

Solution of a Birkhoff Interpolation Problem by a Special Spline Function

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ABSTRACT

In this paper we have discussed a special lacunary interpolation problem in which the function values, first derivatives at the nodes and the third derivatives at any point λ ($0 \leq \lambda \leq 1$) in between the nodes are prescribed. We have solved the unique existence and convergence problems, using spline functions. As this holds for any λ ($0 \leq \lambda \leq 1$) we named it a generalized problem.

General Terms

Your general terms must be any term which can be used for general classification of the submitted material such as Pattern Recognition, Security, Algorithms et. al.

Keywords

Lacunary interpolation, Spline functions

1. INTRODUCTION

Let $\Delta: 0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ be a partition of unit interval $I = [0, 1]$ with

$x_{k+1} - x_k = h_k$, $k = 0, 1, \dots, n-1$. Denote by $S_{n,5}^{(2)}$ the class of quintic splines $s(x)$ satisfying the condition that $s(x) \in C^3(I)$ and is quintic in each subintervals of I . In the past this class of splines is used by various authors with different interpolatory conditions. In [2] this class of splines is used to solve the interpolation problem with following conditions:

$$s_{\Delta}(x_k) = f_k, \quad k = 0, \dots, n;$$

$$s'_{\Delta}(x_k) = f'_k, \quad k = 0, \dots, n;$$

$$s'''_{\Delta}(x_{k+1/3}) = f'''_{k+1/3}, \quad k = 0, \dots, n-1;$$

$$\text{where } x_{k+1/3} = \frac{1}{3}(x_k + x_{k+1})$$

$$s''_{\Delta}(x_0) = f''_0 \quad \text{or} \quad s'''_{\Delta}(x_0) = f'''_0.$$

Some other authors also solved the similar problems with other intermediate points. But the interesting thing is that here in this paper we solved a generalized problem when we take λ

($0 \leq \lambda \leq 1$) as an intermediate point where third derivatives are prescribed. Later we can show that this result holds for any value of λ ($0 \leq \lambda \leq 1$). We proved the unique existence theorem and also shown the convergence.

2. UNIQUE EXISTANCE THEOREM

2.1 Theorem 1

Given a partition Δ of the unit interval $I = [0, 1]$ and the numbers $f_k, f'_k, k = 0, 1, \dots, n-1$;

$f'''_{k+\lambda}$ ($0 \leq \lambda \leq 1$), $k = 0, 1, \dots, n-1$; f''_0, f'''_0 ; there exists a unique spline $s_{\Delta}(x) \in S_{n,5}^{(2)}$ such that

(1.1)

$$\begin{cases} s_{\Delta}(x_k) = f_k, & k = 0, 1, \dots, n; \\ s'_{\Delta}(x_k) = f'_k, & k = 0, 1, \dots, n; \\ s'''_{\Delta}(x_{k+\lambda}) = f'''_{k+\lambda}, & k = 0, 1, \dots, n-1; \\ s''_{\Delta}(x_0) = f''_0 \quad \text{or} \quad s'''_{\Delta}(x_0) = f'''_0. \end{cases}$$

Here $x_{k+\lambda} = \lambda(x_k + x_{k+1})$ and

$$h_k = x_{k+1} - x_k, \quad k = 0, 1, \dots, n-1.$$

2.1.1 Proof of Theorem 1

Here we prove the theorem with the initial condition

$$s''_{\Delta}(x_0) = f''_0 \text{ only, for the condition}$$

$$s'''_{\Delta}(x_0) = f'''_0 \text{ the similar method can be applied.}$$

Let us set

$$(2.1) \quad s_{\Delta}(x) = \begin{cases} s_{\Delta}(x) & \text{when } x_0 \leq x \leq x_1 \\ s_k(x) & \text{when } x_k \leq x \leq x_{k+1}, \quad k = 1, 2, \dots, n-1. \end{cases}$$

$$(2.2) \quad s_{\Delta}(x) = f_0 + (x - x_0)f_0' + \frac{(x-x_0)^2}{2!}f_0'' + \frac{(x-x_0)^3}{3!}a_{0,3} + \frac{(x-x_0)^4}{4!}a_{0,4} + \frac{(x-x_0)^5}{5!}a_{0,5}$$

$$(2.3) \quad s_k(x) = f_k + (x - x_k)f_k' + \frac{(x-x_k)^2}{2!}a_{k,2} + \frac{(x-x_k)^3}{3!}a_{k,3} + \frac{(x-x_k)^4}{4!}a_{k,4} + \frac{(x-x_k)^5}{5!}a_{k,5}$$

