On the Stability of Functional Equations in Random Normed spaces

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ABSTRACT

Let f be a mapping from a linear space X into a complete Random Normed Space Y. In this paper, we prove some results for the stability of Cubic, Quadratic and Jensen-Type Quadratic functional equations in the setting of Random Normed Spaces (RNS).

Keywords

Quadratic functional equation, Cubic functional equation, Jensen-Type Quadratic functional equation, Hyers-Ulam-Rassias stability, Random Normed spaces.

1. INTRODUCTION

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equation is, "How do the solutions of the inequality differ from those of the given functional equation? The stability problem of functional equations originated from a question of S. M. Ulam [19], concerning the stability of group homomorphism:

Let $(G_1, *)$ *be a group and let* $(G_2, \Diamond, d)$ *be a metric group with the metric* $d(.,.)$. *Given* $\varepsilon > 0$, *does there exists a* $\delta(\varepsilon) > 0$ *such that if a mapping* $h: G_1 \longrightarrow G_2$ *satisfies the inequality*

$$d(h(x * y), h(x)\Diamond h(y)) < \delta,$$

for all $x, y \in G_1$, *then there is a homomorphism* $H: G_1 \longrightarrow G_2$ *with* $d(h(x), H(x)) < \varepsilon$, *for all* $x \in G_1$? *If the answer is affirmative, we would say that equation of homomorphism* $H(x\,y) = H(x)\,H(y)$ *is stable.*

In1941, D. H. Hyers [7] gave the first affirmative answer to the question of S. M. Ulam [19] for Banach spaces. Let X and Y be Banach spaces, and let $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and $\varepsilon \ge 0$. Then there exists a unique additive mapping $T: X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon, \text{ for all } x \in X$$

Also, if for each x the function $t \to f(tx)$ from R to Y is continuous at a single point of X, then T is continuous every where in X. In1978, Th. M. Rassias [25] gave the generalization of Hyer's result which allows the Cauchy difference to be unbounded. Let $f : X \to Y$ be a mapping from a normed vector space X into a Banach space Y subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $p<1$. Then the limit

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$ and $C: X \to Y$ is the unique mapping which satisfies

$$\|f(x) - C(x)\| \le \frac{2\varepsilon}{2-2^p}\|x\|^p$$

for all $x \in X$. Also, if for each $x \in X$ the function $f(tx)$ is continuous in $t \in R$, then C is R-linear. The case of the existence of unique additive mapping had been obtained by T. Aoki [28]. In 1994, P. Gavruta [18] following Th. M. Rassias [25] approach for the stability of the linear mapping between Banach spaces further obtained a generalization of Th. M. Rassias in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function $\phi(x, y)$ for the existence of unique linear mapping. The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to a quadratic mapping. A generalized Hyers- Ulam stability problem for the quadratic functional equation was given by F. Skof [11] for the mapping $f : X \to Y$, where X is a normed space and Y is a Banach space. P. Cholewa [17] again generalized the Skof's result for abelian groups. The stability problem of several functional equations have been investigated by a number of authors and there are many interesting results concerning this problem (see [5], [6], [8], [16], [21], [22], [26], [27]). The functional equation

$$f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) = 48f(y) \tag{1.1}$$

is said to be the cubic functional equation since $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability problem of cubic equation (1.1) was proved by Wiwatwanich and Nakmahalhlasint [1] for the mapping $f : X \to Y$, where X and Y are real Banach spaces. Later on, Park and Jung [16] introduced a cubic equation different from the equation (1.1) as follows.

$$f(x+3y) + f(3y-x) = 3f(x+y) + 3f(x-y) + 48f(y)$$

and proved the generalized Hyers-Ulam-Rassias stability for this equation on abelian groups. The functional equation

$$6f(x+y) - 6f(x-y) + 4f(3y) = 3f(x+2y) - 3f(x-2y) + 9f(2y) \tag{1.2}$$

is said to be the cubic - quadratic type functional equation since ax^3+bx^2 is its solution. Chang and Jung [12] established the general solution and generalized Hyers-Ulam stability for the function $f : X \to Y$, where X is a real vector space and Y is a real Banach space. The functional equations

