On Fuzzy Pushdown Automata and Their Covering

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ABSTRACT
Similarity of fuzzy pushdown automata in the sense of transition and output is algebraically embodied by their homomorphism as well as covering. This vary issue is studied in this paper. The ways of obtaining new fuzzy pushdown automata by means of their product is also introduced. Furthermore, we prove that product, homomorphism and covering of fuzzy pushdown automata are internally related. Several algebraic results of homomorphism and covering are also discussed in this paper.

Keywords
Fuzzy automata; Fuzzy pushdown automata; Products; Covering; Homomorphism.

1. INTRODUCTION
Among various generalizations of an automaton, fuzzy automaton was the most widely used generalization introduced by Wee [13], while establishing model of learning system. Since then it has been used in many applications [1, 7, 10, 11, 12]. Many classes of fuzzy automata were introduced and studied in [8], few well known, are Automata with weights, L-semi-group automata, Lattice automata, Boolean automata, Semiring automata and Field automata etc. Reduction/minimization of fuzzy automaton was one the important issue in [3, 9], several methods and algorithms of reduction were proposed [3, 9]. Homomorphism and covering are two algebraic notions related to reduction of automata [4, 5]. Isomorphism gives an equivalent copy of the fuzzy automaton, while covering gives a copy of a fuzzy automaton having fewer states and equally powerful in computing. The problem of reduction of states, in terms of homomorphism and covering, for a fuzzy finite state machines was completely resolved in [6, 7], while for finite automata, in terms of congruences and homomorphism was discussed in [9]. A class, pushdown automata, of automata was generalized to fuzzy pushdown automata by Bucurescu and Pascu [2] and Xing [14]. Xing introduced non-deterministic fuzzy pushdown automata (NFPA) and found that the class of NFPA languages and fuzzy context-free K-grammar languages are equivalent, while Bucurescu and Pascu found that fuzzy pushdown automata accept context sensitive languages by setting a threshold and B-fuzzy automata accept context-free languages.

Closure properties of fuzzy languages accepted by fuzzy automata were discussed by many researchers [7]. We find following a three step mechanism in them:

Step 1: ***Given***: Acceptance of fuzzy languages by fuzzy automaton

Step 2: ***Construction***: Formation of a desired fuzzy automaton

Step 3: ***Verification***: Conclusion Closure property

For the better understanding take the example of a closure property under algebraic union. **Step 1**: Two fuzzy automata M_1 and M_2 accepting languages L_1 and L_2 are given **Step 2**: One construct a desired fuzzy automaton M with the help of M_1 and M_2 **Step 3**: verification of language of M is $L_1 \cup L_2$. One can see that the step 2 (construction) is very important part in the above mechanism for any kind of a closure property. In fact it is the motivating point of defining various products of fuzzy automata. In this paper we take this issue relating to fuzzy pushdown automata, however, in this paper we confine ourselves to step 2 of above mechanism only and postpone the closure properties till the next paper.

The distinctive feature of this work are (i) to construct new fuzzy pushdown automaton from given fuzzy pushdown automata (ii) to discuss decompositions of fuzzy pushdown automaton and (iii) to establish basic foundation for discussing closure properties of fuzzy pushdown automaton languages.

In comparison to existing research on fuzzy automata we have established various algebraic properties such associativity, commutativity, distributivity and certain sort of exchange property of products of fuzzy pushdown automata namely restricted, direct, cascade, cartesian, direct sum and sum.

In section 2 we defined six different products of fuzzy pushdown automata and discuss their interrelations. We use the definition of fuzzy pushdown automata [2] and introduced covering between them. We believe that the notions of covering and homomorphism of fuzzy pushdown automata along with the properties of products will somewhat simplify the construction part for more complex closure properties; this motivates us to introduce covering and homomorphism of fuzzy pushdown automata in section 3. We also establish interrelationship between covering and homomorphism of fuzzy pushdown automata in this section.

2. PRODUCTS OF FUZZY PUSHDOWN AUTOMATA

Recall that any non-empty finite set is called the alphabet. A finite sequence of letters is called a string or word. For simplicity a string $\{a_i\}_{i=1}^{n}$ of alphabets is written as $a_1a_2\cdots a_n$. The set Σ^+ denotes the set of all strings over the alphabet Σ. We shall denote $\Sigma^* = \Sigma^+ \cup \{\Lambda\}$, where Λ is the empty string i.e. string without alphabet letters. The set $\Delta_\Sigma = \{(\sigma,\sigma) | \sigma \in \Sigma\}$ is called the diagonal of Σ. We know that an automaton is a mathematical model of any real life system with finite number of stages/states, those changes by the action of inputs. A pushdown automaton is an extension of an automaton having stack / pushdown symbols in the sense that its adds a memory for an automaton [5]. Fuzzy automaton [7] is an extension of an automaton where the transitions are fuzzy rather than crisp. Therefore, naturally fuzzy pushdown automaton is an extension of fuzzy automaton. Mathematically it is given as:

Definition 2.1 [2] A fuzzy pushdown automaton is a 7- tuple $M = (Q, \Sigma, \Gamma, \mu, H, I, F)$, where Q is a finite non-empty set of states, Σ is a finite non-empty set of inputs, Γ is a finite non-empty set of pushdown symbols, $\mu : Q \times (\Sigma \cup \{\Lambda\}) \times \Gamma \times Q \times \Gamma^* \to [0,\ 1]$, is a fuzzy transition function, $I : Q \to [0,\ 1]$, called a set of initial fuzzy states, $H : \Gamma \to [0,\ 1]$, called a set of pushdown fuzzy symbols and $F : Q \to [0,\ 1]$ called a set of final fuzzy states.

An *FPA* $M = (Q, \Sigma, \Gamma, \mu, H, I, F)$ is called *concentrated* at Γ, if μ is restricted to $Q \times (\Sigma \cup \{\Lambda\}) \times \Gamma \times Q \times \Gamma$.

To elaborate the second step of the mechanism mentioned in the introduction, we begin to define various products of fuzzy pushdown automata.

