New Programming Approach for Steepest Descent Optimization of Rocket Trajectories

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ABSTRACT
In this paper a new programming methodology for optimizing rocket trajectories using steepest descent technique is presented. The programming is done in Matlab platform. At first the dynamical equations of rocket is derived and for the proper derivation and analysis of the equations ,Eulers integration method is used. A new programming approach is designed and the equations are optimized using steepest descent technique, assuming certain boundary conditions. The results obtained are verified and it is seen that the optimal trajectory is attained with all objectives satisfied. This project is done at Vikram Sarabhai Space Centre(VSSC),a constituent of Indian Space Research Organization(ISRO).

General Terms
Rocket dynamics,Euler'smethod,Simulator program.

Keywords
Rocket trajectory,Matlab,Optimization,Steepest descent

1. INTRODUCTION
The optimality of rocket trajectories may be defined according to several objectivesliketransfer time is minimal for a given propellant and payload mass (i.e. for a given launch mass),the required propellant mass is minimal for a given transfer time and launch mass, the required propellant mass is minimal for a given transfer time and payload mass.In practice, the methods of solving deterministic optimal control problems are divided into two categories: direct and indirect methods[9]. Indirect methods proceed by formulating the optimality conditions according to the Pontryagin maximum principle and then numerically solving the resulting two point boundary value problem. Direct methods discretize the original problem in time and solve the resulting parameter optimization problem and thus generate an approximate solution of the original problem. Both methods solve the necessary conditions of optimality and both discretize the problem. The most common objective is to minimize the propellant required or equivalently to maximize the fraction of the rocket that is not devoted to propellant. Of course, as is common in the optimization of continuous dynamical systems, it is usually necessary to provide some practical upper bound for the final time or the optimizer will trade time for propellant. There are also rocket trajectory problems where minimizing flight time is an important, or, for example those using continuous thrust, where minimizing flight time and minimizing propellant use are synonymous.

2. MATHEMATICAL REPRESENTATION OF ROCKETDYNAMICS
Point-mass equations of motion with non-rotating sphericalearth are considered in this paper, which is given by

$$\dot{r} = V sin\gamma \tag{1}$$

$$\dot{V} = \frac{1}{mV}(T\,cos\alpha -- mg\,sin\gamma) \tag{2}$$

$$\dot{\gamma} = \frac{1}{mV}(T\,sin\alpha) + \left(\frac{V}{r} - \frac{g}{V}\right) cos\gamma \tag{3}$$

For a problem without path constraints, with terminal equality constraints, and with an initial state vector whose components are either fixed or optimizable, the first order necessary conditions can be formulated as shown in equations from(4)to(11).

Hamiltonian:

$$H = L + \lambda^T f \tag{4}$$

Dynamics:

$$\dot{X} = f(x,u,t) = \left(\frac{\partial H}{\partial \lambda}\right)^T \tag{5}$$

Adjoint differential equations:

$$\dot{\lambda} = \left(\frac{\partial L}{\partial x}\right)^T - \left(\frac{\partial f}{\partial x}\right)^T \lambda = \left(\frac{\partial H}{\partial x}\right)^T \tag{6}$$

Optimality condition:

$$\left(\frac{\partial H}{\partial u}\right)^T = 0 = \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial L}{\partial x}\right)^T \tag{7}$$

Initial conditions:

$$x_k(t_0)\text{given or } \lambda_k(t_0) = 0 \tag{8}$$

Terminal constraints:

$$\Psi_f(x_f, t_f) = 0 \tag{9}$$

Transversality conditions:

$$\lambda_f = (\frac{\partial \emptyset}{\partial x} + v^T \frac{\partial \Psi_f}{\partial x})_{t=t_f} \qquad (10)$$

Transversality condition for optimizable t_f:

$$\Omega = [(\frac{\partial \emptyset}{\partial t} + v^T \frac{\partial \Psi_f}{\partial t} + H)]_{t=t_f} \qquad (11)$$

Where L = Lagrangefunction

H = Hamilton function

λ = costate variable

Euler-Lagrange [22] equations were extended by the introduction of further necessary conditions, such as the Legendre- Clebsch condition, The Jacobi condition[4] and the Weierstrass condition[4].