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y) \tag{1.3}$$

and

$$f(ax+ay) + f(ax-ay) = 2a^2 f(x) + 2a^2 f(y) \tag{1.4}$$

are said to be Jensen- Type Quadratic functional equations. In 2009, S.Y.Jang, Rye Lee, Choonkil Park, and Dong Yun Shin [24] proved the Fuzzy stability of equation (1.3) and (1.4).
The notion of a Random Normed space in which the values of the norms are probability distribution functions rather than numbers was given by Sherstnev in [2] and again generalized by Alsina, Schweizer and Sklar in [4]

In this paper we adopt the usual terminology, notion and convention of the theory of random normed space. Through out this paper the space of Probabilistic distribution functions is given by Δ^+; that is the space of all mappings F: $R\cup\{+\infty,-\infty\}\to[0,1]$, such that F is left continuous non decreasing and $F(0)=0$, $F(+\infty)=1$ and D^+ is a subset of Δ^+ for which $l^-F(+\infty)=1$ where $l^-F(x)$ denotes the left limit of all function at the point x, that is, $l^-F(x)=Lim_{t\to x^-}F(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e. $F\le G$ if and only if $F(t)\le G(t)$ for all t in R. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0=\begin{cases}0 & if \quad t\le 0\\ 1 & if \quad t>0\end{cases}$$

Finally, we prove the Hyers-Ulam stability of the functional equation (1.1), (1.2), (1.3) and (1.4) respectively in random normed space. We also prove some corollaries in the sense of Hyers-Ulam-Rassias stability.

2. PRELIMINARIES

Definition 2.1. A mapping T: $[0, 1]\times[0, 1]\to[0, 1]$ is a continuous triangular norm (briefly a t – norm) if T satisfies the following conditions :

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a\in[0, 1]$;
(d) $T(a, b)\le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d\in[0, 1]$. (see [3])

Typical examples of continuous t-norm are $T(a, b) = ab$, $T(a, b) = \max(a+b-1,0)$ and $T(a, b) = \min(a, b)$

Definition 2.2. [2] A Random Normed space (briefly RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(PN1) $\mu_x(t)=\varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(PN2) $\mu_{\alpha x}(t)=\mu_x(t/|\alpha|)$ for all x in X, $\alpha\neq 0$ and all $t\ge 0$;

(PN3) $\mu_{x+y}(t+s) \ge T(\mu_x(t),\mu_y(s))$ for all $x, y\in X$ and all $t, s \ge 0$

Definition 2.3. Let (X, μ, T) be an RN- space

(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n-x}(t) > 1-\varepsilon$ whenever $n \ge N$.

(2) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(t) > 1-\varepsilon$ whenever $n \ge m \ge N$.

(3) An RN- space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.4. If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n\to x$, then $Lim_{n\to\infty}\mu_{x_n}(t)=\mu_x(t)$. [3]

Lemma 2.5. Let (X, μ, min) be an RN- space and define $E_{\lambda,\mu}: X\to R^+\cup\{0\}$ by

$$E_{\lambda,\mu}(x) = \inf\{t>0: \mu_x(t)>1-\lambda\}, \tag{2.1}$$

for all $\lambda\in]0,1[$, $x\in X$. Then, one has

$$E_{\lambda,\mu}(x_1-x_n)\le E_{\lambda,\mu}(x_1-x_2)+\ldots+E_{\lambda,\mu}(x_{n-1}-x_n), \tag{2.2}$$

for all $x_1, \ldots, x_n \in X$ and the sequence $\{x_n\}$ is convergent to x with respect to random norm μ if and only if $E_{\lambda,\mu}(x_n-x)\to 0$ as $n\to\infty$. Also, the sequence $\{x_n\}$ is a Cauchy sequence with respect to random norm μ if and only if it is a Cauchy sequence with $E_{\lambda,\mu}$. [10]

3. MAIN RESULTS

In the following theorems, by using the idea of Baktash, Cho, Saadati and Vaezpour (see[10]), we will prove the Hyers – Ulam – Rassias stability of the functional equations (1.1) and (1.2) in Random Normed spaces.

