{u_9\}$. The dominator chromatic number is, $\chi_d[T(P_5)] = 5$.

Theorem 3.4

For wheel graph $W_{1,n}$ of order n ≥ 3,

$$\chi_d[T(W_{1,n})] = n + 2.$$

Proof

Let $W_{1,n}$ be a wheel graph of order $n \ge 3$. Let the vertices of the wheel graph be labeled as follows. The vertex at the centre is labeled by v_1 and the vertices on the rim be labeled consecutively by $v_2, v_3, \ldots, v_{n+1}$. By the definition of total graph, we subdivide each edge exactly once and join all the middle vertices of adjacent edges and join all the adjacent vertices of $W_{1,n}$. Let the middle vertices on the edges v_1v_i, i = 2,…, n+1 of $W_{1,n}$ be c_{i-1} and the middle vertices on v_iv_{i+1}, i = 2, …, n of $W_{1,n}$ be c_{n+i-1} and the middle vertex on $v_{n+1}v_2$ is c_{2n}.

The coloring of $T(W_{1,n})$ can be done in a similar way as described in theorem 2.4. Domination of vertices can also be argued in the same manner. Hence $\chi_d[T(W_{1,n})] = n + 2$.

By combining the observations of [2] and the theorems proved above, we have the following result.

Result 3.5

(i) $\chi_d[M(C_n)] > \chi_d(C_n)$ and $\chi_d[T(C_n)] > \chi_d(C_n)$

(ii) $\chi_d[M(P_n)] > \chi_d(P_n)$ and $\chi_d[T(P_n)] > \chi_d(P_n)$

(iii) $\chi_d[M(W_{1,n})] > \chi_d(W_{1,n})$ and $\chi_d[T(W_{1,n})] > \chi_d(W_{1,n})$.

4. CONCLUSION
In this paper, we obtained the dominator chromatic number of middle and total graphs and compared these parameters with dominator chromatic number of the corresponding graph families. This paper can further be extended by identifying graph families of graphs for which this chromatic number is equal to other kinds of chromatic number.

5. ACKNOWLEDGMENTS
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6. REFERENCES
[1] J. A. Bondy and U.S.R. Murty, Graph theory with Applications, London: MacMillan (1976).

[2] R.M. Gera, On Dominator Colorings in Graphs, Graph Theory Notes of New York LIT, 25-30 (2007)

[3] F. Harary, Graph Theory, Narosa Publishing 1969.

[4] D. Michalak, On Middle and Total graphs with Coarseness Number Equal 1, Lecture Notes in Mathematics, Volume 1018: Graph Theory, Springer-Verlag, Berlin, 139 – 150 (1983).