A New Class of Locally Closed Sets and Locally Closed Continuous Functions in Generalized Topological Spaces

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ABSTRACT
In this paper the idea of μ-locally closed sets in generalized topological space is introduced and study some of their properties. We introduce the notion of (μ,μ')-locally closed continuous functions on generalized topological space and investigate some of their characterizations.

Keywords

μ-LC sets, (μ,μ')–LC continuity, (μ,μ')–LC irresoluteness.

1. INTRODUCTION
Kuratowski and Sierpinski [5] considered the difference of two closed subsets of an n-dimensional Euclidean space. Implicit in their work is the notion of a locally closed subset of a topological space (X,τ). Following Bourbaki [3] we say that a subset of (X, τ) is locally closed in (X, τ) if it is the intersection of an open and closed subset of (X, τ). Stone [8] has used the term FG for a locally closed subset as the spaces that in every embedding are locally closed. The results of Borges [2] show that locally closed sets play an important role in the context of simple extensions. Blumberg [1] introduced the concept of a real valued function on Euclidean space being densely approached at a point of its domain. This notion was generalized in 1958 to general topological spaces by Ptak [7] who used the term nearly continuous function. The concepts of nearly continuous and nearly open functions are important in functional analysis especially in the context of open mapping and closed graph theorems.
Csaszar [4] introduced the concepts of generalized neighborhood systems and generalized topological spaces. And also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, it has been investigated the characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems.
In this paper we introduce the notion of μ-locally closed sets which are denoted by μ-LC sets and study some of the fundamental properties of μ-LC sets in generalized topological spaces. Also we introduce the concept of (μ,μ')-locally closed continuous functions on generalized topological spaces and investigate the results of these functions.

2. PRELIMINARIES

We recall some basic definitions and notations.A generalized topology or simply GT μ [4] on a nonempty set X is a collection of subsets of X such that ϕ ϵ μ and μ is closed under arbitrary union. The elements of μ are called μ-open sets and the complements of μ–open sets are called μ-closed sets. The pair (X,μ) is called a generalized topological space (GTS). If A is a subset of a space μ, then $c_\mu(A)$ is the smallest μ-closed set containing A and $i_\mu(A)$ is the largest μ-open set contained in A. Let (X,μ) and (Y,μ') be generalized topological spaces. A map f: (X,μ) → (Y,μ') is said to be (μ,μ')-continuous if and only if Mϵ μ' implies $f^{-1}(M)$ ϵ μ.

For our analysis, we require the following basic definitions.

2.1 Definition A subset A of a topological space (X,τ) is called locally closed [3], if A = U∩ F, where U ϵ τ and F is closed in (X, τ).

2.2 Definition A topological space (X,τ) is called submaximal [3], if every dense subset is open.

2.3 Theorem [4] Let (X,μ) be a generalized topological space. Then
(i) $c_\mu(A)$ = X- $i_\mu(X-A)$.
(ii) $i_\mu(A)$ = X- $c_\mu(X-A)$.

2.4 Proposition [6] Let (X,μ) be a generalized topological space. For subsets A and B of X, the following properties hold:
1. $c_\mu(X-A)$ = X- $i_\mu(A)$ and $i_\mu(X-A)$ = X- $c_\mu(A)$;
2. If (X-A) ϵ μ, then $c_\mu(A)$ = A and if Aϵ μ, then $i_\mu(A)$ = A;
3. If A⊆B, then $c_\mu(A) \subseteq c_\mu(B)$ and $i_\mu(A) \subseteq i_\mu(B)$;
4. A ⊆ $c_\mu(A)$ and $i_\mu(A) \subseteq$ A;
5. $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$.

3.μ-LOCALLY CLOSED SETS IN GENERALIZED TOPOLOGICAL SPACES

3.1 Definition A subset A of a generalized topological space (X,μ) is called μ-locally closed set (briefly μ-LC set), if A = U ∩ V where U is μ-open in (X, μ) and V is μ-closed in (X, μ).
The collection of all μ-locally closed sets of (X,μ) will be denoted be μ-LC(X,μ).

3.2 Theorem

For a subset A of (X,μ) the following are equivalent:

(i) A is μ-locally closed

(ii) A=U ∩ $c_\mu(A)$, for some μ-open set U.

Proof

(i) ⇒ (ii):

Let A ϵ μ-LC(X,μ). Then there exists a μ-open set U and a μ-closed set V such that A = U∩V.

Since A ⊂ U and A ⊂ $c_\mu(A)$, we have A ⊂ U ∩ $c_\mu(A)$.

Conversely, since $c_\mu(A)$ ⊂ V, U ∩ $c_\mu(A)$ ⊂ U ∩ V = A, which implies that A = U ∩ $c_\mu(A)$.

(ii) ⇒ (i):

Since U is μ-open and $c_\mu(A)$ is μ-closed, U ∩ $c_\mu(A)$ ϵ μ-LC(X,μ).

3.3 Theorem

For a subset A of (X,μ) the following are equivalent:

(i) $c_\mu(A)$-A is μ-closed.

(ii) A ∪ [X-$c_\mu(A)$] is μ–open.