Definition 2.2 Let $M_i = (Q_i, \Sigma, \Gamma, \mu_i, H, I_i, F_i)$ $i = 1,2$ be two fuzzy pushdown automata. Then the *restricted direct product* of M_1 and M_2 is a fuzzy pushdown automaton $M_1 \wedge M_2 = (Q_1 \times Q_2, \Sigma, \Gamma, \mu_1 \wedge \mu_2, H, I_1 \wedge I_2, F_1 \wedge F_2)$ where $\mu_1 \wedge \mu_2 : (Q_1 \times Q_2) \times (\Sigma \cup \{\Lambda\}) \times \Gamma \times (Q_1 \times Q_2) \times \Gamma^* \to [0,\ 1]$

is defined as: $\mu_1 \wedge \mu_2 \left((q_1, q_2), \sigma, z, (p_1, p_2), \alpha\right) =$

$\mu_1(q_1, \sigma, z, p_1, \alpha) \wedge \mu_2(q_2, \sigma, z, p_2, \alpha),\ \forall\ q_1, p_1 \in Q_1,$

$\forall\ q_2, p_2 \in Q_2,\ \sigma \in \Sigma,\ Z \in \Gamma,\ \alpha \in \Gamma^*,$

$I_1 \wedge I_2 : (Q_1 \times Q_2) \to [0,\ 1]$ is defined as

$I_1 \wedge I_2(q_1, q_2) = I_1(q_1) \wedge I_2(q_2), \forall\ (q_1, q_2) \in Q_1 \times Q_2$ and

$F_1 \wedge F_2 : (Q_1 \times Q_2) \to [0,\ 1]$ is defined as

$F_1 \wedge F_2(p_1, p_2) = F_1(p_1) \wedge F_2(p_2),\ \forall\ (p_1, p_2) \in Q_1 \times Q_2$

Definition 2.3 Let $M_i = (Q_i, \Sigma_i, \Gamma_i, \mu_i, H_i, I_i, F_i)$, i = 1, 2 be two fuzzy pushdown automata. Then the *full direct product* of M_1 and M_2 is a fuzzy pushdown automaton $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, \Gamma_1 \times \Gamma_2, \mu_1 \times \mu_2,\ H_1 \times H_2,$ $I_1 \times I_2,\ F_1 \times F_2)$,where

$\mu_1 \times \mu_2 : (Q_1 \times Q_2) \times \left((\Sigma_1 \cup \{\Lambda\}) \times (\Sigma_2 \cup \{\Lambda\})\right) \times$ $(\Gamma_2 \times \Gamma_2) \times (Q_1 \times Q_2) \times (\Gamma_1^* \times \Gamma_2^*) \to [0,\ 1]$ is defined as

$\mu_1 \times \mu_2\left((q_1, q_2), (\sigma_1, \sigma_2), (z_1, z_2), (p_1, p_2), (\alpha_1, \alpha_2)\right)$

$= \mu_1(q_1, \sigma_1, z_1, p_1, \alpha_1) \wedge\ \mu_2(q_2, \sigma_2, z_2, p_2, \alpha_2)$

, $H_1 \times H_2 : (\Gamma_1 \times \Gamma_2) \to [0,\ 1]$ is defined as

$H_1 \times H_2(z_1, z_2) = H_1(z_1) \wedge H_2(z_2),$

$I_1 \times I_2 : Q_1 \times Q_2 \to [0,1]$ is defined as $I_1 \times I_2(q_1, q_2) =$

$I_1(q_1) \wedge I_2(q_2)$ and $F_1 \times F_2 : Q_1 \times Q_2 \to [0,1]$

$F_1(p_1) \wedge F_2(p_2)$, $\forall (q_1, q_2), (p_1, p_2) \in (Q_1 \times Q_2),$

$(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2, (z_1, z_2) \in\ \Gamma_1 \times \Gamma_2, .(\alpha_1, \alpha_2) \in \Gamma_1^* \times \Gamma_2^*.$

Remark 2.4 Definition 2.3 reduces to Definition 2.2, if $\Sigma_1 = \Sigma_2 = \Sigma,\ \Gamma_1 = \Gamma_2 = \Gamma$ and $\mu_1 \times \mu_2$ and $H_1 \times H_2$ are restricted to $(Q_1 \times Q_2) \times \left(\Delta_\Sigma \cup \{\Lambda\}\right) \times \Delta_\Gamma \times (Q_1 \times Q_2) \times (\Delta_{\Gamma^*})$ and (Δ_Γ) respectively. The symbol Δ_A demotes the set $\{(a, a) : a \in A\}$

Definition 2.5 Let $M_i = (Q_i, \Sigma_i, \Gamma_i, \mu_i, H_i, I_i, F_i),\ i = 1, 2$ be two fuzzy pushdown automata. Let $w_\sigma : Q_2 \times \left(\Sigma_2 \cup \{\Lambda\}\right) \to \Sigma_1$, $w_z : Q_2 \times \Gamma_2 \to \Gamma_1$ be mappings. Then the concentrated fuzzy pushdown automaton $M_1 w M_2 = (Q_1 \times Q_2, \Sigma_2, \Gamma_2, \mu_1 w \mu_2, H_2, I_1 \times I_2, F_1 \times F_2)$ at Γ_2 is called the *cascade product* of M_1 and M_2, where $w = (w_\sigma, w_z)$ and $\forall\ q_1, p_1 \in Q_1,\ \forall\ q_2, p_2 \in Q_2,$ $\sigma_2 \in \Sigma_2,\ z_2, z_2' \in \Gamma_2,$ we have

$\mu_1 w \mu_2 : (Q_1 \times Q_2) \times (\Sigma_2 \cup \{\Lambda\}) \times \Gamma_2 \times (Q_1 \times Q_2) \times \Gamma_2 \to [0,\ 1]$ is defined as: $\mu_1 w \mu_2\left((q_1, q_2), \sigma_2, z_2, (p_1, p_2), z_2'\right) =$

$\mu_1(q_1, w_\sigma(q_2, \sigma_2), w_z(q_2, z_2), p_1, w_z(q_2, z_2')) \wedge \mu_2(q_2, \sigma_2, z_2, p_2, z_2'),$

$I_1 \times I_2 : Q_1 \times Q_2 \to [0,1]$ is defined as

$I_1 \times I_2(q_1, q_2) = I_1(q_1) \wedge I_2(q_2)$ and $F_1 \times F_2 : Q_1 \times Q_2 \to [0,1]$

is defined as $F_1 \times F_2(p_1, p_2) = F_1(p_1) \wedge F_2(p_2)$.