$$\frac{\partial^2 H}{\partial u^2} \geq 0 \qquad (12)$$

A detailed discussion on these conditions can be found in Ref. [22].In this part optimization of the three variables take place, they are radial distance from the center of earth r, rocket velocity v and the flight path angle γ.The equations $(1) - (3)$ can be described in generic form as,

$$\dot{X} = f(X, U) \qquad (13)$$

$$\begin{bmatrix} \dot{r} \\ \dot{v} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} V sin\gamma \\ \frac{1}{mV}(Tcos\alpha - mgsin\gamma) \\ \frac{1}{mV}(Tsin\alpha) + (\frac{V}{r} - \frac{g}{V}) cos\gamma \end{bmatrix} \qquad (14)$$

Where the state and control vectors are defined as

$$\dot{X} = [x_1 x_2 x_3]^T \qquad (15)$$

That is

$$\dot{X} = [rv\gamma]^T \qquad (16)$$

$$U = \alpha \qquad (17)$$

In this a trajectory optimization problem of a single stage launch vehicle is considered. The objective here is to generate the guidance command history $\alpha(t), t \in (t_0, t_f)$such that the following concerns are taken care of,

(a) At the final time t_f, the specified terminal constraints must meet as accurately. The terminal constraints include constraints on altitude, velocity and flight path angle (which is the angle made by the velocity vector with respect to the local horizontal).

(b) The system should demand minimum guidance command, which can be ensured by formulating a 'minimum time' problem.

To achieve the above objectives, the following cost function is selected, which consists of terminal penalty terms and a dynamic control minimization term.

$$J = (r(t_f) - r_f)^2 s_r + (v(t_f) - v_f)^2 s_v + (row(t_f) - row_f)^2 s_\gamma \qquad (18)$$

Where $s_r, s_v, and s_\gamma$ weighing factors. The Hamiltonian (Ref equation (4)) is defined as ,

$$H = J + \lambda^T \qquad (19)$$

Where $\lambda^T = [\lambda_1 \lambda_2 \lambda_3]$, costate variables.

$$H = J - \lambda_1 (Vsin\gamma) - \lambda_2 \left(\frac{1}{mV}(Tcos\alpha - mgsin\gamma) \right) - \lambda_3 \left(\frac{1}{mV}(Tsin\alpha) \left(\frac{V}{r} - \frac{g}{V} \right) cos\gamma \right) \qquad (20)$$

From equation (18) we get

$$H = (r(t_f) - r_f)^2 s_r + (v(t_f) - v_f)^2 s_v + (row(t_f) - row_f)^2 s_\gamma - \lambda_1 (Vsin\gamma) - \lambda_2 \left(\frac{1}{mV}(Tcos\alpha - mgsin\gamma) \right) - \lambda_3 \left(\frac{1}{mV}(Tsin\alpha) + \left(\frac{V}{r} - \frac{r}{V} \right) cos\gamma \right) \qquad (21)$$

Equation (21) gives the final Hamiltonian function. This function has solve to get the optimum solution .Therefore

$$\frac{\partial H}{\partial r} = \lambda_3 \frac{v}{r^2} cos\gamma = -\dot{\lambda}_1 \qquad (22)$$

$$\frac{\partial H}{\partial v} = -\lambda_1 sin\gamma + \lambda_3 - \left(\frac{1}{mv^2}(Tsin\alpha) - \left(\frac{1}{r} - \frac{g}{v^2} \right) cos\gamma \right) = -\dot{\lambda}_2 \qquad (23)$$

$$\frac{\partial H}{\partial \gamma} = -\lambda_1 cos\gamma - \lambda_2 gcos\gamma + \lambda_3 \left(\frac{v}{r} - \frac{g}{v} \right) sin\gamma = -\dot{\lambda}_3 \qquad (24)$$

$$\frac{\partial H}{\partial \alpha} = \lambda_2 \frac{T}{m} sin\alpha - \lambda_3 \frac{T}{mv} cos\alpha \qquad (25)$$

