International Journal of Computer Applications (0975 – 8887)
Volume 46– No.21, May 2012

Fuzzification of Heat Equation

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ABSTRACT
In this paper we are in search of fuzzy solution of one dimensional Heat Equation. Here we have observed the findings if we use fuzzy intervals at different grid points as per finite difference method of numerical solution of partial differential equation. Bender-Schmidt Recurrence Scheme is used in solving the one dimensional Heat Equation numerically.

Keywords
fuzzy membership function(f.m.f.), triangular fuzzy number(t.f.n.), , α-cut, finite difference, fuzzy interval(F.I.)

1. INTRODUCTION
The concept of Fuzzy differential equation was first introduced by Chang and Zadeh [1]. Dubois and Prade [2] has given the extension principle. In this paper, we first applied finite difference method to solve one dimensional heat equation numerically ,then fuzzified.

2. BASIC CONCEPT AND DEFINITIONS
A triangular Fuzzy number μ is defined by three real numbers with base as the interval [a,c] and b as the vertex of the triangle.The membership functions are defined as follows:

$$\mu(x)(=)\begin{cases} \frac{x-a}{b-a}; & where\ a \le x \le b \\ \frac{x-c}{b-c}; & where\ b \le x \le c \\ 0 & ; \quad otherwise \end{cases}$$ where α-cuts are given by

$$\Delta_L(\alpha)(=)a+\alpha(b-a) \qquad and \qquad \Delta_R(\alpha)(=)c+\alpha(b-c)$$

3. PRELIMINARIES OF HEAT QUATION
The one dimensional Heat Equation, which is parabolic in nature is $\frac{\partial u}{\partial t}(=)\alpha^2\frac{\partial^2 u}{\partial x^2}$ -------------------------------------(1)
whose solution is $u(x,t)$ with x and t as space and time coordinates respectively within the domain $0 \le x \le l; 0 \le t \le \infty$. Solution of (1) is not defined in a closed form but propagates in an open ended region from initial values satisfying the prescribed boundary conditions as shown in Fig.1.

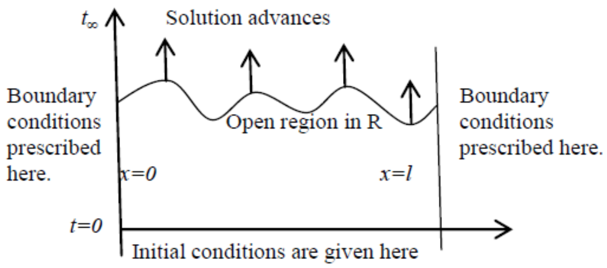

Fig-1

3.1 Bender-Schmidt Method
To solve Heat Equation (1) subject to the boundary conditions
$u(0,t)(=)T_0$, $u(l,t)(=)T_l$ and initial condition $u(x,0)(=)f(x)$, hence using finite difference method and substituting

$$u_{xx}(=)\frac{1}{h^2}(u_{i+1,j}-2u_{i,j}+u_{i-1,j}) \qquad \text{and}$$

$$u_t(=)\frac{1}{k}(u_{i,j+1}-u_{i,j})$$

we get $u_{i,j+1}(=)\lambda u_{i+1,j}+(1-2\lambda)u_{i,j}+\lambda u_{i-1,j}$ -------------(2)

where $\lambda = k/h^2 a$ and $a = 1/\alpha^2$.

If h and k are so choosen that coefficient of $u_{i,j}$ vanishes, i.e. λ = ½. Hence (2) becomes

$u_{i,j+1}(=)\frac{1}{2}[u_{i+1,j}+u_{i-1,j}]$ --------------------------------------(3)

which is called Bender-Schmidt recurrence equation.

