On The Fractional Systems' Fault Detection: Evaluation of Fractional Residual

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ABSTRACT
The paper deals with fractional residual evaluation. Three methods to evaluate fractional residuals generated by dynamic parity space method are presented. They are based on the fractional derivative approximations: the Grunwald, the pole-zero and the Diethelm approximations. They are compared in order to select the best method in terms of precision and minimum detection time delay. The selected method is used to evaluate residual of a real system.

Keywords
Fractional residual, residual evaluation,
fractional derivative approximation, electronic device.

1. INTRODUCTION
Recent years have witnessed a particular attention to the use of fractional calculus in various fields. Many diffusive phenomena can be described by fractional models (ie. involving non integer differentiation orders). In electrochemistry for instance, diffusion of charges in acid batteries is governed by Randles models [1] that involve Warburg impedance with an integrator of order 0.5. Electrochemical diffusion showed to have a tight relation with derivatives of order 0.5 [2]. In thermal diffusion of a semi infinite homogeneous medium, Battaglia et al. [3] have shown that the solution for the heat equation links thermal flux to a half order derivative of the surface temperature on which the flux is applied. The fractional differentiation models are used in the automatic field through many applications as control [4, 5, 6, 7], identification [1, 8, 9, 10] and diagnosis [11].

In the diagnosis field, the use of classical method, based on rational models, is inappropriate in diagnosing systems described by fractional models. Therefore, a model based method called fractional dynamic parity space has been extended to systems described by fractional models [11].

The latter method is based on two successive steps: Residual generation and residual evaluation.

Fractional residuals generated by fractional dynamic parity space method depend on fractional input and output derivatives. They can be evaluated using many approximations.

Despite the simplicity of Grunwald fractional derivative approximation [12], it seems inappropriate in the fault detection context. Due to its high computational time [13], the Grunwald approximation used in fractional residual evaluation can introduce a delay in fault detection. Thus, It is recommended to find alternative methods. This paper presents and compares three methods to evaluate fractional residual: The Gr¨unwald's approximation, the Diethelm's approximation and the recursive pole-zero approximation. It is organized as following.

Section 2 presents some definitions and properties of fractional calculus. Section3 exposes residual generation through the dynamic parity space fault detection method and presents residual evaluation through three methods Grunwald, Diethelm and recursive pole-zero approximations. Section 4 presents an illustrative example comparing the latter fractional residual evaluation methods. In section 5, the best selected method will be used to evaluate fractional residual of a real system.

2. MATHEMATICAL BACKGROUND
2.1. Fractional derivative
The Caputo fractional derivative at a positive real order $n \in R_+$ of a function *f(t)* is defined by [14] according to:

$$D^{\upsilon} f(t) = \frac{1}{\Gamma(\lceil \upsilon \rceil - \upsilon)} \int_0^t \frac{f^{\lceil \upsilon \rceil}(\tau)}{(t-\tau)^{\upsilon+1-\lceil \upsilon \rceil}} d\tau \quad (1)$$

The Laplace transform of the fractional derivative at order υ of a relaxed function *f(t)* at *t = 0* (*i. e. f(t) =0* ∀ *(t<0)*is:

$$\mathcal{L}\left(D^{\upsilon} f(t)\right) = s^{\upsilon} \mathcal{L}\left(f(t)\right) \quad (2)$$

where *s* is the Laplace variable. s^{υ} is called fractional differentiation operator.

2.2. Approximations of the fractional derivative
The fractional derivative may be numerically evaluated using many approximations such as Grunwald, Diethelm and pole-zero approximations.

2.2.1. The Grunwald's derivative approximation
The fractional derivative is numerically evaluated using

the Grunwald's approximation for small sampling time h

[15]:

$$D^{\upsilon} f(t) \approx \frac{1}{h^{\upsilon}} \sum_{j=0}^{\left[\frac{t}{h}\right]} (-1)^j \binom{\upsilon}{j} f(t - jh) \qquad (3)$$

where:

$$\binom{\upsilon}{j} = \frac{\Gamma(\upsilon + 1)}{\Gamma(\upsilon + 1)\Gamma(\upsilon - j + 1)} \qquad (4)$$

Equation (3) shows that fractional derivative is not a local operator, because quantity (4) is not zero as $j > \upsilon$ and $\upsilon \notin \mathbb{N}$, whereas it is zero when $j > \upsilon$ and $\upsilon \in \mathbb{N}$. Hence, $D^{\upsilon} f(t)$ depends on the whole past of $f(t)$, $t \in [0,t]$, unless differentiation order υ is integer.