3.1 Stability of Cubic Functional Equation (1.1) in RN- space

Theorem 3.1:- Let X be a linear space, (Z, μ^1, min) an RN-space. Let $\phi: X\times X\to Z$ be a function such that for some $0<\alpha<27$

$$\mu^1_{\phi(0,3y)}(t) \ge \mu^1_{\alpha\phi(0,y)}(t) \tag{3.1}$$

$f(0)=0$ and $Lim_{n\to\infty}\mu^1_{\phi(3^n x,3^n y)}27^n(t)=1$, for all $x, y\in X$ and all $t > 0$. Let (Y, μ, min) be a complete RN-space. If $f: X\to Y$ is a mapping such that

$$\mu_{f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)-48f(y)}(t) \ge \mu^1_{\phi(x,y)}(t), \tag{3.2}$$

for all $x, y\in X$ and all $t > 0$

Then there exists a unique cubic mapping $C: X\to Y$ such that

$$\mu_{f(y)-C(y)}(t)\ge\mu^1_{\phi(0,y)(27+\alpha)}(48(27-\alpha)t) \tag{3.3}$$

for all $y\in X$ and all $t > 0$.

Proof: - By using the Lemma 2.5, equation (3.2) implies that,

$$E_{\lambda,\mu}(f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)-48f(y))$$
$$= \inf\{t > 0;\ \mu_{f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)-48f(y)}(t) > 1-\lambda\}$$
$$\le \inf\{t > 0;\ \mu^1_{\phi(x,y)}(t) > 1-\lambda\}$$
$$= E_{\lambda,\mu^1}(\phi(x,y)) \text{ for all } x, y\in X, \lambda\in(0, 1) \tag{3.4}$$

Now substituting $y = -y$, we have

$$E_{\lambda,\mu}(f(x-3y)-3f(x-y)+3f(x+y)-f(x+3y)-48f(-y))$$
$$\le E_{\lambda,\mu^1}(\phi(x,-y)) \tag{3.5}$$

Adding (3.4) and (3.5), we get

$$E_{\lambda,\mu}(-48f(y)-48f(-y))$$

$\leq E_{\lambda,\mu^1}\ (\phi(x,y)) + E_{\lambda,\mu^1}\ (\phi(x,-y))$

$\leq 2\, E_{\lambda,\mu^1}\ (\phi(x,y))\ E_{\lambda,\mu}\ (f(y)+f(-y)) \qquad \leq$

$$\frac{1}{24}\ E_{\lambda,\mu^1}\ (\phi(x,y)) \qquad (3.6)$$

From equation (3.2) and (3.4). Let us fix $x = 0$, then we get

$\mu_{f(3y)-3f(y)+3f(-y)-f(-3y)-48f(y)}(t) \geq \mu^1_{\phi(0,y)}(t)$,

for all $y \in X$ and all $t > 0$.

$E_{\lambda,\mu}\ (f(3y)-3f(y)+3f(-y)-f(-3y)-48f(y))$

$$\leq E_{\lambda,\mu^1}\ (\phi(0,y)) \qquad (3.7)$$

Solving (3.6) and (3.7), we get

$E_{\lambda,\mu}\ (2f(3y)-54f(y))$

$\leq E_{\lambda,\mu}\ (f(3y)+f(-3y))-3(f(y)+f(-y)) + E_{\lambda,\mu^1}\ (\phi(0,y))$

$\leq E_{\lambda,\mu^1}\ (\phi(0,y)) + E_{\lambda,\mu}\ (f(3y)+f(-3y))$

$\qquad + E_{\lambda,\mu}\ 3(f(y)+f(-y))$

$\leq E_{\lambda,\mu^1}\ (\phi(0,y)) + \frac{1}{24}\ E_{\lambda,\mu^1}\ (\phi(0,3y)) + \frac{3}{24}\ E_{\lambda,\mu^1}\ (\phi(0,y))$

from (3.6)

$\leq \frac{27}{24}\ E_{\lambda,\mu^1}\ (\phi(0,y)) + \frac{1}{24}\ E_{\lambda,\mu^1}\ (\phi(0,3y))$