The above equations $(22) - (25)$ to be solved to get the solutions of costate variables, that is $\lambda_1, \lambda_2 and \lambda_3$. In an optimal control formulation, the necessary conditions of optimality include the state equation, costate equation, optimal control equation and appropriate boundary conditions.An steepest descent method needs accurate initial

conditions. The equations (1)(2)&(3) in the simulator program is developed initial steering profile for steepest descent optimization is obtained.For finding the accurate steering profile taking an assumption that is acceleration is a linearly increasing quantity. So rate of change of velocity is approximating as linear function. This linear function is selected by polynomial approximation[17] that is by changing the functions randomly to fit the curve properly.

$$\alpha = cos^{-1}\left(\frac{c_1 t + c_2 + g \, sin\gamma}{T/m}\right) \qquad (26)$$

Now knowing the values of $\dot{\lambda}_1$, $\dot{\lambda}_2 and \dot{\lambda}_3$, so to get $\lambda_1, \lambda_2 and \lambda_3$ have to integrate. Therefore the final values of $\lambda^T$is $\lambda_{1f}, \lambda_{2f} and \lambda_{3f}$. To find the final values of λ.

$$\lambda_{1f} = \frac{\partial J}{\partial r} = 2 \times s_r \times \left(r(t_f) - r_f\right) \qquad (27)$$

$$\lambda_{2f} = \frac{\partial J}{\partial v} = 2 \times s_v \times \left(v(t_f) - v_f\right) \qquad (28)$$

$$\lambda_{3f} = \frac{\partial J}{\partial \gamma} = 2 \times s_\gamma \times \left(\gamma(t_f) - \gamma_f\right) \qquad (29)$$

The integration method used in the optimal control problem is Eulers method. It is the usual basic method.

2.1 Implementation of Euler's method

Euler's method can be implemented both in simulator program and the optimal control problem. In simulator program the integration of $r, v \, and \, \gamma$ happens. That is given as follows.

$$r = r_0 + \dot{r} \, dt \qquad (30)$$

$$v = v_0 + \dot{v} \, dt \qquad (31)$$

$$\gamma = \gamma_0 + \dot{\gamma} \, dt \qquad (32)$$

In optimal control the λ^T is integrating to get final λ values.

$$\lambda_1 = \lambda_{10} + \dot{\lambda}_1 \, dt \qquad (33)$$

$$\lambda_2 = \lambda_{20} + \dot{\lambda}_2 \, dt \qquad (34)$$

$$\lambda_3 = \lambda_{30} + \dot{\lambda}_3 \, dt \qquad (35)$$

Where dt is the integration step size.$r_0, v_0 \& \gamma_0$are the initial values of $r, v \, and \, \gamma$. Similarly $\lambda_{10}, \lambda_{20} \& \lambda_{30}$ are the initial values of $\lambda_1, \lambda_2 \& \lambda_3$ .Thus gets the solution of equation (21) For optimal solutions, The optimality condition (Ref equation (26)) is given by

$$\frac{\partial H}{\partial \alpha} = 0 \qquad (36)$$

The weighting factor tow (τ) should select according to the steepest descent method . That is

$$toww = \frac{\partial H}{\partial \alpha} \times \left(\frac{\partial H}{\partial \alpha}\right)^T \qquad (37)$$

$$tow(\tau) = \frac{1}{\sqrt[2]{toww}} \qquad (38)$$

Thus weighting factor tow (τ) is obtained. Then it is applied to the control variable alpha (α) ,toget the new alphaprofile.

$$\alpha = \alpha_0 - \frac{\partial H}{\partial \alpha} \times tow(\tau) \qquad (39)$$

Where α_0 = privious computed α

Equation (21) is the gradiant function, optimizes the objective function. An optimal control problem needs accurate initial conditions. From the equations (1) − (3) the simulator program is developeand initial steering profile for steepest descent optimization is obtained.For finding the accurate steering profile acceleration is assumed to be a linearly increasing quantity. So rate of change of velocity is approximated as linear function. This linear function is selected by polynomial approximation that is by changing the functions randomly to fit the curve properly. The vehicle acceleration is a linear function of time as given in the equation. The constants are obtained by varying initial and final steering angle value. Thus tuning of constants$c_1 \, and \, c_2$are done. The initial and final α is thus obtained to be3.5 deg⬜and 14 deg respectively.Therefore by solving the equation (21)using the initial and final values the appropriates values for $c_1 \, and \, c_2$ can be find out;

$$c_1 = 6.4217 \times 10^{-3}$$

$$c_2 = 1.918235$$

3. SIMULATOR PROGRAM STEPS AND RESULTS

The simulator program is mainly used for generating the initial steering angle. This steering is given as the input thrust angle for trajectory optimization problem. Table (1)shows travelling time and the initial steering angle profile. The main programming steps are given below:-

a)-Develop rocket dynamics equations.