For determining the coefficients we apply the interpolatory condition (1.1) and the continuity requirements that

$s_{\Delta}(x_k) \in C^2(I)$. Then we have

$$(2.4) \quad \begin{cases} f_1 = f_0 + h_0 f_0' + \frac{(h_0)^2}{2!}f_0'' + \frac{(h_0)^3}{3!}a_{0,3} + \frac{(h_0)^4}{4!}a_{0,4} \\ f_1' = f_0' + h_0 f_0'' + \frac{(h_0)^2}{2!}a_{0,3} + \frac{(h_0)^3}{3!}a_{0,4} + \frac{(h_0)^4}{4!}a_{0,5} \\ f_1''' = a_{0,3} + \lambda h_0 a_{0,4} + \frac{(\lambda h_0)^2}{2!}a_{0,5} \end{cases}$$

$$(2.5) \quad \begin{cases} f_{k+1} = f_k + h_k f_k' + \frac{(h_k)^2}{2!}a_{k,2} + \frac{(h_k)^3}{3!}a_{k,3} + \frac{(h_k)^4}{4!}a_{k,4} \\ f_{k+1}' = f_k' + h_k a_{k,2} + \frac{(h_k)^2}{2!}a_{k,3} + \frac{(h_k)^3}{3!}a_{k,4} + \frac{(h_k)^4}{4!}a_{k,5} \\ f_{k+\lambda}''' = a_{k,3} + \lambda h_k a_{k,4} + \frac{(\lambda h_k)^2}{2!}a_{k,5} \end{cases}$$

$$k = 1, 2, \dots, n-2$$

and (2.6)

$$\begin{cases} a_{k+1} = a_k + h_k a_{k,3} + \frac{(h_k)^2}{2!}a_{k,4} + \frac{(h_k)^3}{3!}a_{k,5} \\ a_{1,2} = f_0'' + h_0 a_{0,3} + \frac{(h_0)^2}{2!}a_{0,4} + \frac{(h_0)^3}{3!}a_{0,5} \end{cases}$$

$$(2.7) \quad a_{0,5} = \frac{1}{(10\lambda^2 - 8\lambda + 1)} \left[\frac{480(3\lambda - 1)}{h_0^5} (f_1 - f_0 - h_0 f_0' - \frac{h_0^2}{2!}f_0'') - \frac{120(4\lambda - 1)}{h_0^4} (f_1' - f_0' - h_0 f_0'') + \frac{20f_1'''}{h_0^3} \right]$$

$$(2.8) \quad a_{0,4} = \frac{1}{(10\lambda^2 - 8\lambda + 1)} \left[\frac{-120(6\lambda^2 - 1)}{h_0^4} (f_1 - f_0 - h_0 f_0' - \frac{h_0^2}{2!}f_0'') + \frac{24(10\lambda^2 - 1)}{h_0^3} (f_1' - f_0' - h_0 f_0'') - \frac{8f_1'''}{h_0^2} \right]$$

$$(2.9) \quad a_{0,3} = \frac{1}{(10\lambda^2 - 8\lambda + 1)} \left[\frac{120(2\lambda - 1)}{h_0^3} (f_1 - f_0 - h_0 f_0' - \frac{h_0^2}{2!}f_0'') - \frac{12(5\lambda - 2)}{h_0^2} (f_1' - f_0' - h_0 f_0'') + f_1''' \right]$$