Definition 2.6 Let $M_i = (Q_i, \Sigma_i, \Gamma_i, \mu_i, H_i, I_i, F_i),\quad i = 1,2$ be two fuzzy pushdown automata such that $\Sigma_1 \cap \Sigma_2 = \varnothing$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. Then the *cartesian product* of M_1 and M_2 is fuzzy pushdown automaton $M_1 \bullet M_2 = (Q_1 \times Q_2, \Sigma_1 \cup \Sigma_2, \Gamma_1 \cup \Gamma_2,\ \mu_1 \bullet \mu_2, H_1 \bullet H_2, I_1 \bullet I_2, F_1 \bullet F_2)$, where

$\mu_1 \bullet \mu_2 : (Q_1 \times Q_2) \times (\Sigma_1 \cup \Sigma_2 \cup \{\Lambda\}) \times (\Gamma_1 \cup \Gamma_2) \times (Q_1 \times Q_2) \times$ $\Gamma_1{}^* \ \cup\ \Gamma_2{}^* \to [0,\ 1]$ is defined as:

$$\mu_1 \bullet \mu_2((q_1, q_2), \sigma, z, (p_1, p_2), \alpha) = \begin{cases} \mu_1(q_1, \sigma, z, p_1, \alpha), & \text{if } \sigma \in \Sigma_1,\ z \in \Gamma_1, \alpha \in \Gamma_1^*, q_2 = p_2 \\ \mu_2(q_2, \sigma, z, p_2, \alpha), & \text{if } \sigma \in \Sigma_2,\ z \in \Gamma_2, \alpha \in \Gamma_2^*, q_1 = p_1 \\ 0, & \text{Otherwise} \end{cases}$$

and define

$H_1 \bullet H_2 : \Gamma_1 \cup \Gamma_2 \to [0,\ 1]$ by $H_1 \bullet H_2(z) = \begin{cases} H_1(z), & \text{if } z \in \Gamma_1 \\ H_2(z), & \text{if } z \in \Gamma_2, \end{cases}$

$I_1 \times I_2 : Q_1 \times Q_2 \to [0,1]$ is defined as $I_1 \times I_2(q_1, q_2) = I_1(q_1) \wedge I_2(q_2)$

and $F_1 \times F_2 : Q_1 \times Q_2 \to [0,1]$. is defined as

$F_1 \times F_2(p_1, p_2) = F_1(p_1) \wedge F_2(p_2)$

Definition 2.7 Let $M_i = (Q_i, \Sigma_i, \Gamma_i, \mu_i, H_i, I_i, F_i),$ i = 1, 2 be two fuzzy pushdown automata such that $Q_1 \cap Q_2 = \phi,\ \Sigma_1 \cap \Sigma_2 = \phi$ and $\Gamma_1 \cap \Gamma_2 = \phi$. Then the *direct sum* of M_1 and M_2 is a fuzzy pushdown automaton

$M_1 \oplus M_2 = (Q_1 \cup Q_2, \Sigma_1 \cup \Sigma_2, \Gamma_1 \cup \Gamma_2,$

$\mu_1 \oplus \mu_2, H_1 \oplus H_2, I_1 \oplus I_2, F_1 \oplus F_2)$, where

$\mu_1 \oplus \mu_2 : (Q_1 \cup Q_2) \times (\Sigma_1 \cup \Sigma_2 \cup \{\Lambda\}) \times$

$(\Gamma_1 \cup \Gamma_2) \times (Q_1 \cup Q_2) \times \Gamma_1{}^* \cup \Gamma_2{}^* \to [0, 1]$

is defined as :

$$\mu_1 \oplus \mu_2(q,\sigma,z,p,\alpha) = \begin{cases} \mu_1(q,\sigma,z,p,\alpha), & \text{if } q,p \in Q_1, \sigma \in \sum_1, z \in \Gamma_1, \alpha \in \Gamma^*{}_1 \\ \mu_2(q,\sigma,z,p,\alpha), & \text{if } q,p \in Q_2, \sigma \in \sum_2, z \in \Gamma_2, \alpha \in \Gamma^*{}_2 \\ 1, & \text{if } (q,\sigma,z) \in Q_1 \times \sum_1 \times \Gamma_1, (p,\alpha) \in Q_2 \times \Gamma^*{}_2 \\ & \text{or } (q,\sigma,z) \in Q_2 \times \sum_2 \times \Gamma_2, (p,\alpha) \in Q_1 \times \Gamma^*{}_1 \\ 0, & \text{Otherwise} \end{cases}$$

$H_1 \oplus H_2 : \Gamma_1 \cup \Gamma_2 \to [0, 1]$ is defined by $H_1 \oplus H_2(z) = \begin{cases} H_1(z), & \text{if } z \in \Gamma_1 \\ H_2(z), & \text{if } z \in \Gamma_2, \end{cases}$

$I_1 \oplus I_2 : Q_1 \cup Q_2 \to [0,1]$ is defined as :

$$I_1 \oplus I_2(q) = \begin{cases} I_1(q), & q \in Q_1 \\ I_2(q), & q \in Q_2 \end{cases}$$

and $F_1 \oplus F_2 : Q_1 \cup Q_2 \to [0,1]$ is defined as :

$$F_1 \oplus F_2(p) = \begin{cases} F_1(p), & p \in Q_1 \\ F_2(p), & p \in Q_2 \end{cases}$$