$E_{\lambda,\mu}\left(\frac{f(3y)}{27} - f(y)\right)$

$$\leq \frac{1}{48}\,[\, E_{\lambda,\mu^1}\ (\phi(0,y)) + \frac{1}{27}\ E_{\lambda,\mu^1}\ (\phi(0,3y))] \qquad (3.8)$$

Now, it follows from

$$\frac{f(3^n y)}{27^n} - f(y) = \sum_{k=0}^{n-1}\left(\frac{f(3^{k+1}y)}{27^{k+1}} - \frac{f(3^k y)}{27^k}\right),$$

$E_{\lambda,\mu}\left(\frac{f(3^n y)}{27^n} - f(y)\right) \leq E_{\lambda,\mu} \sum_{k=0}^{n-1}\left(\frac{f(3^{k+1}y)}{27^{k+1}} - \frac{f(3^k y)}{27^k}\right)$

$\leq \sum_{k=0}^{n-1} E_{\lambda,\mu}\left(\frac{f(3^{k+1}y)}{27^{k+1}} - \frac{f(3^k y)}{27^k}\right)$

$$\leq \frac{1}{48}\sum_{k=0}^{n-1}\frac{1}{27^k}\,[\, E_{\lambda,\mu^1}\ (\phi(0,3^k y)) + \frac{1}{27}\ E_{\lambda,\mu^1}\ (\phi(0,3^{k+1}y))\,] \qquad (3.9)$$

Again for any positive integer m, dividing (3.9) by 27^m and replacing y with $3^m y$ to obtain that

$E_{\lambda,\mu}\left(\frac{f(3^{n+m}y)}{27^{n+m}} - \frac{f(3^m y)}{27^m}\right)$

$$\leq \frac{1}{48}\sum_{k=0}^{n-1}\frac{1}{27^{k+m}}\,[\, E_{\lambda,\mu^1}\ (\phi(0,3^{k+m}y)) + \frac{1}{27}\ E_{\lambda,\mu^1}\ (\phi(0,3^{k+m+1}y))\,] \qquad (3.10)$$

This shows that $\{f(3^n y)/27^n\}$ is a Cauchy sequence in (Y, μ, *min*) because the right hand side of (3.10) converges to zero when $m \to \infty$. Since (Y, μ, *min*) is a complete RN-space , this sequence converges to some points $C(y) \in Y$. Fix $y \in X$ and put $m = 0$ in (3.10) then we obtain

$E_{\lambda,\mu}\left(\frac{f(3^n y)}{27^n} - f(y)\right)$

$$\leq \frac{1}{48}\sum_{k=0}^{n-1}\frac{1}{27^k}\,[\, E_{\lambda,\mu^1}\ (\phi(0,3^k y)) + \frac{1}{27}\ E_{\lambda,\mu^1}\ (\phi(0,3^{k+1}y))\,] \qquad (3.11)$$

So, we get

$E_{\lambda,\mu}(C(y) - f(y))$

$\leq E_{\lambda,\mu}\left(C(y) - \frac{f(3^n y)}{27^n}\right) + E_{\lambda,\mu}\left(\frac{f(3^n y)}{27^n} - f(y)\right)$

$$\leq E_{\lambda,\mu}\left(C(y) - \frac{f(3^n y)}{27^n}\right) + \frac{1}{48}\sum_{k=0}^{n-1}\frac{1}{27^k}\,[\, E_{\lambda,\mu^1}\ (\phi(0,3^k y)) + \frac{1}{27}\ E_{\lambda,\mu^1}\ (\phi(0,3^{k+1}y))\,] \qquad (3.12)$$

Taking the limit as $n \to \infty$, we get

$E_{\lambda,\mu}(C(y) - f(y))$

$\leq \frac{1}{48}\sum_{k=0}^{\infty}\frac{1}{27^k}\,[\, E_{\lambda,\mu^1}\ (\phi(0,3^k y)) + \frac{1}{27}\ E_{\lambda,\mu^1}\ (\phi(0,3^{k+1}y))\,]$