4. FUZZIFICATION OF HEAT EQUATION
Let $u_{i-1,j}(=)[E_{01},E_{02},E_{03}]$ and $u_{i+1,j}(=)[E_{21},E_{22},E_{23}]$

Hence $[u_{i,j+1}]^{(F.I.)}(=)\left[\frac{E_{01}+E_{21}}{2},\frac{E_{02}+E_{22}}{2},\frac{E_{03}+E_{23}}{2}\right]$

Now fuzzy membership function for $u_{i-1,j}$ is

$$\mu_{u_{i-1,j}}(X)(=)\begin{cases} \frac{X-E_{01}}{E_{02}-E_{01}}; & where\ E_{01} \le X \le E_{02} \\ \frac{-X+E_{03}}{E_{03}-E_{02}}; & where\ E_{02} \le X \le E_{03} \\ 0 & ; \ Otherwise \end{cases}$$

whose interval of confidence is

$$[u_{i-1,j}]^{(\alpha)}(=)[E_{01}+(E_{02}-E_{01})\alpha, E_{03}-(E_{03}-E_{02})\alpha]$$

Next, fuzzy membership function for $u_{i+1,j}$ is

$$\mu_{u_{i+1,j}}(X)(=)\begin{Bmatrix} \dfrac{X-E_{21}}{E_{22}-E_{21}};\ where\ E_{21}\le X\le E_{22} \\ \dfrac{-X+E_{23}}{E_{23}-E_{22}};\ where\ E_{22}\le X\le E_{23} \\ 0 \qquad ;\ Otherwise \end{Bmatrix}$$

whose interval of confidence is

$$[u_{i+1,j}]^{(\alpha)}(=)[E_{21}+(E_{22}-E_{21})\alpha, E_{23}-(E_{23}-E_{22})\alpha]$$

Hence α-cut for $u_{i,j+1}$ is

$$[u_{i,j+1}]^{(\alpha)}(=)\frac{1}{2}\begin{bmatrix}(E_{01}+E_{21})+\{(E_{02}-E_{01})+(E_{22}-E_{21})\}\alpha, \\ (E_{03}+E_{23})-\{(E_{03}-E_{02})+(E_{23}-E_{22})\}\alpha\end{bmatrix}$$

Now to retain two roots α ϵ [0,1]

Let

$$X_1=\frac{(E_{01}+E_{21})+\{(E_{02}+E_{22})-(E_{01}+E_{21})\}\alpha}{2}$$

$$\Rightarrow \alpha=\frac{2X_1-(E_{01}+E_{21})}{(E_{02}+E_{22})-(E_{01}+E_{21})}$$

And

$$X_2=\frac{(E_{03}+E_{23})-\{(E_{03}+E_{23})-(E_{02}+E_{22})\}\alpha}{2}$$

$$\Rightarrow \alpha=\frac{-2X_2+(E_{03}+E_{23})}{(E_{03}+E_{23})-(E_{02}+E_{22})}$$

Hence, fuzzy membership function for $u_{i,j+1}$ is

$$\mu_{u_{i,j+1}}(X)(=)\begin{Bmatrix} \dfrac{2X-(E_{01}+E_{21})}{(E_{02}+E_{22})-(E_{01}+E_{21})}; \\ where\ \dfrac{E_{01}+E_{21}}{2}\le X\le\dfrac{E_{02}+E_{22}}{2} \\ \dfrac{-2X+(E_{03}+E_{23})}{(E_{03}+E_{23})-(E_{02}+E_{22})}; \\ where\ \dfrac{E_{02}+E_{22}}{2}\le X\le\dfrac{E_{03}+E_{23}}{2} \\ 0 \qquad ;\ Otherwise \end{Bmatrix} \text{-----}(4)$$

With the help of this we can find out f.m.f. of different *u's* of j+1th level in terms of its preceding level as per Bender-Schmidt recurrence equation.

5. A NUMERICAL EXAMPLE

Let us consider the one dimensional Heat Equation as

$$2\frac{\partial u}{\partial t}(=)\frac{\partial^2 u}{\partial x^2} \text{--}(5)$$

with the conditions $u(x,0)=4x-x^2$

$u(0,t)=0=u(4,t)$

Here $u_{0,0}=u(0,0)=0$, $u_{1,0}=u(1,0)=3$, $u_{2,0}=u(2,0)=4$
$U_{3,0}=u(3,0)=3$, $u_{4,0}=u(4,0)=0$.