2.2.2. The Diethelm's derivative approximation

The Diethelm's derivative is based on the approximation of the Caputo fractional derivative based on the trapezoidal rule in [16].

$$D^{\upsilon} y(t) = \frac{h^{-\upsilon}}{\Gamma(2-\upsilon)} \sum_{j=0}^{N} \beta_{j,N} \left(y_{N-j} - \sum_{k=0}^{\lfloor \upsilon \rfloor} \frac{(N-j)^k}{k!} \right) + O\left(h^{2-\upsilon}\right) \qquad (5)$$

with $N = t/h$ and

$$\beta_{i,j} = \begin{cases} 1, j = 1 \\ (j+1)^{1-\upsilon} - 2j^{1-\upsilon} + (j-1)^{1-\upsilon}, 0 < j < 1 \\ (1-\upsilon) N^{-\upsilon} - N^{1-\upsilon} + (N-1)^{1-\upsilon}, j = N \end{cases}$$

2.2.3. The pole- zero approximation

The pole zero approximation consists in approximating the fractional differentiation operator s^{υ} with rational model.

As s can not be approximated within an infinite frequency band,Oustaloup proposed a recursive approximation within a limited frequency band[17].

$$s^{\upsilon} \approx s^{\upsilon}_{[w_A, w_B]} \approx \left(\prod_{k=1}^{N} \frac{1 + \frac{s}{w'_k}}{1 + \frac{s}{w_k}} \right) \qquad N \in \mathbb{N} \qquad (6)$$

The recursivity of zeros and poles is realized through a distribution of transitional frequencies ω_A et ω_B according to these relationships:

$$\frac{w_k}{w_{k+1}} = \frac{w'_k}{w'_{k+1}} = \alpha\eta \qquad (7)$$

$$\frac{w'_{k+1}}{w_k} = \eta; \qquad \frac{w_k}{w'_k} = \alpha \qquad (8)$$

where α and η are real parameters which depend on the differentiation order υ and the number of zeros and poles. The bigger N, the better the approximation of the differentiators is.

$$\alpha = \left(\frac{w_h}{w_b}\right)^{\upsilon/N} \quad and \quad \eta = \left(\frac{w_h}{w_b}\right)^{(1-\upsilon)/N} \qquad (9)$$

2.3. Fractional systems' representation

2.3.1. Differential equation

A Single Input Single Output (SISO) Linear Time Invariant (LTI) fractional system H, relaxed at $t = 0$ is described

by a differential equation:

$$a_1 D^{\upsilon_{a_1}} y(t) + a_2 D^{\upsilon_{a_2}} y(t) + \cdots + a_L D^{\upsilon_{a_L}} y(t) = b_1 D^{\upsilon_{b_1}} u(t) + \cdots + b_M D^{\upsilon_{b_M}} u(t) \qquad (10)$$

where : $a_1, a_2, \ldots, a_L, b_1, \ldots, b_M$ are real numbers;

$u(t)$, $y(t)$ are respectively the input and the output signals;

$\upsilon_{a1}, \ldots, \upsilon_{aL}$ and $\upsilon_{b1}, \ldots, \upsilon_{bM}$ are real positive

differentiation orders such that $\frac{\upsilon_{a_i}}{\nu} \in \mathbb{Q}$, $i = 1,\ldots,L$ and $\frac{\upsilon_{b_j}}{\nu} \in \mathbb{Q}$, $j = 1,\ldots,M$ are integers.

υ is called commensurate order [18].