From (2.5) we have,

$$(2.10) \quad a_{k,5} = \frac{1}{(30\lambda^2 - 24\lambda + 3)} \left[\frac{1440(3\lambda - 1)}{h_k^5} (f_{k+1} - f_k - h_k f_k' - \frac{h_k^2}{2!}f_k'') - \frac{20(72\lambda - 18)}{h_k^4} (f_{k+1}' - f_k' - h_k f_k'') + \frac{20(18 - 36\lambda)}{h_k^3} a_{k,2} + \frac{60}{h_k^2} f_{k+\lambda}''' \right]$$

$$(2.11) \quad a_{k,4} = \frac{1}{(30\lambda^2 - 24\lambda + 3)} \left[\frac{(-2160\lambda^2 + 360)}{h_k^4} (f_{k+1} - f_k - h_k f_k') + \frac{(720\lambda^2 - 72)}{h_k^3} (f_{k+1}' - f_k') + \frac{(360\lambda^2 - 108)}{h_k^2} a_{k,2} - \frac{24}{h_k} f_{k+\lambda}''' \right]$$

(2.12)

$$a_{k,3} = \frac{1}{(30\lambda^2 - 24\lambda + 3)} \left[\frac{(720\lambda^2 - 360\lambda)}{h_k^3} (f_{k+1} - f_k - h_k f'_k) + \frac{(72\lambda - 180\lambda^2)}{h_k^2} (f'_{k+1} - f'_k) + \frac{(108\lambda - 180\lambda^2)}{h_k} a_{k,2} + 3f''_{k+\lambda} \right]$$

Using values of these coefficients in (2.6) we get

(2.13)

$$a_{1,2} = \frac{1}{(10\lambda^2 - 8\lambda + 1)} \left[\frac{(20\lambda^3 - 34\lambda^2 + 18\lambda - 3)}{(2\lambda - 1)} f''_0 + \frac{1}{3} h_0 f'''_0 + \frac{20(-6\lambda^2 + 6\lambda - 1)}{h_0^2} (f_1 - f_0 - h_0 f'_0) + \frac{(120\lambda^3 - 172\lambda^2 + 72\lambda - 1)}{(2\lambda - 1)h_0} f''_0 \right]$$

 (2.14) $a_{k+1,2}$

$$+ \frac{(-30\lambda^2 + 36\lambda - 9)}{(30\lambda^2 - 24\lambda + 3)} a_{k,2} = \frac{1}{(30\lambda^2 - 24\lambda + 3)} \left[\frac{(-360\lambda^2 - 360\lambda - 60)}{h_k^2} (f_{k+1} - f_k - h_k f'_k) + \frac{(180\lambda^2 - 168\lambda + 24)}{h_k} (f'_{k+1} - f'_k) + h_k f''_k \right]$$

The coefficient matrix of the system of equations (2.13) and

 (2.14) in the unknowns $a_{k,2}$, $k = 1, 2, \dots, n-1$ is seen to be

 nonsingular and hence the coefficients $a_{k,2}$, $k = 1, 2, \dots, n-1$, are uniquely determined and so are, therefore, the coefficients $a_{k,3}$, $a_{k,4}$, $a_{k,5}$, $k = 1, 2, \dots, n-1$.

3. THEOREM OF CONVERGENCE

Let $f \in C^l(I)$, $l = 5, 6$. Then for the unique spline $s_\Delta(x)$ of Theorem 1 associated with the function f , we have

(3.1)

$$\|s_\Delta^{(5)}(x) - f^{(5)}(x)\| \begin{cases} O(\omega_5(H)), & \text{if } f \in C^5(I) \\ K_3 H \|f^{(6)}\| + O(\omega_5(H)), & \text{if } f \in C^6(I) \end{cases}$$

And if $\frac{\max h_k}{\min h_k} \leq \lambda \leq \infty$ and $H =$

$\max_{0 \leq k \leq n-1} h_k$, then

(3.2)

$$\|s_\Delta^{(q)}(x) -$$

$$f^{(q)}(x)\| \begin{cases} O(H^{4-q} \omega_5(H)), & \text{if } f \in C^5(I), \\ K_2 H^{6-q} \|f^{(6)}\| + O(H^{5-q} \omega_5(H)), & \text{if } f \in C^6(I), \end{cases}$$

$q = 0, 1, 2, 3, 4$.