Also, u(0,1)=0=u(4,1), u(0,2)=0=u(4,2), u(0,3)=0=u(4,3), u(0,4)=0=u(4,4) ,u(0,5)=0=u(4,5).

Let us consider fuzzy intervals for *u(0,0) and u(2,0) as*
$u_{0,0}(=)[-0.001,\ 0,\ 0.001]$ and
$u_{2,0}(=)[3.999,\ 4,\ 4.001]$
Therefore as per Bender-Schmidt scheme
$[u_{1,1}]^{(F.I.)}(=)[1.999,\ 2,\ 2.001]$

Now fuzzy membership function for $u_{0,0}$ is

$$\mu_{u_{0,0}}(X)(=)\begin{Bmatrix} \dfrac{X+0.001}{0.001};\ where\ -0.001\le X\le 0 \\ \dfrac{-X+0.001}{0.001};\ where\ 0\le X\le 0.001 \\ 0 \qquad ;\ Otherwise \end{Bmatrix}$$

Hence α-cut for $u_{0,0}$ is

$$[u_{0,0}]^{(\alpha)}(=)[0.001\alpha-0.001\ ,\ -0.001\alpha+0.001]$$

Next fuzzy membership function for $u_{2,0}$ is

$$\mu_{u_{2,0}}(X)(=)\begin{Bmatrix} \dfrac{X-3.999}{0.001};\ where\ 3.999\le X\le 4 \\ \dfrac{-X+4.001}{0.001};\ where\ 4\le X\le 4.001 \\ 0 \qquad ;\ Otherwise \end{Bmatrix}$$

Hence α-cut for $u_{2,0}$ is

$$[u_{2,0}]^{(\alpha)}(=)[3.999+0.001\alpha,\ 4.001-0.001\alpha]$$

Hence α-cut for $u_{1,1}$ is

$$[u_{1,1}]^{(\alpha)}(=)[1.999+0.001\alpha,\ 2.001-0.001\alpha]$$

Now to retain two roots α ϵ [0,1]
Let

$$X_1=1.999+0.001\alpha \Rightarrow \alpha=\frac{X_1-1.999}{0.001}$$

$$X_2=2.001-0.001\alpha \Rightarrow \alpha=\frac{-X_2+2.001}{0.001}$$

Hence fuzzy membership function for $u_{1,1}$ is

$$\mu_{u_{1,1}}(X)(=)\begin{Bmatrix} \dfrac{X-1.999}{2-1.999};\ where\ 1.999\le X\le 2 \\ \dfrac{-X+2.001}{2.001-2};\ where\ 2\le X\le 2.001 \\ 0 \qquad ;\ Otherwise \end{Bmatrix} \text{---}(6)$$

Next for the grid point $u_{2,1}$,

let us consider fuzzy intervals for *u(1,0) and u(3,0) as*

$u_{1,0}(=)[2.999,\ 3,\ 3.001]$ and $u_{3,0}(=)[2.999,\ 3,\ 3.001]$

Therefore as per Bender-Schmidt scheme

$[u_{2,1}]^{(F.I.)}(=)[2.999,\ 3,\ 3.001]$

Now fuzzy membership function for $u_{1,0}$ is

$$\mu_{u_{1,0}}(X)(=)\begin{Bmatrix} \dfrac{X-2.999}{3-2.999};\ where\ 2.999 \le X \le 3 \\ \dfrac{-X+3.001}{3.001-3};\ where\ 3 \le X \le 3.001 \\ 0\quad ;\ Otherwise \end{Bmatrix}$$

Hence α-cut for $u_{1,0}$ is

$[u_{1,0}]^{(\alpha)}(=)[2.999+0.001\alpha,\ 3.001-0.001\alpha]$

Next fuzzy membership function and α-cut for $u_{3,0}$ are same as that of $u_{1,0}$