b)-To get initial alpha profile develop a simulator program that has minimum objective function values.

c)-Put this alpha profile in the trajectory optimization problem.

d)By varying costate variables optimization takes place

e)Alpha profile is updated by using steepest descent technique.

f) Alpha profile update is done by tuning of multiplication factor.

g) The optimized result will be obtained after 82 iterations.

The results obtained are shown below:-

Table 1 Travelling time and the initial steering angle profile

| Time in seconds	initial thrust angle α values in radian
1	0.275101
50	0.279912
100	0.271316
150	0.255160
200	0.235788
250	0.217232
300	0.203625
350	0.198925
400	0.206037
450	0.226118
500	0.258920
510	0.266945

Figure (1) shows the graph obtained for the simulation program.The figure indicates the radial distance from earth centre.The symbol,' * ' indicates the target, $rf = 6862.41573\ km$. By this method, the rocket achieved the desired position, but not the desired velocity. Therefore the above method is not an optimized one. So has to be optimized.
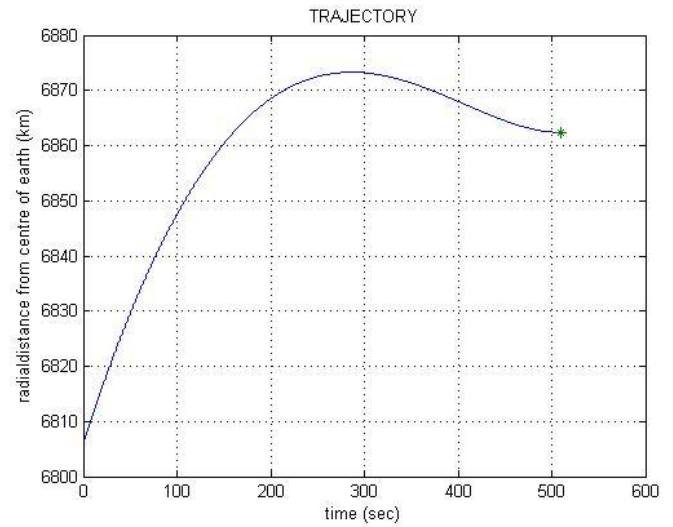
Fig 1 Radial distance from earth centreVs time
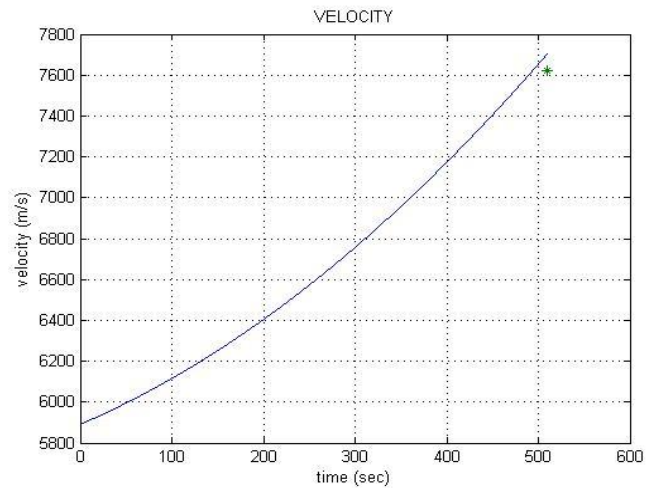
Fig 2 Velocity Vs time
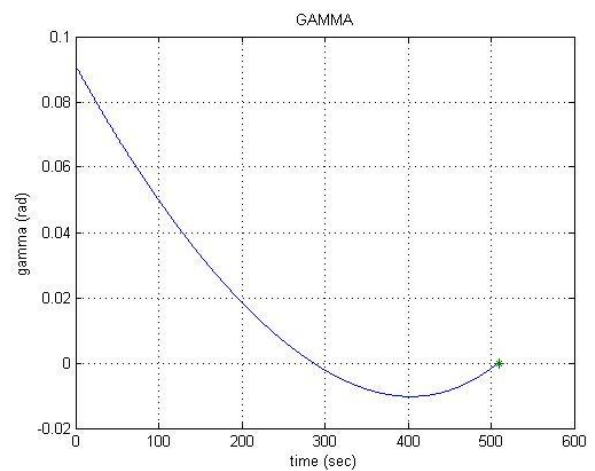
Fig 3: Flight path angle Vs time