Then, equation (10) can be rewritten as follows [19]:

$$a_1 \underbrace{D^{\nu}(\ldots(D^{\nu} y(t)))}_{\frac{\upsilon_{a_1}}{\nu} times} + \ldots + a_L \underbrace{D^{\nu}(\ldots(D^{\nu} y(t)))}_{\frac{\upsilon_{a_L}}{\nu} times} = b_1 \underbrace{D^{\nu}(\ldots(D^{\nu} u(t)))}_{\frac{\upsilon_{b_1}}{\nu} times} + \ldots + b_M \underbrace{D^{\nu}(\ldots(D^{\nu} u(t)))}_{\frac{\upsilon_{b_M}}{\nu} times} \qquad (11)$$

By changing the notation, equation (11) is equivalent to:

$$a'_1 D^{\nu} y(t) + \ldots + a'_{L'} \underbrace{D^{\nu}(\ldots(D^{\nu} y(t)))}_{L' times} = b'_1 D^{\nu} u(t) + \ldots + b'_{M'} \underbrace{D^{\nu}(\ldots(D^{\nu} u(t)))}_{M' times} \qquad (12)$$

where : $L' = \frac{\upsilon_{a_L}}{\nu} \in \mathbb{Q}$ and $M' = \frac{\upsilon_{bM}}{\nu} \in \mathbb{Q}$ and $a'_{L'} = 1$.

2.3.2. Transfer function

Applying Laplace transform to (10) yields to irrational transfer function:

$$F(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^{\upsilon_{b_0}} + b_1 s^{\upsilon_{b_1}} + \ldots + b_M s^{\upsilon_{b_M}}}{1 + a_1 s^{\upsilon_{a_1}} + \ldots + a_L s^{\upsilon_{a_L}}}. \quad (13)$$

If the corresponding differential equation is commensurable then the transfer function is so. In this case,

$$F(s) = \frac{Y(s)}{U(s)} = \frac{b_0 (s^{\upsilon})^{\frac{\upsilon_{b_0}}{\nu}} + b_1 (s^{\upsilon})^{\frac{\upsilon_{b_1}}{\nu}} + \ldots + b_M (s^{\nu})^{\frac{\upsilon_{b_M}}{\nu}}}{1 + a_1 (s^{\nu})^{\frac{\upsilon_{a_1}}{\nu}} + \ldots + a_L (s^{\nu})^{\frac{\upsilon_{a_L}}{\nu}}}, \quad (14)$$

where $\frac{\upsilon_{a_i}}{\nu} \in \mathbb{Z}$, $i = 1,\ldots,L$ and $\frac{\upsilon_{b_j}}{\nu} \in \mathbb{Z}$, $j = 1,\ldots,M$.

When $\nu = 1$ (*i. e.* all orders are integers), the

Obtained $F(s)$ is the classical rational transfer function.

2.3.3. State space representation

A fractional system can also be described with a fractional

state space representation where *A*, *B*, and *C* are complexes coefficients fulls matrix with appropriate dimensions.

$$\begin{cases} x^{(\nu)}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (15)$$

A companion fractional state space representation can

be easily obtained from (12)

where:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a'_1 & -a'_2 & \cdots & \cdots & -a'_{L'-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b'_{M'} \\ \vdots \\ b'_1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note that the evolution matrix' s dimension of a system

described by differential equation (10) is inversely proportional to the commensurate order *v*.

Indeed, $size(A) = L' = \frac{\upsilon_{a_L}}{\nu}$. Thus, the commensurate order *v* as it is defined previously yields to the smallest state space representation.

2.4. Numerical simulation

The fractional systems simulation methods are based on numerical evaluation of derivatives. They are classified into methods based on discrete time numerical approximation of fractional differentiation such as the Grunwald and the Diethelm [13] approximations and methods based on continuous time approximations such as poles and zeros approximation [17].

3. FRACTIONAL RESIDUAL GENERATION AND EVALUATION

3.1. Residual generation using fractional dynamic parity space method

Let consider a fractional system affected with a fault and

described by the state space representation:

$$\begin{cases} x^{(\nu)}(t) = Ax(t) + Bu(t) + F_1 d(t) \\ y(t) = Cx(t) + F_2 d(t) \end{cases} \quad (16)$$

where $x(t) \in \mathbb{R}^n$, $x^{(\nu)}(t) \in \mathbb{R}^n$, $u(t) \in R^q$, $y(t) \in R^m$

and $d(t) \in \mathbb{R}^r$ are respectively the state, the derivative of the state at order *v*, the input, the output and the fault vectors. *A*, *B* and *C* are constant matrices of appropriate dimensions. F_1 and F_2 are matrices reflecting the faults directions. The fault vector $d(t)$ is assumed to be unknown.