Where K_2 and K_3 are some constants involving λ

$(0 \leq \lambda \leq 1)$.

3.1 Auxiliary Lemmas

Now we give three lemmas that are used to obtain the proof of the Theorem of convergence theorem.

3.1.1 Lemma

Let $A_{k,2} = a_{k,2} - f''_k$.

Then we have for $k = 1, 2, \dots, n-1$.

$$|A_{k,2}| = \begin{cases} O\left(\sum_{v=0}^{k-1} h_v^3 \omega_5(h_v)\right), & \text{if } f \in C^5(I) \\ K_1 h_k^4 f^{(6)} + O\left(\sum_{v=0}^{k-1} h_v^4 \omega_6(h_v)\right), & \text{if } f \in C^6(I) \end{cases}$$

$$\text{Where } K_1 = \frac{(20\lambda^3 - 30\lambda^2 + 12\lambda - 1)}{120(30\lambda^2 - 24\lambda + 3)}.$$

Proof From (2.14) we have

$$(3.1.1) \quad A_{k+1,2} + \frac{(-30\lambda^2 + 36\lambda - 9)}{(30\lambda^2 - 24\lambda + 3)} A_{k,2} = (a_{k+1,2} - f''_{k+1}) + \frac{(-30\lambda^2 + 36\lambda - 9)}{(30\lambda^2 - 24\lambda + 3)} (a_{k,2} - f''_k) = \alpha_k \text{ (say), } k = 1, 2, \dots, n-2.$$

$$\alpha_k =$$

$$\frac{1}{(30\lambda^2 - 24\lambda + 3)} \left[\frac{(-360\lambda^2 + 360\lambda - 60)}{h_k^2} (f_{k+1} - f_k - h_k f'_k) + \frac{(180\lambda^2 - 168\lambda + 24)}{h_k} (f'_{k+1} - f'_k) + h_k f''_k \right] - \left[f''_{k+1} + \frac{(-30\lambda^2 + 36\lambda - 9)}{(30\lambda^2 - 24\lambda + 3)} f''_k \right]$$

If $f \in C^5(I)$ then by Taylor's formula (3.1.2)

$$\alpha_k = O(h_k^3 \omega_5(h_k)).$$

Similarly if $f \in C^5(I)$, then (3.1.3)

$$\alpha_k = K_1 h_k^4 f^{(6)} + O(h_k^4 \omega_6(h_k)).$$

Also from (2.13)

$$(3.1.4) \begin{cases} |A_{1,2}| = \\ |a_{1,2} - f_1''| = \\ \begin{cases} O(h_0^3 \omega_5(h_v)), & \text{if } f \in C^5(I) \\ K_2 h_0^4 f^{(6)} + O(h_0^4 \omega_6(h_v)), & \text{if } f \in C^6(I) \end{cases} \end{cases} \text{ Where}$$

From (3.1.1) and (3.1.2) and the derivatives for α_k we have

$$|A_{k,2}| = \begin{cases} O(\sum_{v=0}^{k-1} h_v^3 \omega_5(h_v)), & \text{if } f \in C^5(I) \\ K_1 h_k^4 f^{(6)} + O(h_k^4 \omega_6(h_k)), & \text{if } f \in C^6(I) \end{cases}$$

This proves the assertion of lemma.

3.1.2 Lemma

Let $A_{k,4} = a_{k,4} - f_k^{(4)}$ and $\frac{\max h_k}{\min h_k} \leq$

$$\lambda \leq \infty, \quad H = \max_{0 \leq k \leq n-1} h_k.$$

Then we have for $k = 0, 1, \dots, n-1$.

$$|A_{k,4}| = \begin{cases} O(\omega_5(H)), & \text{if } f \in C^5(I) \\ K_2 H^2 \|f^{(6)}\| + O(H \omega_6(H)), & \text{if } f \in C^6(I) \end{cases}$$

Where

$$K_2 = \frac{-40\lambda^3 + 30\lambda^2 - 1}{10(30\lambda^2 - 24\lambda + 3)}.$$