$\leq \frac{1}{48}\sum_{k=0}^{\infty}\left(\frac{\alpha}{27}\right)^k E_{\lambda,\mu^1}\ (\phi(0,y)) + \frac{1}{48}\sum_{k=0}^{\infty}\left(\frac{\alpha}{27}\right)^{k+1} E_{\lambda,\mu^1}\ (\phi(0,y))$

$\leq \frac{1}{48}\ \frac{27}{27-\alpha}\ E_{\lambda,\mu^1}\ (\phi(0,y)) + \frac{1}{48}\ \frac{\alpha}{27-\alpha}\ E_{\lambda,\mu^1}\ (\phi(0,y))$

$$\leq E_{\lambda,\mu^1}\ (\phi(0,y))\left(\frac{27+\alpha}{48(27-\alpha)}\right) \qquad (3.13)$$

that is,

$\inf\{t > 0\ ;\ \mu_{C(y)-f(y)}(t) > 1-\lambda\}$

$$\leq \inf\{t > 0\ ;\ \mu^1_{\phi(0,y)(27+\alpha)}(48(27-\alpha)t) > 1-\lambda\} \qquad (3.14)$$

Then, we get

$$\mu_{C(y)-f(y)}(t) \geq \mu^1_{\phi(0,y)(27+\alpha)}(48(27-\alpha)t) \qquad (3.15)$$

Now, for uniqueness let there exist a cubic mapping $D: X \to Y$ which satisfies (3.3) then, clearly $C(3^n y) = 27^n C(y)$ and $D(3^n y) = 27^n D(y)$ for all $n \in N$. It follows from (3.3) that

$\mu_{C(y)-D(y)}(t) = Lim_{n\to\infty}\mu_{(C(3^n y)/27^n)-(D(3^n y)/27^n)}(t)$

$\mu_{(C(3^n y)/27^n)-(D(3^n y)/27^n)}(t)$

$\geq \min\{\ \mu_{(C(3^n y)/27^n)-(f(3^n y)/27^n)}(t/2)\ ,\ \mu_{(D(3^n y)/27^n)-(f(3^n y)/27^n)}(t/2)\ \}$

$\geq \mu^1_{\phi(0,3^n y)}(27^n(27-\alpha)t) \geq \mu^1_{\phi(0,y)}\left(\frac{(27^n(27-\alpha)t)}{\alpha^n}\right)$

Since $Lim_{n\to\infty}(27^n(27-\alpha)t)/\alpha^n = \infty$, we get $Lim_{n\to\infty}\mu^1_{\phi(0,y)}(27^n(27-\alpha)t)/\alpha^n = 1$. Therefore, it follows that $\mu_{C(y)-D(y)}(t) = 1$ for all $t > 0$ and so $C(y) = D(y)$. This completes the proof.

Corollary 3.2:- Let X be a linear space, (Z, μ^1, *min*) an RN-space and (Y, μ, *min*) a complete RN-space. Let p, q be non-negative real numbers and let $z_0 \in Z$. If $f: X \to Y$ is a mapping such that

$$\mu_{f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)-48f(y)}(t) \geq \mu^1_{(\|x\|^p+\|y\|^q)}(t)\ , \qquad (3.16)$$

for all $x, y \in X$ and all $t > 0$ $f(0)=0$ and p, q<3, then there exists a unique cubic mapping $C: X \to Y$ such that

$$\mu_{f(y)-C(y)}(t) \geq \mu^1_{\|y\|^q z_0 (27+\alpha)}(48(27-\alpha)t) , \quad (3.17)$$

for all $y \in X$ and all $t > 0$

Proof: Define $\phi(x,y) = \theta(\|x\|^p + \|y\|^q)$ and applying the Theorem (3.1) we get the desired result where $\alpha = 3^q$.