To generate analytical redundancy equations, starting from the output equation:

$$y(t) = Cx(t) + F_2 d(t) \quad (17)$$

The first step consists in calculating the time derivative of

the output to an order *v* (*v* is the commensurate order).

$$y^{(\nu)}(t) = Cx^{(\nu)}(t) + F_2 d^{(\nu)}(t) \quad (18)$$

Using the state equation (16) leads to a differential equation which links the output, the input, the state and the fault:

$$y^{(\nu)}(t) = CAx(t) + CBu(t) + CF_1 d(t) + F_2 d^{(\nu)}(t) \quad (19)$$

In a second step, the time derivative of the output to order

2ν is calculated:

$$y^{(2\nu)}(t) = Cx^{(2\nu)}(t) + F_2 d^{(2\nu)}(t) \quad (20)$$

Referring to equations (16) and (19), equation (20) can be

rewritten as:

$$y^{(2\nu)}(t) = CA^2 x(t) + CABu(t) + CBu^{(\nu)}(t) + CAF_1 d(t) + CF_1 d^{(\nu)} + F_2 d^{(2\nu)}(t) \quad (21)$$

So on, successive time derivatives of the output $y(t)$ are calculated up to a given order $k\nu$, where *k* denotes the number of derivatives. In most cases, it is sufficient to choose *k* equal to the size of the evolution matrix *A*.

$$y^{(k\nu)} = CA^k x(t) + CA^{k-1}Bu(t) + ...+ CBu^{((k-1)\nu)}(t) + CA^{k-1}F_1 d(t) + ...+ CF_1 d^{((k-1)\nu)}(t) + F_2 d^{(k\nu)}(t) \quad (22)$$

Then, using (17), (19), (21) and (22):

$$Y(t,k) - G(k)U(t,k) = H(k)\,x(t) + E(k)D(t,k) \quad (23)$$

where:

$$Y(t,k) = \left[\; y^T(t) \quad (y^{(\nu)}(t))^T \quad \cdots \quad (y^{(k\nu)}(t))^T \;\right]^T \quad (24)$$

$$U(t,k) = \left[\; u^T(t) \quad (u^{(\nu)}(t))^T \quad \cdots \quad (u^{(k\nu)}(t))^T \;\right]^T \quad (25)$$

$$D(t,k) = \left[\; d^T(t) \quad (d^{(\nu)}(t))^T \quad \cdots \quad (d^{(k\nu)}(t))^T \;\right]^T \quad (26)$$

$$H(k) = \left[\; C \quad CA \quad \cdots \quad CA^k \;\right]^T \quad (27)$$

$$E(k) = \begin{bmatrix} F_2 & 0 & \cdots & \cdots & 0 \\ CF_1 & F_2 & 0 & \cdots & 0 \\ CAF_1 & CF_1 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ CA^{k-1}F_1 & CA^{k-2}F_1 & \cdots & CF_1 & F_2 \end{bmatrix} \quad (28)$$

$$G(k) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ CA^{k-1}B & CA^{k-2}B & \cdots & CB & 0 \end{bmatrix} \quad (29)$$

To eliminate unknown states let multiply equation (23) by a matrix Ω called parity matrix such that:

$$\Omega H = 0 \quad \text{and} \quad \Omega \mathrm{E} \neq 0 \quad (30)$$

Thus, two expressions of the residual vector are obtained.

The first is expressed as a function of the fault. It is called internal form:

$$R(t) = \Omega E(k) D(t,k) \quad (31)$$

The second is expressed as a function of the input and the output signals. It is called external or computational form:

$$R(t) = \Omega(Y(t,k) - G(k)U(t,k)) \quad (32)$$

Equation (31) shows that the residual vector *R (t)* is zero in the fault-free case and non-zero when a fault occurs.