Definition 2.8 Let $M_i = (Q_i, \sum_i, \Gamma_i, \mu_i, H_i, I_i, F_i)$, $i = 1,2$ be two fuzzy pushdown automata such that $Q_1 \cap Q_2 = \phi$, $\sum_1 \cap \sum_2 = \phi$ and $\Gamma_1 \cap \Gamma_2 = \phi$. Then the *sum* of M_1 and M_2, is a fuzzy pushdown automaton $M_1 + M_2 = (Q_1 \cup Q_2, \sum_1 \cup \sum_2, \Gamma_1 \cup \Gamma_2, \mu_1 + \mu_2, H_1 + H_2,$ $I_1 + I_2,\ F_1 + F_2)$, where

$\mu_1 + \mu_2 : (Q_1 \cup Q_2) \times (\sum_1 \cup \sum_2 \cup \{\Lambda\}) \times (\Gamma_1 \cup \Gamma_2) \times (Q_1 \times Q_2) \times$ $\Gamma_1{}^* \cup \Gamma_2{}^* \to [0, 1]$ is defined as :

$$\mu_1 + \mu_2(q,\sigma,z,p,\alpha) = \begin{cases} \mu_1(q,\sigma,z,p,\alpha), & \text{if } q,p \in Q_1, \sigma \in \sum_1, z \in \Gamma_1, \alpha \in \Gamma^*{}_1 \\ \mu_2(q,\sigma,z,p,\alpha), & \text{if } q,p \in Q_2, \sigma \in \sum_2, z \in \Gamma_2, \alpha \in \Gamma^*{}_2 \\ 0, & \text{Otherwise} \end{cases}$$

and define $H_1 + H_2 : \Gamma_1 \cup \Gamma_2 \to [0, 1]$ by

$$H_1 + H_2(z) = \begin{cases} H_1(z), & \text{if } z \in \Gamma_1 \\ H_2(z), & \text{if } z \in \Gamma_2 \end{cases}$$

$I_1 + I_2 : Q_1 \cup Q_2 \to [0,1]$ is defined as :

$$I_1 + I_2(q) = \begin{cases} I_1(q), & q \in Q_1 \\ I_2(q), & q \in Q_2 \end{cases}$$

and $F_1 + F_2 : Q_1 \cup Q_2 \to [0,1]$ is defined as :

$$F_1 + F_2(p) = \begin{cases} F_1(p), & p \in Q_1 \\ F_2(p), & p \in Q_2 \end{cases}$$

2. **Remark 2.9** If $\mu_1 \oplus \mu_2 |_{Q_1 \times \sum_1 \times \Gamma_1 \times Q_2 \times \Gamma_2{}^*} = \phi$ $= \mu_1 \oplus \mu_2 |_{Q_2 \times \sum_2 \times \Gamma_2 \times Q_1 \times \Gamma_1{}^*}$, then the Definition 2.7 and 2.8 coincides.

3. COVERING AND HOMOMORPHISM OF FUZZY PUSHDOWN AUTOMATA

In this section the notion of covering of FPA is introduced. We have established relation between homomorphism of *FPA* and covering, beside various properties of covering.

Definition 3.1 Let $M_i = (Q_i, \sum_i, \Gamma_i, \mu_i, H_i, I_i, F_i)$, $i = 1,\ 2$ be two fuzzy pushdown automata. Let $\eta : Q_2 \to Q_1$ be a surjective partial function, let $\psi : \sum_1 \to \sum_2$ and $\delta : \Gamma_1 \to \Gamma_2$ be functions. Then the triplet (η, ψ, δ) is called the covering from M_1 to M_2, symbolically $(\eta, \psi, \delta) : M_1 \to M_2$, if for all $\sigma \in \sum_1, z \in \Gamma_1, \alpha \in \Gamma_1{}^*$ and q, p belongs to domain of η, we have

(i) $\mu_1(\eta(q), \sigma, z, \eta(p), \alpha) \leq \mu_2(q, \psi(\sigma), \delta(z), p, \delta(\alpha))$
(ii) $I_1(\eta(q)) \leq I_2(q)$
(iii) $F_1(\eta(p)) \leq F_2(p)$ and
(iv) $H_1(z) \leq H_2(\delta(z))$

Theorem 3.2 The covering relation is reflexive and transitive, but not symmetric.

The inter relationship between products of fuzzy pushdown automata enable us to ensure that

Theorem 3.3 Let $M_i = (Q_i, \sum_i, \Gamma_i, \mu_i, H_i, I_i, F_i)$ be fuzzy pushdown automata, $i = 1, 2$. Then

(a) $M_1 \wedge M_2 \leq M_1 \times M_2$, when $\sum_1 = \sum_2 = \sum$, $\Gamma_1 = \Gamma_2 = \Gamma$ and $H_1 = H_2 = H$

(b) $M_1 + M_2 \leq M_1 \oplus M_2$, when $Q_1 \cap Q_2 = \phi$, $\sum_1 \cap \sum_2 = \phi$ and $\Gamma_1 \cap \Gamma_2 = \phi$

Proof (a) Let η be the identity function and define $\psi : \sum \to \sum \times \sum$ by $\psi(\sigma) = (\sigma, \sigma)$ and

$\delta : \Gamma \to \Gamma \times \Gamma$ by $\delta(z) = (z, z)$ Then,

$\mu_1 \wedge \mu_2(\eta(q_1, q_2), \sigma, z, \eta(p_1, p_2), \alpha)$

$= \mu_1 \times \mu_2((q_1, q_2), (\sigma, \sigma), (z, z), (p_1, p_2), (\alpha, \alpha))$

$= \mu_1 \times \mu_2((q_1, q_2), \psi(\sigma), \delta(z), (p_1, p_2), \delta(\alpha)),$

Clearly, $I_1 \wedge I_2(\eta(q_1, q_2)) = I_1 \wedge I_2(q_1, q_2)$

$= I_1(q_1) \wedge I_2(q_2) = I_1 \times I_2(q_1, q_2)$

Similarly, $F_1 \wedge F_2(\eta(p_1, p_2)) = F_1 \times F_2(p_1, p_2)$
and $H(z) = H(z) \wedge H(z) = H \times H(z, z) = H \times H(\delta(z))$ Hence, (η, ψ, δ) is a covering.