Example 3.1:- Let (X, ‖.‖) be a Banach Algebra and

$$\mu_x(t) = \begin{cases} \max\left\{1 - \frac{\|x\|}{t}, 0\right\} & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases}$$

for every $x, y \in X$, let

$$\mu^1_{\phi(x,y)}(t) = \begin{cases} max\left\{1 - \frac{48\|x\| + 48\|y\|}{t}, 0\right\} if & t > 0 \\ 0 & if \quad t \leq 0 \end{cases}$$

We know that norm is a distribution function and $Lim_{n\to\infty}\mu^1_{\phi(3^n x, 3^n y)} 27^n(t) = 1$ for every $x, y \in X$ and $t > 0$. By the definition (2.2) (X, μ, T) is a RN-space. In fact, $\mu_x(t) = 1$ for all $t>0 \Rightarrow \frac{\|x\|}{t} = 0$ for all $t > 0 \Rightarrow x = 0$ and certainly (PN2) $\mu_{\lambda x}(t) = \mu_x\left(\frac{t}{\lambda}\right)$ for all $x \in X$ and $t > 0$. Therefore, for every x, $y \in X$ and $t, s > 0$, we obtain

$$\mu_{x+y}(t+s) = \max\left\{1 - \frac{\|x+y\|}{t+s}, 0\right\} = \max\left\{1 - \left\|\frac{x+y}{t+s}\right\|, 0\right\} =$$
$$\max\left\{1 - \left\|\frac{x}{t+s} + \frac{y}{t+s}\right\|, 0\right\} \geq \max\left\{1 - \left\|\frac{x}{t}\right\| - \left\|\frac{y}{t}\right\|, 0\right\} \geq T(\mu_x(t), \mu_y(s))$$

Also RN- space (X, μ, T) is complete for

$$\mu_{x-y}(t) \geq 1 - \frac{\|x-y\|}{t} \qquad (x, y \in X, t > 0)$$

and hence (X, ‖.‖) is complete.

Let us define a mapping $f : X \to X$, $f(x) = x^3 + \|x\| x_0$, where x_0 is a unit vector in X. Now by using a simple calculation, we get

$$\|f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y)\|$$
$$= \big\|\|x+3y\| - 3\|x+y\| + 3\|x-y\| - \|x-3y\| - 48\|y\|\big\|\|x_0\|$$
$$\leq 48\|y\| \leq 48\|x\| + 48\|y\| \text{ for all } x, y \in X.$$

Hence $\mu_{f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)-48f(y)}(t) \geq \mu^1_{\phi(x,y)}(t)$ for all $x, y \in X$ and $t>0$

Now let

$$\mu^1_{\phi(0,3^n y)} 27^n(27-\alpha)t = \max\left\{1 - \frac{48\|y\|}{3^{2n}(27-\alpha)t}, 0\right\} ,$$

where $0 < \alpha < 27 \Rightarrow Lim_{n\to\infty}\mu^1_{\phi(0,3^n y)} 27^n(27-\alpha)t = 1$ which implies that all the conditions of Theorem (3.1) hold.

Since

$$\mu^1_{\phi(0,y)} \frac{48(27-\alpha)t}{(27+\alpha)} = \max\left\{1 - \frac{(27+\alpha)\|y\|}{(27-\alpha)t}, 0\right\},$$

We deduce that $C(x) = x^3$ is the unique cubic mapping C: $X \to X$ such that

$$\mu_{f(y)-C(y)}(t) \geq \max\left\{1 - \frac{(27+\alpha)\|y\|}{(27-\alpha)}, 0\right\}, \text{ for all } y \in X \text{ and } t > 0.$$

4.1 Stability of Cubic- Quadratic Type Functional Equation (1.2) in RN -spaces

Theorem 4.1:- Let X be a real linear space, (Z, μ^1, min) an RN-space. Let $\phi: X \times X \to Z$ be a function such that for some $0<\alpha<4$

$$\mu^1_{\phi(2x,2x)}(t) \geq \mu^1_{\alpha\phi(x,x)}(t) \quad (4.1)$$

and

$$\mu^1_{\phi(0,2x)}(t) \geq \mu^1_{\alpha\phi(0,x)}(t) \quad (4.2)$$

$f(0) = 0$ and $Lim_{n\to\infty}\mu^1_{\phi(2^n x, 2^n y)} 4^n(t) = 1$, for all $x, y \in X$ and all $t > 0$. Let (Y, μ, min) be a complete RN-space. If $f: X \to Y$ an even function satisfying the inequality

$$\mu_{6f(x+y)-6f(x-y)+4f(3y)-3f(x+2y)+3f(x-2y)-9f(2y)}(t) \geq \mu^1_{\phi(x,y)}(t) , \quad (4.3)$$

for all $x, y \in X$ and all $t > 0$.Then there exists a unique quadratic mapping $C: X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu^1_{4\phi(x,x)}(3(4-\alpha)t) + \mu^1_{\phi(0,x)}(3(4-\alpha)t) , \quad (4.4)$$

for all $x \in X$ and all $t > 0$.