3.2. Residual evaluation

In practice, the residual is evaluated using the computational form (32). The two residual forms (31) and (32) are identical to those obtained for rational models. The differences are behind *U(t, k), Y(t, k)* and *D(t, k)* which depend on fractional derivatives of the input, the output and the fault. These derivatives may be evaluated using approximations in section (2.2).

3.3. Residual evaluation using Grunwald's approximation

Residual evaluation using Grunwald's derivative consists in evaluating the components of the vectors (24) (*i. e.* $y^{(iv)}$ *(t) (i=1…k)* and (25) *(i. e. u (t), i = 1...k,)* using (3).

Thus,

$$y^{(i\nu)}(t) = D^{i\nu}y(t) = \frac{1}{h^{i\nu}} \sum_{j=0}^{\left[\frac{t}{h}\right]} (-1)^j \binom{\nu}{j} y(t-jh),\ i = 1...k \quad (33)$$

$$u^{(i\nu)}(t) = D^{i\nu}u(t) = \frac{1}{h^{i\nu}} \sum_{j=0}^{\left[\frac{t}{h}\right]} (-1)^j \binom{\nu}{j} u(t-jh),\ i = 1...k \quad (34)$$

where *h* is the sampling time, *v* is the derivation order and *k* is the number of derivatives.

Sums in (34) and (33) have growing dimensions with time. This proves the global nature of fractional differentiation. This residual evaluation method is relatively easy to implement but it has high computation time. This can introduce, in real time application, a considerable delay in fault detection and have bad impact in the diagnosis procedure.

3.4. Residual evaluation using Diethelm's approximation

Residual evaluation using Diethelm's approximation consists in evaluating the components of the vectors (24)

(i. e. $y^{(iv)}$ *(t)(i=1…k)* and (25) (*ie. u (t), i = 1...k*) using (5). Thus,

$$u^{(i\nu)}(t) = D^{i\nu}u(t) = \frac{h^{-i\nu}}{\Gamma(2-i\nu)} \sum_{j=0}^{N} \beta_{j,N} \left(u_{N-j} - \sum_{n=0}^{\lfloor i\nu \rfloor} \frac{(N-j)^n}{n!} \right) + O\left(h^{2-i\nu}\right),\ i = 1...k \quad (35)$$

$$y^{(i\nu)}(t) = D^{i\nu}y(t) = \frac{h^{-i\nu}}{\Gamma(2-i\nu)} \sum_{j=0}^{N} \beta_{j,N} \left(y_{N-j} - \sum_{n=0}^{\lfloor i\nu \rfloor} \frac{(N-j)^n}{n!} \right) + O\left(h^{2-i\nu}\right),\ i = 1...k \quad (36)$$

where *h* is the sampling time, *v* is the derivation order and *k* is the number of derivatives.

3.5. Residual evaluation through recursive distribution of zeros and poles

Residual evaluation through recursive distribution of zeros and poles consists in:

-replacing vectors (24) and (25) by their Laplace transform.

$$Y(s,k) = [\ y^T(s) \quad s^\nu y^T(s) \quad \cdots \quad s^{k\nu} y^T(s)\ ]^T \tag{37}$$

$$U(s,k) = [\ u^T(s) \quad s^\nu u^T(s) \quad \cdots \quad s^{k\nu} u^T(s)\ ]^T \tag{38}$$

- replacing each differentiation operator s^{ν}, $i = 1,...,k$ in (37) and (38) by their recursive distribution of N zeros and poles described in section (2.2.3).

4. APPLICATION TO A REAL TIME SYSTEM

4.1. Description

The considered system is an electronic device designed on the basis of a recursive zeros and poles distribution [17].

Figure (1) shows the scheme of the circuit used on the experimental setup [20].