Proof

(i) ⇒ (ii):

Let V = $c_\mu(A)$-A.

Then V is μ–closed by the assumption.

Then X-V = X- [$c_\mu(A)$ -A]

= X ∩ - [$c_\mu(A)$ -A]^c

= A∪ [X-$c_\mu(A)$].

But X-V is μ–open.

This shows that A ∪ [X-$c_\mu(A)$] is μ–open.

(ii) ⇒ (i):

Let P = A∪ [X- $c_\mu(A)$].

Then P is μ–open.

This implies that X-P is μ–closed.

Then X-P = X - [A ∪ [X - $c_\mu(A)$]]

= $c_\mu(A)$ ∩ [X-A]

= $c_\mu(A)$ - A.

Thus $c_\mu(A)$ - A is μ–closed.

3.4 Theorem

For a subset A of (X,μ) the following are equivalent:

(i) A ∪ [X - $c_\mu(A)$] is μ–open.

(ii) A = U ∩ $c_\mu(A)$, for some μ–open set U.

Proof

(i) ⇒ (ii):

Let P = A ∪ [X-$c_\mu(A)$]. Then P is μ–open.

Hence we prove that A = P ∩ $c_\mu(A)$, for some μ–open set P.

Then P ∩ $c_\mu(A)$ = [A ∪ [X- $c_\mu(A)$]] ∩ $c_\mu(A)$

= [$c_\mu(A)$∩ A] ∪ [$c_\mu(A)$ ∩ [X- $c_\mu(A)$]]

= A ∪ ϕ = A.

Therefore A = P ∩ $c_\mu(A)$.

(ii)⇒ (i):

Let A = U ∩ $c_\mu(A)$, for some μ–open set U.

Then prove that A∪ [X- $c_\mu(A)$] is μ-open.

A ∪ [X- $c_\mu(A)$] = [U ∩ $c_\mu(A)$] ∪ [X- $c_\mu(A)$]

= U∩X = X, which is μ–open.

Thus A ∪ [X- $c_\mu(A)$] is μ–open.

3.5 Remark

Submaximaity, in the topological space exist whereas in the generalized topological space does not exist as seen from the following examples

3.6 Example

Let X= {a, b, c} with the topology [It is also a generalized topology], (X,τ) = {ϕ, X, {a}, {a, b}, {a, c}}. This topology is submaximal, because all the dense subsets are open.

3.7 Example

Let X = {a, b, c}. Consider the generalized topological spaces (i) (X,μ') = {ϕ , X, {a, b}, {a, c}} and (ii) (X,μ'') = {ϕ,{a},{a,b}}.These generalized topological spaces are not submaximal, since the dense subsets {a} and {b, c} of (X,μ') are not μ–open and also the dense subsets X and {a, c } of (X,μ'') are not μ-open.

3.8 Theorem

Let A and Z be subsets of a generalized topological space (X,μ) and let A ⊆ Z. If Z is μ-open in (X,μ) and A ϵ μ-LC(Z, μ/Z), then A ϵ μ-LC(X, μ).

Proof

Suppose A is μ-LC set, then there exists a μ-open set G of μ-LC(Z, μ/Z) such that

A= G ∩ $(c_\mu)_Z$(A).

But $(c_\mu)_Z$(A) = Z ∩ $c_\mu(A)$.

Therefore A = G ∩ Z ∩ $c_\mu(A)$, where G ∩ Z is μ-open.

Thus A ϵ μ-LC(X,μ).

3.9 Remark

The following example shows the assumption that z is μ-open cannot be removed from the above theorem.

3.10 Example

Let X= {a, b, c, d}. Consider (X,μ) = { ϕ, {b}, {c, d}, {b, c, d}}. The elements of (X, μ) are μ-open sets. Put Z = A = {a, b, c}. Then Z is not μ–open and A ϵ μ-LC(Z, μ/Z). However A∉ μ-LC(X, μ). Since μ-LC(X, μ) = { ϕ, {b}, {c}, {b,c}, {c, d}, {b, c, d}}.

4.(μ,μ')-LOCALLY CLOSED CONTINUOUS FUNCTIONS

4.1 Definition

A function f: X → Y between the generalized topological spaces (X,μ) and (Y,μ') is called (μ,μ')-locally closed continuous function (briefly, (μ,μ')-LC continuous function), if f^{-1}(A) ϵ μ-LC(X,μ), for each A ϵ (Y, μ').

4.2 Definition

A function f: X → Y between the generalized topological spaces (X, μ) and (Y, μ') is said to be (μ,μ')-locally closed irresolute (briefly, (μ,μ')-LC irresolute), if f^{-1}(A) ϵ μ-LC(X, μ), for each A ϵ μ'-LC (Y, μ').

4.3 Theorem

Let f: (X, μ) → (Y, μ') be a function. If f is (μ,μ')-LC irresolute, then it is (μ,μ')-LC continuous.

Proof

By the definitions the proof is immediate.

4.4 Theorem

If f: $(X,\mu) \to (Y, \mu')$ is (μ,μ')-continuous, then f is (μ,μ')-LC irresolute.