Figure (3)& (2) indicates the flight path angle and velocity variation with respect to time.1n figure (4.8), the symbol' * ' indicates the value of final velocity, $Vf\ =\ 7623.5321\ m/s$

and in figure (3), it indicates the value of final gamma, $\gamma = 0\ deg$. In figure (2) the desired target-velocity is not achieved. The simulation program provides the basic model trajectory and the initial steering angle. The trajectory optimization problem is explained in the following sections. Euler's method of integration can be used for optimization. In Euler's method of integration the integration step size is same. Iteration results are shown below. Total iteration number = 82

Table 2 Itration number and the objective function value

Iteration number	Objective function values
1	64.0930
5	93581
10	22177
15	76128
20	19333
25	53712
30	20685
35	28653
40	20762
45	16666
50	12255
55	81863
60	4699.7
65	2037.8
70	418.6931
71	237.3277
73	30.6289
75	26.113
77	2.289
79	0.0002527

80	8.3802×10^{-6}
81	0.0002527
82	8.3802×10^{-6}

Table (2) shows the objective function values for each iteration. By analyzing the objective function values, it seems to be decreasing and tends to a minimum value. Table (3) shows final radial distance from center of earth, final velocity of rocket and final flight path angle in each iteration.

Table 3 Iterationnumber, final radial distance, final velocity and final flight path angle

Iteration number	final radial distance from centre of earth (km)	final velocity of rocket (m/s)	Final flight path angle (rad)
1	6862.4	7703.6	0.0002116
5	68655	7696.6	0.0015444
10	6863.9	7700.7	-0.0002827
15	6865.2	7697.5	-0.0004971
20	6863.8	7700.2	-0.0024106
25	6864.7	7696.5	-0.0030378
30	6863.9	7696.0	-0.0049032
35	6864.1	7691.4	-0.0061224
40	6863.9	7686.7	-0.0076483
45	6863.7	7680.5	-0.0090698
50	6863.5	7673.3	-0.010482
55	6863.3	7664.9	-0.011885
60	6863.1	7655.5	-0.013288
65	6862.9	7645.0	-0.014698
70	6862.6	7633.5	-0.016121
71	6862.6	7631.0	-0.016407
73	6862.5	7626.1	-0.016981
75	6862.4	7623.6	-0.01713

77	6862.4	7623.6	-0.017293
79	6862.4	7623.5	-0.017286
80	6862.4	7623.5	-0.017286
81	6862.4	7623.5	-0.017286
82	6862.4	7623.5	-0.017286

Table 4 Itration number and weighting factor (τ)

Iteration number	Weighting factor (τ)
1	0.0031
5	0.0044
10	0.0122
15	0.0052
20	0.0088
25	0.0059
30	0.0069
35	0.0060
40	0.0056
45	0.0054
50	0.0055
55	0.0060
60	0.0070
65	0.0096
70	0.0191
71	0.0250
73	0.0367
75	5.0936×10^{-6}
77	1.7206×10^{-5}
79	0.001649