The system is described by a fractional transfer function:

$$G(s) = \frac{3.5714}{3.7514 + s^{1.55}} \tag{39}$$

4.2. Residual generation

Assuming that an additive fault can affect the sensor, the system can be modeled with the following fractional state space representation:

$$\begin{cases} x^{(1.55)}(t) = -3.7514x(t) + u(t) \\ y(t) = 3.7514x(t) + f(t) \end{cases} \tag{40}$$

Initially the fault vector *f(t)* is considered zero. According to equation (23), the analytic redundancy equation can be written as:

$$Y(t,1) - G(1)U(t,1) = H(1)\,x(t) \tag{41}$$

where:

$$H(1) = [3.7514 \quad -14.073]^T \tag{42}$$

$$G(1) = \begin{bmatrix} 0 & 0 \\ 3.5714 & 0 \end{bmatrix} \tag{43}$$

$$E(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{44}$$

The parity matrix Ω verifying (30) can be easily obtained

$$\Omega = [3.5714 \quad 1] \tag{45}$$

Then, according to (32),the computation form of the residual is:

$$R(t) = 3.5714 \quad y(t) + y^{(1.55)}(t) - 3.5714 \quad u(t) \tag{46}$$

Now, *f (t)* is an additive fault affecting the sensor at instant

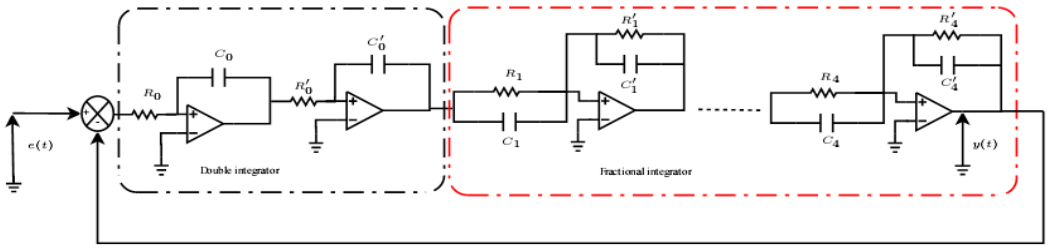

Figure 1. The symbolic scheme of the circuit used on the experimental setup

t = *15.5s*. In the following section, residuals evaluated through Grunwald's approximation, Diethelm's approximation and pole-zero approximation are noted respectively: R_G , R_D and R_{Apz}.

4.3. Residual evaluation

Evaluating residual consists in evaluating the fractional (1.55) derivative *y (t)* in (46) using the three approximations detailed in section (3.2). Fractional residuals R_G , R_D and R_{Apz} are plotted for two cases without fault in figure (2) and with fault in figure (3) R_G ,R_D and R_{Apz} are compared on the basis of two criteria:

-The nullity of the fractional approximated residual in steady state and in faultless case.

-The computational time taken to evaluate fractional residual. In steady state and in absence of fault, R G is perfectly null but R_D and R_{Apz} have low values different from zero.

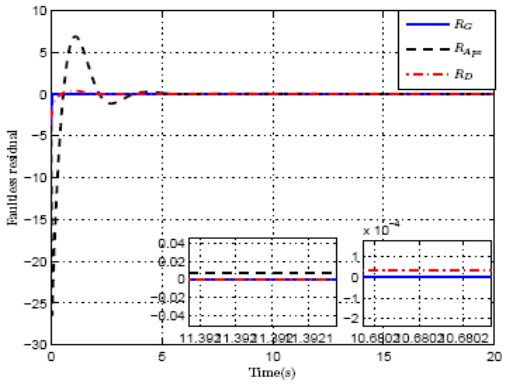

Figure 2. faultless simulated residuals

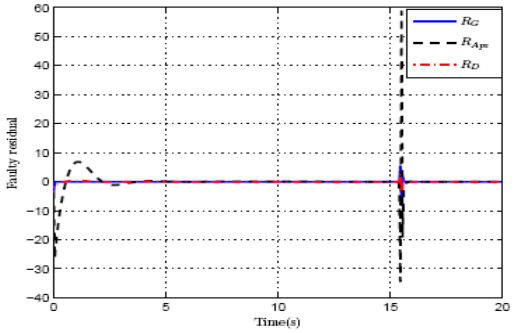

Figure 3. faulty simulated residuals

The figure shows that, in steady state, R_D is more close to R_G than R_{Apz}. R_G is used as reference to compare R_D and R_{Apz}. Let $E_D = R_D - R_G$ be the error between R_D and R_G and $E_{Apz} = R_{Apz} - R_G$ the error between R_G and R_{Apz}.
The table 1 shows that the mean square error (MSE) of E_D is lower than which of E_{Apz}. So, R_D is more close to R_G than R_{Apz} . In fact, R_D is practically null.