Hence α-cut for $u_{1,1}$ is

$[u_{2,1}]^{(\alpha)}(=)[2.999+0.001\alpha,\ 3.001-0.001\alpha]$

Now to retain two roots α ϵ [0,1]

Let

$$X_1 = 2.999+0.001\alpha \Rightarrow \alpha = \frac{X_1-2.999}{0.001}$$

$$X_2 = 3.001-0.001\alpha \Rightarrow \alpha = \frac{-X_2+3.001}{0.001}$$

Hence fuzzy membership function for $u_{2,1}$ is

$$\mu_{u_{2,1}}(X)(=)\begin{Bmatrix} \dfrac{X-2.999}{3-2.999};\ where\ 2.999 \le X \le 3 \\ \dfrac{-X+3.001}{3.001-3};\ where\ 3 \le X \le 3.001 \\ 0\quad ;\ Otherwise \end{Bmatrix} \text{---(7)}$$

Next for the grid point $u_{3,1}$,

let us consider fuzzy intervals for *u(2,0) and u(4,0) as*

$u_{2,0}(=)[3.999,\ 4,\ 4.001]$ and

$u_{4,0}(=)[-0.001,\ 0,\ 0.001]$

Therefore as per Bender-Schmidt scheme

$[u_{3,1}]^{(F.I.)}(=)[1.999,\ 2,\ 2.001]$

Now fuzzy membership function for $u_{4,0}$ is

$$\mu_{u_{4,0}}(X)(=)\begin{Bmatrix} \dfrac{X+0.001}{0.001};\ where\ -0.001 \le X \le 0 \\ \dfrac{-X+0.001}{0.001};\ where\ 0 \le X \le 0.001 \\ 0\quad ;\ Otherwise \end{Bmatrix}$$

Hence α-cut for $u_{4,0}$ is

$[u_{4,0}]^{(\alpha)}(=)[0.001\alpha-0.001,\ -0.001\alpha+0.001]$

Next fuzzy membership function for $u_{2,0}$ is

$$\mu_{u_{2,0}}(X)(=)\begin{Bmatrix} \dfrac{X-3.999}{0.001};\ where\ 3.999 \le X \le 4 \\ \dfrac{-X+4.001}{0.001};\ where\ 4 \le X \le 4.001 \\ 0\quad ;\ Otherwise \end{Bmatrix}$$

Hence α-cut for $u_{2,0}$ is

$[u_{2,0}]^{(\alpha)}(=)[3.999+0.001\alpha,\ 4.001-0.001\alpha]$

Hence α-cut for $u_{3,1}$ is

$[u_{3,1}]^{(\alpha)}(=)[1.999+0.001\alpha,\ 2.001-0.001\alpha]$

Now to retain two roots α ϵ [0,1]

Let

$$X_1 = 1.999+0.001\alpha \Rightarrow \alpha = \frac{X_1-1.999}{0.001}$$

$$X_2 = 2.001-0.001\alpha \Rightarrow \alpha = \frac{-X_2+2.001}{0.001}$$

Hence fuzzy membership function for $u_{3,1}$ is

$$\mu_{u_{3,1}}(X)(=)\begin{Bmatrix} \dfrac{X-1.999}{2-1.999};\ where\ 1.999 \le X \le 2 \\ \dfrac{-X+2.001}{2.001-2};\ where\ 2 \le X \le 2.001 \\ 0\quad ;\ Otherwise \end{Bmatrix} \text{---(8)}$$

Similarly we can find out fuzzy membership function at different grid points which are $u_{1,2}$; $u_{2,2}$; $u_{3,2}$ $u_{1,3}$; $u_{2,3}$; $u_{3,3}$ $u_{1,4}$; $u_{2,4}$; $u_{3,4}$ $u_{1,5}$; $u_{2,5}$; $u_{3,5}$ etc.

6. CONCLUSION

Here we can go forward in the open end of time variable as t goes to infinity and find solution of heat equation in numerical fuzzified form.

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