(b) Set η,ψ and δ as identity functions.

Case(i) If $q,p \in Q_1$, $\sigma \in \sum_1, z \in \Gamma_1, \alpha \in \Gamma_1{}^*$, then $\mu_1 + \mu_2(\eta(q),\sigma,z,\eta(p),\alpha) = \mu_1(q,\sigma,z,p,\alpha)$

$= \mu_1 \oplus \mu_2(q,\psi(\sigma),\delta(z),p,\delta(\alpha))$,

$H_1 + H_2(z) = H_1(z) = H_1 \oplus H_2(z) = H_1 \oplus H_2(\delta(z))$

Similarly, $I_1 + I_2(\eta(q_1,q_2)) = I_1 \oplus I_2(q_1,q_2)$

and $F_1 + F_2(\eta(p_1,p_2)) = F_1 \oplus F_2(p_1,p_2)$

Case(ii) If $q,p \in Q_2$, $\sigma \in \sum_2, z \in \Gamma_2, \alpha \in \Gamma_2{}^*$, then $\mu_1 + \mu_2(\eta(q),\sigma,z,\eta(p),\alpha) = \mu_2(q,\sigma,z,p,\alpha) =$

$\mu_1 \oplus \mu_2(q,\psi(\sigma),\delta(z),p,\delta(\alpha))$,

$H_1 + H_2(z) = H_2(z) = H_1 \oplus H_2(z) = H_1 \oplus H_2(\delta(z))$

Similarly,

$I_1 + I_2(\eta(q_1,q_2)) = I_1 \oplus I_2(q_1,q_2)$ and
$F_1 + F_2(\eta(p_1,p_2)) = F_1 \oplus F_2(p_1,p_2)$

Case(iii) If $(q,\sigma,z) \in Q_1 \times \sum_1 \times \Gamma_1$ and $(p,\alpha) \in Q_2 \times \Gamma_2{}^*$ or $(q,\sigma,z) \in Q_2 \times \sum_2 \times \Gamma_2$, and $(p,\alpha) \in Q_1 \times \Gamma_1{}^*$,
then $\mu_1 + \mu_2(\eta(q),\sigma,z,\eta(p),\alpha) = 0 < 1$
$= \mu_1 \oplus \mu_2(q,\psi(\sigma),\delta(z),p,\delta(\alpha))$,

In all other cases $\mu_1 + \mu_2(\eta(q),\sigma,z,\eta(p),\alpha)$

$= 0 = \mu_1 \oplus \mu_2(q,\psi(\sigma),\delta(z),p,\delta(\alpha))$,

Theorem 3.4 Let $M_i = (Q_i,\sum_i,\Gamma_i,\mu_i,H_i,I_i,F_i)$ be fuzzy pushdown automata, $i = 1,2$, $Q_1 \cap Q_2 = \phi$, $\sum_1 \cap \sum_2 = \phi$ and $\Gamma_1 \cap \Gamma_2 = \phi$. Then

(a). $M_1 \le M_1 \oplus M_2$ **(b)** $M_2 \le M_1 \oplus M_2$

(c). $M_1 \le M_1 + M_2$ **(d)**. $M_2 \le M_1 + M_2$

Proof We discuss (a) only. Let $\eta : Q_1 \cup Q_2 \to Q_1$ be a partial surjective mapping defined by $\eta(q) = q,\ \forall\ q \in Q_1$. Consider $\psi : \sum_1 \to \sum_1 \cup \sum_2$ and $\delta : \Gamma_1 \to \Gamma_1 \cup \Gamma_2$ as the inclusion mappings. Clearly , (η,ψ,δ) is a covering.

Theorem 3.5 Let $M_i = (Q_i,\sum_i,\Gamma_i,\mu_i,H_i,I_i,F_i)$ be a fuzzy pushdown automata, $i = 1,2,3$ such that $M_1 \le M_2$. Then

a. $M_1 \wedge M_3 \le M_2 \wedge M_3$ **b.** $M_3 \wedge M_1 \le M_3 \wedge M_2$, if $\sum_1 = \sum_2 = \sum_3$, $\Gamma_1 = \Gamma_2 = \Gamma_3$ and $H_1 = H_2 = H_3$

c. $M_1 \times M_3 \le M_2 \times M_3$ **d.** $M_3 \times M_1 \le M_3 \times M_2$

e. $M_1\,\omega^1\,M_3 \le M_2\,\omega^2\,M_3$ **f.** $M_3\omega^1\,M_1 \le M_3\,\omega^2 M_2$, if ω^2 is determined by ω^1 in a natural way.

g. $M_1 \oplus M_3 \le M_2 \oplus M_3$ **h.** $M_3 \oplus M_1 \le M_3 \oplus M_2$,
If $Q_i \cap Q_j = \phi$, $\sum_i \cap \sum_j = \phi$ and $\Gamma_i \cap \Gamma_j = \phi$
for $i,j = 1,2,3$

i. $M_1 + M_3 \le M_2 + M_3$ **j.** $M_3 + M_1 \le M_3 + M_2$,
If $Q_i \cap Q_j = \phi$, $\sum_i \cap \sum_j = \phi$
and $\Gamma_i \cap \Gamma_j = \phi$ for $i,j = 1,2,3$

Proof Let $\eta : Q_2 \to Q_1$ and $\psi : \sum_1 \to \sum_2$, $\delta : \Gamma_1 \to \Gamma_2$ be such that $\mu_1(\eta(q),\sigma,z,\eta(p),\alpha) \le \mu_2(q,\psi(\sigma),\delta(z),p,\delta(\alpha))$,
$I_1(\eta(q)) \le I_2(q)$,
$F_1(\eta(p)) \le F_2(p)$ and $H_1(z) \le H_2(\delta(z))$

(a) Define $\eta' : Q_2 \times Q_3 \to Q_1 \times Q_3$ by .

$\eta'(q_2,q_3) = (\eta(q_2),q_3)$. Consider ψ',δ' as identity functions on $\sum$, Γ respectively.