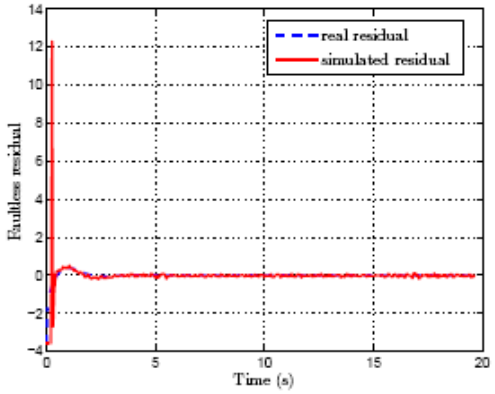

Figure 4. Real and simulated faultless residuals

Table 1. Mean square error

	E_{Apz}	E_D
MSE	0.052	1.72110^{-4}

Concerning the computational time, it is calculated using Matlab 2008 with an AMD Athlon Dual-Core having a performance index 4.6 clocked at 2GHz.The table 2 shows that *R* has the higher computational time which is relatively important in a fault detection context.

Table 2. Computational time

	R_G	R_{Apz}	R_D
Execution time	2.77s	2.01s	0.197s

Residual evaluation through pole-zero approximation reduces considerably the computational time compared to which made to evaluate R_G. So, the pole-zero approximation prevails sufficient results in term of cost. The best quality for cost is registered for R_D which has very low computational time.

Using pole-zero approximation to evaluate fractional residual instead of Grunwald's approximation can be beneficial, since it minimizes the computational time and presents acceptable results in term of the nullity of the residual. But, all depend on the number of poles and zeros chosen for the approximation.Seeing the importance of minimizing fault detection delay in a diagnosis context, using the Diethelm's approximation seems the most suitable and sufficient to evaluate fractional residual. In fact, residual evaluation using the Diethelm's approximation has the lowest computational time which minimizes hugely the fault detection delay also it is available to decide about the system state since R_D is practically null in steady state and in absence of fault. In worst case, the non nullity of R_D can be overcame using a low threshold.

Since the Diethelm approximation is selected as the best in the evaluation of fractional residual in terms of precision and cost, it is used to evaluate residual (46) using real measured input/output.

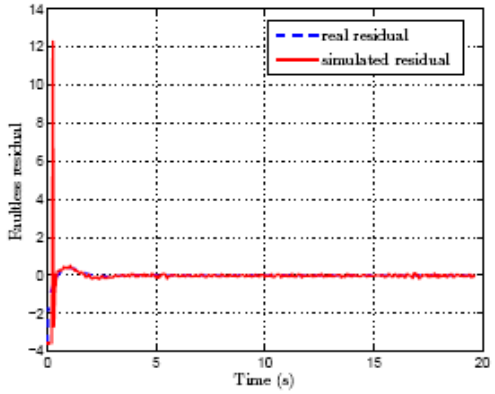

Figure 4. Real and simulated faultless residuals

The simulated and the real residuals are indistinguishable.
As expected in simulation, when the fault affects the output, the real residual (figure (5)) is sensitive to the fault and residual's peak corresponds to the instant when adding the fault *(ie. 15.5s).* So, the real residual evaluated using the Diethelm's approximation is efficient and relevant to decide about the system state.

5. CONCLUSION

This paper presents three methods to evaluate fractional residual: using the Gr¨unwald's approximation, the Diethelm's approximation and the recursive pole-zero approximation. A comparative study shows that the Diethelm's approximation presents the best performances in residual evaluation in terms of precision and the nullity of the residual in steady state and in fault free case. The selected method is then used to evaluate real residual of a real system. Experimental
results have shown the effectiveness of the Diethelm's approximation in evaluating residuals.

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