Proof From (2.8) and (2.11) we see $A_{0,2} = 0$, then

$$(3.1.5) \quad A_{k,4} = a_{k,4} - f_k^{(4)} = \frac{(360\lambda^2 - 108)}{(30\lambda^2 - 24\lambda + 3)h_k^2}$$

$$A_{k,2} + \beta_k, \quad k = 0, 1, \dots, n-1$$

Where $\beta_k =$

$$\frac{1}{(30\lambda^2 - 24\lambda + 3)} \left[\frac{(-2160\lambda^2 + 360)}{h_k^4} (f_{k+1} - f_k - h_k f_k') + \frac{(-2160\lambda^2 + 360)}{h_k^4} (f_k - f_{k-1} - h_k f_{k-1}') \right]$$

$$(3.1.6) \quad \beta_k = O(h_k \omega_5(H)), \text{ if } f \in C^5(I).$$

If $f \in C^6(I)$, then (3.1.7)

$$\beta_k = K_2 h_k^2 f_k^{(6)} + O(h_k^2 \omega_6(h_k)), \text{ where } K_2 = \frac{(-40\lambda^3 + 30\lambda^2 - 1)}{10(30\lambda^2 - 24\lambda + 3)}.$$

Using Lemma 3.1, we have for $k = 0, 1, \dots, n-1$.

$$|A_{k,4}| = \begin{cases} O\left(\frac{1}{h_k^2} \sum_{v=0}^{k-1} h_v^3 \omega_5(h_v)\right) + O(h_k \omega_5(h_k)), & \text{if } f \in C^5(I) \\ K_2 h_k^2 f_k^{(6)} + O(h_k^2 \omega_6(h_k)), & \text{if } f \in C^6(I) \end{cases}$$

The result clearly holds for $k=0$. Hence if $\frac{\max h_k}{\min h_k} \leq \lambda$

$$\leq \infty, \quad H = \max_{0 \leq k \leq n-1} h_k,$$

we have from (3.1.5) to (3.1.7)

$$|A_{k,4}| = \begin{cases} O(\omega_5(H)), & \text{if } f \in C^5(I) \\ K_2 H \|f^{(6)}\| + O(H \omega_6(H)), & \text{if } f \in C^6(I) \end{cases} \quad k = 0, 1, \dots, n-1.$$

This proves Lemma 3.1.2.

3.1.3 Lemma

Let $A_{k,5} = a_{k,5} - f_k^{(5)}$

Then we have for $k = 0, 1, \dots, n-1$

$$|A_{k,5}| = \begin{cases} O(\omega_5(H)), & \text{if } f \in C^5(I) \\ K_3 H \|f^{(6)}\| + O(H \omega_6(H)), & \text{if } f \in C^6(I) \end{cases}$$

$$\text{Where } K_3 = \frac{(10\lambda^3 - 6\lambda + 1)}{10(30\lambda^2 - 24\lambda + 3)}.$$

Proof Following similar method we can get the results for $|A_{k,5}|$ hence we omitted the proof.

4. PROOF OF THEOREM 2

Let $x \in [x_k, x_{k+1}]$, $k = 0, 1, \dots, n-1$

$$\text{Then from (2.3) we have} \quad (4.1) \quad s_k^{(5)}(x)$$

$$= a_{k,5}$$

$$\text{and} \quad (4.2) \quad s_k^{(5)}(x)$$

$$= a_{k,4} + (x - x_k) a_{k,5}$$

Therefore

$$\begin{aligned} & |s_k^{(5)}(x) - f^{(5)}(x)| \\ &= |s_k^{(5)}(x) - f_k^{(5)} + f_k^{(5)} - f^{(5)}(x)| \end{aligned}$$

$$\leq \left| a_{k,5} - f_k^{(5)} \right| + \left| f_k^{(5)} - f^{(5)}(x) \right|.$$