Proof:- By using Lemma 2.5, equation (4.3) follows that

$E_{\lambda,\mu}(6f(x+y) - 6f(x-y) + 4f(3y) - 3f(x+2y) + 3f(x-2y) - 9f(2y))$

$= \inf\{t > 0: \mu_{6f(x+y)-6f(x-y)+4f(3y)-3f(x+2y)+3f(x-2y)-9f(2y)}(t) > 1-\lambda\}$

$\leq \inf\{t > 0; \mu^1_{\phi(x,y)}(t) > 1-\lambda\}$

$= E_{\lambda,\mu^1}(\phi(x,y))$, for all $x, y \in X$, $\lambda \in (0, 1)$ (4.5)

Putting $x = 0$, and then replace y by x we get

$E_{\lambda,\mu}(6f(x) - 6f(-x) + 4f(3x) - 3f(2x) + 3f(-2x) - 9f(2x)) \leq E_{\lambda,\mu^1}(\phi(0,x))$

$E_{\lambda,\mu}(4f(3x) - 9f(2x)) \leq E_{\lambda,\mu^1}(\phi(0,x))$, for all $x \in X$ (4.6)

Putting $y = x$ in (4.5) we get

$$E_{\lambda,\mu}(f(3x) + 3f(x) - 3f(2x)) \leq E_{\lambda,\mu^1}(\phi(x,x)) \quad (4.7)$$

Now from (4.6) and (4.7), we get

$E_{\lambda,\mu}\ (3f(2x)-12f(x))$

$\leq\ E_{\lambda,\mu}\ (4f(3x)+12f(x)-12f(2x)\ +4f(3x)-9f(2x))$

$\leq\ E_{\lambda,\mu}\ (4f(3x)+12f(x)-12f(2x)) + E_{\lambda,\mu}\ (4f(3x)-9f(2x))$

$\leq\ E_{\lambda,\mu}\ 4(f(3x)+3f(x)-3f(2x)) +\ E_{\lambda,\mu}\ (4f(3x)-9f(2x))$

$\leq\ 4E_{\lambda,\mu}\ (f(3x)+3f(x)-3f(2x)) + E_{\lambda,\mu}\ (4f(3x)-9f(2x))$

$\leq 4\ E_{\lambda,\mu^1}\ \phi(x,x)\ +\ E_{\lambda,\mu^1}\ \phi(0,x)$

$$E_{\lambda,\mu}\left(\frac{f(2x)}{4}-f(x)\right) \leq \frac{1}{12}\,[4\,E_{\lambda,\mu^1}\ \phi(x,x)\ +\ E_{\lambda,\mu^1}\ \phi(0,x)\ ] \tag{4.8}$$

Now replacing x by $2x$ in (4.8) and then dividing by 4, the resulting inequality with (4.8) gives

$$E_{\lambda,\mu}\left(\frac{f(2^2x)}{4^2}-\frac{f(2x)}{4}\right)$$

$$\leq \frac{1}{48}\,[4\,E_{\lambda,\mu^1}\ \phi(2x,2x) + E_{\lambda,\mu^1}\ \phi(0,2x)\ ]$$

$$E_{\lambda,\mu}\left(\frac{f(2^2x)}{4^2}-f(x)\right)$$

$$\leq \frac{1}{12}\left(\frac{E_{\lambda,\mu^1}4\phi(2x,2x)+E_{\lambda,\mu^1}\phi(0,2x)}{4}\right)+$$

$$\frac{1}{12}\left(E_{\lambda,\mu^1}4\phi(x,x)+E_{\lambda,\mu^1}\phi(0,x)\right)$$