Proof

The proof is obvious from the definitions.

4.5 Remark

Converse of theorem 4.3 need not be true as seen from the following example.

4.6 Example

Let X = {a, b, c} with the generalized topology $(X,\mu) = \{\phi, X, \{a\}, \{a, b\}\}$ and Y={1, 2, 3} with the generalized topology $(Y,\mu') = \{\phi, Y, \{1\}\}$. We define a mapping f: $(X, \mu) \to (Y, \mu')$ such that f(a) = 2, f(b) = 1 and f(c) = 3. Hence f is (μ,μ')–LC continuous but not (μ,μ')–LC irresolute.

4.7 Remark

The following example provides a function which is (μ,μ')–LC irresolute but not (μ,μ')–continuous.

4.8 Example

Let X = Y = {a, b, c} and f: $(X,\mu) \to (Y, \mu')$ be a function defined by f(a) = a, f(b) = b and f(c) = c. Let $(X,\mu) = \{\phi, \{a\}, \{a, b\}\}$ and $(Y, \mu') = \{\phi, \{b\}\}$. Then f is (μ,μ')–LC irresolute, but not (μ,μ')–continuous, since $f^{-1}(\{b\}) \in \mu$-LC(X, μ), for $\{b\} \in \mu'$-LC (Y, μ') and $f^{-1}(\{b\}) \notin (X, \mu)$, for $\{b\} \in (Y, \mu')$.

4.9 Remark

From the above theorems and examples, we obtain the following relations:

$$(\mu,\mu')\text{–continuity} \;\underset{\not\longleftarrow}{\longrightarrow}\; (\mu,\mu')\text{–LC irresoluteness} \;\underset{\not\longleftarrow}{\longrightarrow}\; (\mu,\mu')\text{–LC continuity}$$

where $A \longrightarrow B$ (resp.,$A \not\longrightarrow B$) represents that A implies B (resp., A does not imply B)

4.10 Theorem

If g: $(X,\mu) \to (Y, \mu')$ is (μ,μ')-LC continuous and h : $(Y, \mu') \to (Z, \mu'')$ is (μ',μ'')-continuous, then $h \circ g$: $(X, \mu) \to (Z, \mu'')$ is (μ,μ'')-LC continuous.

Proof

Given g: $X \to Y$ is (μ,μ')-LC continuous and h: $Y \to Z$ is (μ',μ'')-continuous. By the definitions, $g^{-1}(V) \in \mu$-LC (X), $V \in Y$ and $h^{-1}(W) \in Y$, $W \in Z$. Let $W \in Z$.

Then $(h \circ g)^{-1}(W) = (g^{-1} h^{-1})(W) = g^{-1}(h^{-1}(W)) = g^{-1}(V)$, for $V \in Y$.

This implies $(h \circ g)^{-1}(W) = g^{-1}(V) \in \mu$-LC (X), $W \in Z$.

Therefore, $h \circ g$ is (μ,μ'')-LC continuous.

4.11 Theorem

If g: $(X,\mu) \to (Y, \mu')$ is (μ,μ')-LC irresolute and h: $(Y, \mu') \to (Z, \mu'')$ is (μ',μ'')-LC continuous, then
$h \circ g$: $(X,\mu) \to (Z, \mu'')$ is (μ,μ'')-LC continuous.

Proof

Let g : $X \to Y$ is (μ,μ')-LC-irresolute and h : $Y \to Z$ is (μ',μ'')-LC-continuous.

By definitions, $g^{-1}(V) \in \mu$-LC(X), for $V \in \mu'$-LC(Y) and $h^{-1}(W) \in \mu'$-LC(Y), for $W \in Z$. Let $W \in Z$. Then $(h \circ g)^{-1}(W) = g^{-1}(h^{-1}(W)) = g^{-1}(V)$, for $V \in \mu'$-LC(Y).

This implies $(h \circ g)^{-1}(W) = g^{-1}(V) \in \mu$-LC(X), $W \in Z$. Therefore $h \circ g$ is (μ,μ'')-LC continuous.

4.12 Theorem

If g: $(X,\mu) \to (Y, \mu')$ is a (μ,μ')-LC irresolute and h : $(Y, \mu') \to (Z, \mu'')$ is a (μ,μ')-LC irresolute, then $h \circ g$: $(X, \mu) \to (Z, \mu'')$ is also a (μ,μ'')-LC irresolute.

Proof

By the hypothesis and definition we have, $g^{-1}(V) \in \mu$-LC(X), for $V \in \mu'$-LC(Y) and $h^{-1}(W) \in \mu'$-LC(Y), for $W \in \mu''$-LC(Z).

Let $W \in \mu''$-LC(Z).

Then $(h \circ g)^{-1}(W) = g^{-1}(h^{-1}(W)) = g^{-1}(V)$, for $V \in \mu'$-LC(Y).

This implies $(h \circ g)^{-1}(W) = g^{-1}(V) \in \mu$-LC(X), $W \in \mu''$-LC (Z). Therefore $h \circ g$ is (μ,μ'') –LC irresolute.

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