80	0.011844
81	0.001649
82	0.011843

Table (4) shows weighting factor (τ) variation in each iteration. Weighting factor has a significant major role in optimizing the trajectory. Figure(4), (5) & (6) shows the graphs obtained for Steepest Descent method with Euler's integration. The figure (4)indicates the radial distance from earth center. The symbol, ' * ' indicates the target,$rf = 6862415.73\ m$. The above method is optimized with minimum objective function value. From the graph it can be seen that the rocket achieved the target-position .A higher altitude is considered in this method, sothat it can achieve the target-velocity.
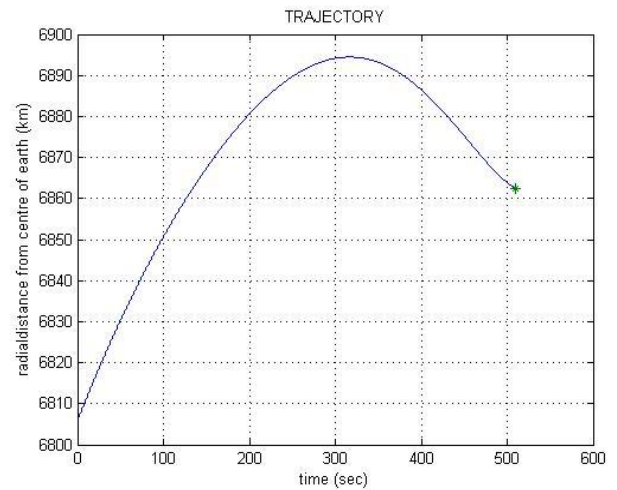
Fig.4 Radial distance from the centre of earth Vs time

Figure (5)&(6)indicates the flight path angle and velocity variation with respect to time.1n figure (5)the symbal' * ' indicate the final velocity, $Vf = 7623.5321\ m/s$ and in figure (6)the symbal' * ' indicates final gamma ,$\gamma = 0.0002deg$. In figure (4), graph achieving the target-velocity while comparing with the simulator program results (refer figure (2)).The graph shows that the desired gamma value is not achieved. But it is the optimal solution since the objective function has the minimum value.
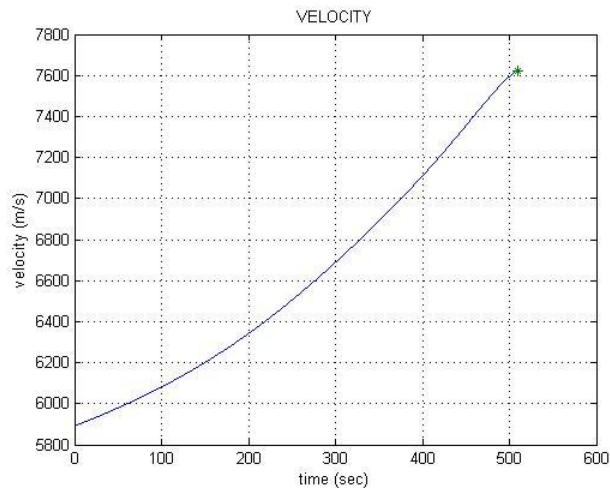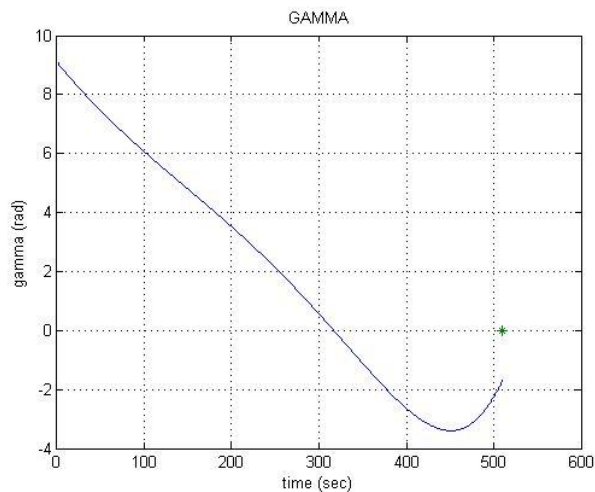
Fig 5 VelocityVstime

Fig .6 Flight path angle Vs time

5 .CONCLUSIONS

A new programming approach for steepest descent optimization of rocket trajectories is presented in this paper. From the results it is clear that the final objectives are achieved with optimum resources and time. All the constraints are satisfied and the trajectory is a optimal one. This methodology can be implemented in future missions and the results]can be verified and compared with the real time outputs. Comparison of the results with real time results and formulation of more efficient algorithms will be the future work.

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