(b) Similar to **(a)**

(c) Define $\eta' : Q_2 \times Q_3 \to Q_1 \times Q_3$,
by $\eta'(q_2,q_3) = (\eta(q_2),q_3)$
$\psi' : \sum_1 \times \sum_3 \to \sum_2 \times \sum_3$ by $\psi'(\sigma_1,\sigma_3) = (\psi(\sigma_1),\sigma_3)$
and $\delta' : \Gamma_1 \times \Gamma_3 \to \Gamma_2 \times \Gamma_3$
by $\delta'(z_1,z_3) = (\delta(z_1),z_3)$.

One can verify easily that

$\mu_1 \times \mu_3(\eta'(q_2,q_3),(\sigma_1,\sigma_3),$
$(z_1,z_3),\eta'(p_2,p_3),(\alpha_1,\alpha_3))$
$\le \mu_2 \times \mu_3((q_2,q_3),\psi'(\sigma_1,\sigma_3),$

$\delta'(z_1,z_3),(p_2,p_3),\delta'(\alpha_1,\alpha_3))$
and $I_1 \times I_3(\eta'(q_2,q_3)) \le I_2 \times I_3(q_2,q_3)$;

$F_1 \times F_3(\eta'(p_2,p_3)) = F_2 \times F_3(p_2,p_3)$
and $H_1 \times H_3(z_1,z_3) \le H_2 \times H_3(\delta'(z_1,z_3))$

Hence, (η',ψ',δ') is a required covering.

(d) Define $\eta' : Q_3 \times Q_2 \to Q_3 \times Q_1$,

by $\eta'(q_3,q_2)=(q_3,\eta(q_2))$
$\psi':\Sigma_3\times\Sigma_1\to\Sigma_3\times\Sigma_2$ by $\psi'(\sigma_3,\sigma_1)=(\sigma_3,\psi(\sigma_1))$ and $\delta':\Gamma_1\times\Gamma_3\to\Gamma_2\times\Gamma_3$ by $\delta'(z_3,z_1)=(z_3,\delta(z_1))$.

Then (η',ψ',δ') is a covering

(e) For given $w^1=(w^1{}_\sigma,w^1{}_z)$,

where $w^1{}_\sigma:Q_3\times\Sigma_3\to\Sigma_1$, $w^1{}_z:Q_3\times\Gamma_3\to\Gamma_1$ we denote $w^2=(w^2{}_\sigma,w^2{}_z)$,
where $w^2{}_\sigma:Q_3\times\Sigma_3\to\Sigma_2$, $w^2{}_z:Q_3\times\Gamma_3\to\Gamma_2$ such that $\psi\circ w_\sigma{}^1=w_\sigma{}^2$ and $\delta\circ w_z{}^1=w_z{}^2$

Define $\eta':Q_2\times Q_3\to Q_1\times Q_3$ *by* $\eta'(q_2,q_3)=(\eta(q_2),q_3)$ and take ψ',δ' as identity functions on Σ_3, Γ_3 respectively.

Then

$$\mu_1w^1\mu_3(\eta'(q_2,q_3),\sigma_3,z_3,\eta'(p_2,p_3),z_3')=$$
$$\mu_1w^1\mu_3((\eta(q_2),q_3),\sigma_3,z_3,(\eta(p_2),p_3),z_3')$$

$$=\mu_1(\eta(q_2),w^1{}_\sigma(q_3,\sigma_3),w^1{}_z(q_3,z_3),\eta(p_2),w^1{}_z(q_3,z_3'))\wedge\mu_3(q_3,\sigma_3,z_3,p_3,z_3')$$
$$\le\mu_2(q_2,\psi(w^1{}_\sigma(q_3,\sigma_3)),\delta(w^1{}_z(q_3,z_3)),p_2,\delta(w^1{}_z(q_3,z_3')))\wedge\mu_3(q_3,\sigma_3,z_3,p_3,z_3')$$
$$=\mu_2(q_2,w^2{}_\sigma(q_3,\sigma_3),w^2{}_z(q_3,z_3),p_2,w^2{}_z(q_3,z_3'))\wedge\mu_3(q_3,\sigma_3,z_3,p_3,z_3')$$
$$=\mu_2w^2\mu_3((q_2,q_3),\sigma_3,z_3,(p_2,p_3),z_3')$$
$$=\mu_2w^2\mu_3((q_2,q_3),\psi'(\sigma_3),\delta'(z_3),(p_2,p_3),\delta'(z_3')),$$

Clearly, $I_1\wedge I_3(\eta'(q_2,q_3))=I_1\wedge I_3(\eta(q_2),q_3)=I_1(\eta(q_2))\wedge I_3(q_3)\le I_2(q_2)\wedge I_3(q_3)=I_2\wedge I_3(q_2,q_3)$ and $F_1\wedge F_3(\eta'(p_2,p_3))=F_2\wedge F_3(p_2,p_3)$

Hence, (η',ψ',δ') is a required covering.

(f) Given $w^1=(w^1{}_\sigma,w^1{}_z)$, where $w^1{}_\sigma:Q_1\times\Sigma_1\to\Sigma_3$, $w^1{}_z:Q_1\times\Gamma_1\to\Gamma_3$ define functions

$w^2{}_\sigma:Q_2\times\Sigma_2\to\Sigma_3$ and $w^2{}_z:Q_2\times\Gamma_2\to\Gamma_3$ as $w^1{}_\sigma(\eta(q_2),\sigma_1)=w^2{}_\sigma(q_2,\psi(\sigma_1))$ and $w^1{}_z(\eta(q_2),z_1)=w^2{}_z(q_2,\delta(z_1))$ .Denote $w^2=(w^2{}_\sigma,w^2{}_z)$.

Since η is a surjective function, ψ,δ are functions and Σ,Γ are finite, such w^2 exists, but not unique.

Clearly $\eta':Q_3\times Q_2\to Q_3\times Q_1$ defined by $\eta'(q_3,q_2)=(q_3,\eta(q_2))$ defines a required covering (η',ψ,δ) .