If $f \in C^5(I)$ then using Lemma 3.1.3, we have

$$\left| s_k^{(5)}(x) - f^{(5)}(x) \right| = O(\omega_5(H)). \quad (4.3)$$

Again from (4.2)

$$\begin{aligned} (4.4) \quad s_k^{(4)}(x) - f^{(4)}(x) &= (a_{k,4} - f^{(4)}) + \\ &+ (x - x_k)(a_{k,5} - f_k^{(5)}) - [\\ &f^{(4)}(x) - f_k^{(4)} - (x - x_k)f_k^{(5)}] \\ &= A_{k,4} \\ &+ (x - x_k)A_{k,5} - (x - x_k)(f^{(4)}(\eta_k) - f_k^{(5)}), \\ &x_k \leq \eta_k \leq x \end{aligned}$$

Thus,

$$\left| s_k^{(5)}(x) - f^{(5)}(x) \right| \leq |A_{k,4}| + H|A_{k,5}| + H\omega_5(H)$$

Now applying Lemma 3.1.2 and 3.1.3 we get,

$$(4.5) \quad \left| s_k^{(4)}(x) - f^{(4)}(x) \right| = O(\omega_5(H)) + H O(\omega_5(H)) = O(\omega_5(H)).$$

Now, $\left| s_k'''(x) - f'''(x) \right| =$

$$\left| \int_{x_{k+\lambda}}^x [s_k^{(4)}(t) - f^{(4)}(t)] dt \right| \leq (x - x_{k+\lambda}) \left| s_k^{(4)}(x) - f^{(4)}(x) \right|$$

$$(4.6) \quad \left| s_k'''(x) - f'''(x) \right| = (H\omega_5(H)).$$

Set $h(x_k) = h(x_{k+1}) = 0$.

So by Rolle's theorem, there exists a μ_k ,

$$x_k < \mu_k < x_{k+1}, \text{ such that } h'(\mu_k) = s_k''(\mu_k) - f''(\mu_k) = 0.$$

This gives $\left| s_k''(x) - f''(x) \right| =$

$$\left| \int_{\mu_k}^x [s_k'''(t) - f'''(t)] dt \right| \leq (x - \mu_k) \left| s_k'''(x) - f'''(x) \right| = O(HH\omega_5(H))$$

$$(4.7) \quad \left| s_k''(x) - f''(x) \right| = (H^2\omega_5(H)).$$

Again using interpolatory conditions (1.1) we can write

$$\begin{aligned} &\left| s_k'(x) - f'(x) \right| = \left| \int_{x_k}^x [s_k''(t) - f''(t)] dt \right| \\ (4.8) \quad &\left| s_k'(x) - f'(x) \right| = (H^3\omega_5(H)). \end{aligned}$$

Similarly

$$(4.9) \quad \left| s_k(x) - f(x) \right| = \left| \int_{x_k}^x [s_k'(t) - f'(t)] dt \right| = O(H^4\omega_5(H)).$$

This proves the theorem for $f \in C^5(I)$. Next we consider the case when $f \in C^6(I)$. Then from Lemma 3.1.3

$$\begin{aligned} &\left| s_k^{(5)}(x) - f^{(5)}(x) \right| = \left| (a_{k,5} - f^{(5)}) + (x - x_k)f_k^{(5)}(\xi_k) \right|, \\ &x_k \leq \xi_k \leq x \\ &\leq K_3 H \|f^{(6)}\| + O(H\omega_6(H)). \end{aligned}$$

Again

$$\begin{aligned} &s_k^{(4)}(x) - f^{(4)}(x) = A_{k,4} + (x - x_k)A_{k,5} + \frac{(x - x_k)^2}{2} f^{(5)}(\xi_k), \\ &x_k \leq \xi_k \leq x \end{aligned}$$

Which on using Lemma 3.2 and Lemma 3.3, gives

$$(4.10) \quad \left| s_k^{(4)}(x) - f^{(4)}(x) \right| \leq K_3 H \|f^{(6)}\| + (H\omega_6(H)).$$

From (4.10) on using method of successive integration we at once have

$$(4.11) \quad \left| s_k^{(q)}(x) - f^{(q)}(x) \right| \leq K_3 H^{6-q} \|f^{(6)}\| + (H^{6-q}\omega_6(H)), \quad q = 1, 2, 3, 4.$$

This proves the theorem of convergence for $f \in C^6(I)$.

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