(g) Define

$\eta':Q_2\cup Q_3\to Q_1\cup Q_3$ by $\eta'(q)=\begin{cases}\eta(\text{q}), & \text{if } q\in Q_2\\ q, & \text{otherwise}\end{cases}$ and
$\psi':\Sigma_1\cup\Sigma_3\to\Sigma_2\cup\Sigma_3$ and $\delta':\Gamma_1\cup\Gamma_3\to\Gamma_2\cup\Gamma_3$ respectively as $\psi'(\sigma)=\begin{cases}\psi(\sigma), & \text{if } \sigma\in\Sigma_1\\ \sigma, & \text{otherwise}\end{cases}$ and

$\delta'(\text{z})=\begin{cases}\delta(\text{z}), & \text{if z}\in\Gamma_1\\ z, & \text{otherwise}\end{cases}$ Then

for $q,p\in Q_2,\sigma\in\Sigma_2,z\in\Gamma_2,\alpha\in\Gamma^*{}_2$

or $q,p\in Q_3,\sigma\in\Sigma_3,z\in\Gamma_3,\alpha\in\Gamma^*{}_3$, we have

$$\mu_1\oplus\mu_3(\eta'(q),\sigma,z,\eta'(p),\alpha)$$
$$\le\mu_2\oplus\mu_3(q,\psi'(\sigma),\delta'(z),p,\delta'(\alpha))$$

If $(q,\sigma,z)\in Q_1\times\Sigma_1\times\Gamma_1$, $(\text{p},\alpha)\in\text{Q}_3\times\Gamma_3{}^*$

or $(q,\sigma,z)\in Q_3\times\Sigma_3\times\Gamma_3$, $(\text{p},\alpha)\in\text{Q}_1\times\Gamma_1{}^*$ and $(q,\sigma,z)\in Q_2\times\Sigma_2\times\Gamma_2$, $(\text{p},\alpha)\in\text{Q}_3\times\Gamma_3{}^*$

or $(q,\sigma,z)\in Q_3\times\Sigma_3\times\Gamma_3$, $(\text{p},\alpha)\in\text{Q}_2\times\Gamma_2{}^*$, then

$$\mu_1\oplus\mu_3(\eta'(q),\sigma,z,\eta'(p),\alpha)=1$$

$$=\mu_2\oplus\mu_3(q,\psi'(\sigma),\delta'(z),p,\delta'(\alpha)).$$

In all other remaining cases we have equality follows directly by the definition.

Clearly, $I_1\oplus I_3(\eta'(q))\le I_2\oplus I_3(q)$,
$F_1\oplus F_3(\eta'(p))\le F_2\oplus F_3(p)$ and
$H_1\oplus H_3(z)\le H_2\oplus H_3(\delta(z))=H_2\oplus H_3(\delta'(z))$

One can prove (h), (i) and (j) similar to (g).

We now define homomorphism of fuzzy pushdown automata and find its relation with the covering.

Definition 3.6 Let $M_i=(Q_i,\Sigma_i,\Gamma_i,\mu_i,H_i,I_i,F_i)$, $i=1,2$ be two fuzzy pushdown automata. Let $\delta_1:Q_1\to Q_2$, $\delta_2:\Sigma_1\to\Sigma_2$ and $\delta_3:\Gamma_1\to\Gamma_2$ be functions. Then the triplet $(\delta_1,\delta_2,\delta_3)$ is called a *fuzzy pushdown automaton homomorphism* from M_1 to M_2 , symbolically $(\delta_1,\delta_2,\delta_3):M_1\to M_2$, if for all $q,p\in Q_1,\sigma\in\Sigma_1,z\in\Gamma_1,\alpha\in\Gamma_1{}^*$,

(i) $\mu_1(q,\sigma,z,p,\alpha)\le\mu_2(\delta_1(q),\delta_2(\sigma),\delta_3(z),\delta_1(p),\delta_3(\alpha))$
(ii) $I_1(q)\le I_2(\delta_1(q))$
(*iii*) $F_1(p)\le F_2(\alpha(p))$ and
(*iv*) $H_1(z)\le H_2(\delta_3(z))$

The homomorphism $(\delta_1,\delta_2,\delta_3)$ is called *monomorphism* (*epimorphism*, *isomorphism*), if the functions δ_1,δ_2 and δ_3

are injective (surjective, bijective respectively). In the case of an isomorphism, we shall write $M_1 \cong M_2$.

If equality holds simultaneously in all the conditions of the above definition, then $(\delta_1, \delta_2, \delta_3)$ is termed as a *strong homomorphism*.

Theorem 3.7 Let $M_i = (Q_i, \sum_i, \Gamma_i, \mu_i, H_i, I_i, F_i)$, $i = 1,2$ be two fuzzy pushdown automata. If $(\alpha, \beta, \gamma) : M_1 \to M_2$ is a homomorphism with α injective, then $M_1 \le M_2$.

Proof Denote $\eta = \alpha^{-1}, \psi = \beta \text{ and } \delta = \gamma$. Then (η, ψ, δ) is the required covering.

The above theorem lead to
Corollary 3.8 Isomorphic fuzzy pushdown automata covers each other.

Theorem 3.9 Let $M_i = (Q_i, \sum_i, \Gamma_i, \mu_i, H_i, I_i, F_i)$, $i = 1,2$ be two fuzzy pushdown automata. If $(\alpha, \beta, \gamma) : M_1 \to M_2$ is a strong epimorphism, then $M_2 \le M_1$.

Proof Since β and γ are surjective, for $\sigma_2 \in \sum_2, z_2 \in \Gamma_2, \alpha_2 \in {\Gamma_2}^*$ there exists at least one $\sigma_1 \in \sum_1, z_1 \in \Gamma_1, \alpha_1 \in {\Gamma_1}^*$ such that $\beta(\sigma_1) = \sigma_2, \gamma(z_1) = z_2 \text{ and } \gamma(\alpha_1) = \alpha_2$. By axiom of choice select one such $\sigma_1 \in \sum_1, z_1 \in \Gamma_1$ and define functions $\psi : \sum_2 \to \sum_1$ by $\psi(\sigma_2) = \sigma_1$ and $\delta : \Gamma_2 \to \Gamma_1$ by $\delta(z_2) = z_1$.

Now, letting $\eta = \alpha$, one can easily checked that ((η, ψ, δ) is the covering $M_2 \le M_1$.

We now state few more properties of isomorphism, those are consequences of the definition only and hence we omit their proofs.

Theorem 3.10 Let $M_i = (Q_i, \sum_i, \Gamma_i, \mu_i, H_i, I_i, F_i)$, $i = 1,2,3$, be a fuzzy pushdown automata. Then following are true whenever the products are defined

a. $M_1 \wedge (M_2 \oplus M_3) \cong (M_1 \wedge M_2) \oplus (M_1 \wedge M_3)$

b. $M_1 \wedge (M_2 + M_3) \cong (M_1 \wedge M_2) + (M_1 \wedge M_3)$

c. $M_1 \times (M_2 \oplus M_3) \cong (M_1 \times M_2) \oplus (M_1 \times M_3)$

d. $M_1 \times (M_2 + M_3) \cong (M_1 \times M_2) + (M_1 \times M_3)$

g. $M_1 \bullet (M_2 \oplus M_3) \cong (M_1 \bullet M_2) \oplus (M_1 \bullet M_3)$

h. $M_1 \bullet (M_2 + M_3) \cong (M_1 \bullet M_2) + (M_1 \bullet M_3)$

Further,

e. $M_1 \omega (M_2 \oplus M_3) \cong (M_1 \omega^1 M_2) \oplus (M_1 \omega^2 M_3)$

f. $M_1 \omega (M_2 + M_3) \cong (M_1 \omega^1 M_2) + (M_1 \omega^2 M_3)$,

where ω^1 and ω^2 are determined by ω in a natural way.

Following theorem is exchange type of products in relation to covering

Theorem 3.11 Let $M_i = (Q_i, X_i, Y_i, \delta_i, \sigma_i)$ be a fuzzy pushdown automata, $i = 1,2,3,4$. Then

a. $(M_1 \wedge M_2) \times (M_3 \wedge M_4) \le (M_1 \times M_3) \wedge (M_2 \times M_4)$,
if $\sum_1 = \sum_2,\ \sum_3 = \sum_4$;
$\Gamma_1 = \Gamma_2, \Gamma_3 = \Gamma_4$ and $H_1 = H_2, H_3 = H_4$

Further, for given ω^1 and ω^2 one can naturally define ω^3 such that

b. $(M_1 \omega^1 M_2) \times (M_3 \omega^2 M_4) \le (M_1 \times M_3) \omega^3 (M_2 \times M_4)$

c. $(M_1 w^1 M_2) \wedge (M_3 w^2 M_4) \le (M_1 \times M_3) w^3 (M_2 \times M_4)$

d. $(M_1 \omega^1 M_2) \wedge (M_3 \omega^2 M_4) \le (M_1 \wedge M_3) \omega^3 (M_2 \wedge M_4)$

e. $(M_1 \omega^1 M_2) \bullet (M_3 \omega^2 M_4) \le (M_1 \bullet M_3) \omega^3 (M_2 \bullet M_4)$,
if $\sum_i$'s and Γ_i's are pair - wise disjoint.

Proof: We prove (b) and (e) only.

Proof: Let $w^1 = ({w^1}_\sigma, {w^1}_z)$ and $w^2 = ({w^2}_\sigma, {w^2}_z)$.

(b) Define
${w^3}_\sigma : (Q_2 \times Q_4) \times (\sum_2 \times \sum_4) \to \sum_1 \times \sum_3$ and
${w^3}_z : (Q_2 \times Q_4) \times (\Gamma_2 \times \Gamma_4) \to \Gamma_1 \times \Gamma_3$ as

${w^3}_\sigma((q_2, q_4), (\sigma_2, \sigma_4))$.

$$= ({w^1}_\sigma(q_2, \sigma_2), {w^2}_\sigma(q_4, \sigma_4), {w^3}_z((q_2, q_4), (z_2, z_4))$$

$$= ({w^1}_z(q_2, z_2), {w^2}_z(q_4, z_4)$$

Clearly $w^3 = ({w^3}_\sigma, {w^3}_z)$,
$\eta((q_1, q_3), (q_2, q_4)) = ((q_1, q_2), (q_3, q_4))$ and ψ, δ as identity functions on $\sum_2 \times \sum_4$, $\Gamma_2 \times \Gamma_4$ respectively determines a required covering.

(e) Define ${w^3}_\sigma : (Q_2 \times Q_4) \times (\sum_2 \cup \sum_4) \to \sum_1 \cup \sum_3$ and ${w^3}_z : (Q_2 \times Q_4) \times (\Gamma_2 \cup \Gamma_4) \to \Gamma_1 \cup \Gamma_3$

Respectively as

$${w^3}_\sigma((q_2, q_4), \sigma) = \begin{cases} {w^1}_\sigma(q_2, \sigma), & \text{if } \sigma \in \sum_2 \\ {w^2}_\sigma(q_4, \sigma), & \text{if } \sigma \in \sum_4 \end{cases} \text{ and}$$

$${w^3}_z((q_2, q_4), z) = \begin{cases} {w^1}_z(q_2, z), & \text{if } z \in \Gamma_2 \\ {w^2}_z(q_4, z), & \text{if } z \in \Gamma_4 \end{cases}$$

Denote $w^3 = (w^3{}_\sigma, w^3{}_z)$, Clearly $\eta((q_1,q_3),(q_2,q_4)) = ((q_1,q_2),(q_3,q_4))$ and ψ, δ as identity functions on $\sum_2 \cup \sum_4$, $\Gamma_2 \cup \Gamma_4$ respectively, determines a required covering.

4. CONCLUSION

Algebraic counterparts namely covering and homomorphism of reduction and equivalent respectively of fuzzy pushdown automata are discussed. Covering between various products of fuzzy pushdown automata are established. It is expected that the theory developed in this paper will be helpful in establishing equivalence between the languages accepted by fuzzy pushdown automata and fuzzy context-free languages.

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