for all $x\in X$. By induction we can write

$$E_{\lambda,\mu}\left(\frac{f(2^nx)}{4^n}-f(x)\right)$$

$$\leq \frac{1}{12}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix)+E_{\lambda,\mu^1}\phi(0,2^ix)}{4^i}\right) \tag{4.9}$$

for all $x\in X$. We divide (4.9) by 4^m and replacing x with 2^mx, we get

$$E_{\lambda,\mu}\left(\frac{f(2^{n+m}x)}{4^{n+m}}-\frac{f(2^mx)}{4^m}\right)$$

\

$$\leq \frac{1}{12}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^i2^mx,2^i2^mx)+E_{\lambda,\mu^1}\phi(0,2^i2^mx)}{4^{i+m}}\right)$$

$$\leq \frac{1}{12}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^{i+m}x,2^{i+m}x)+E_{\lambda,\mu^1}\phi(0,2^{i+m}x)}{4^{i+m}}\right) \tag{4.10}$$

This implies that $\{f(2^nx)/4^n\}$ is a Cauchy sequence in X by taking the Limit $m\to\infty$ since (Y, μ, min) is a complete RN –Space it follows that the sequence $\{f(2^nx)/4^n\}$ converges in (Y, μ, min). Taking $m = 0$ in the equation (4.10), we get

$$E_{\lambda,\mu}\left(\frac{f(2^nx)}{4^n}-f(x)\right)$$

$$\leq \frac{1}{12}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix)+E_{\lambda,\mu^1}\phi(0,2^ix)}{4^i}\right)$$

So that

$E_{\lambda,\mu}\ (C(x)-f(x))$

$$\leq\ E_{\lambda,\mu}\left(C(x)-\frac{f(2^nx)}{4^n}\right) +\ E_{\lambda,\mu}\left(\frac{f(2^nx)}{4^n}-f(x)\right)$$

$$\leq\ E_{\lambda,\mu}\left(C(x)-\frac{f(2^nx)}{4^n}\right) + \frac{1}{12}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix)+E_{\lambda,\mu^1}\phi(0,2^ix)}{4^i}\right)$$

Now taking the limit as $n\to\infty$, we get

$$E_{\lambda,\mu}\ (C(x)-f(x)) \leq \frac{1}{12}\sum_{i=0}^{\infty}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix)+E_{\lambda,\mu^1}\phi(0,2^ix)}{4^i}\right)$$

$$\leq \frac{1}{12}\sum_{i=0}^{\infty}\frac{4\alpha^i}{4^i}E_{\lambda,\mu^1}\phi(x,x) + \frac{1}{12}\sum_{i=0}^{\infty}\frac{\alpha^i}{4^i}E_{\lambda,\mu^1}\phi(0,x)$$

$$\leq\ E_{\lambda,\mu^1}\ \phi(x,x)\ \frac{4}{3(4-\alpha)} +\ E_{\lambda,\mu^1}\ \phi(0,x)\ \frac{1}{3(4-\alpha)} \tag{4.11}$$

that is

$\inf\{\ t>0\ ;\ \mu_{C(x)-f(x)} > 1\text{-}\lambda\}$

$\leq \inf\ \{t>0\ ;\ \mu^1_{4\phi(x,x)}3(4-\alpha)(t)+\mu^1_{\phi(0,x)}3(4-\alpha)(t) > 1\text{-}\lambda\}$ (4.12)

then, we get

$$\mu_{C(x)-f(x)}(t) \geq\ \mu^1_{4\phi(x,x)}3(4-\alpha)(t)+\mu^1_{\phi(0,x)}3(4-\alpha)(t) \tag{4.13}$$

Now, for uniqueness let there exist a quadratic mapping $T: X\to Y$ which satisfies (4.3) then, clearly $C(2^nx) = 4^nC(x)$ and $T(2^nx) = 4^nT(x)$, for all $n\in N$. It follows from (4.3) that

$\mu_{(C(x)-T(x))}(t)\ \ =\ Lim_{n\to\infty}\mu_{(C(2^nx)/4^n)-(T(2^nx)/4^n)}(t)$

$\mu_{(C(2^nx)/4^n)-(T(2^nx)/4^n)}(t)$

$\geq \min\{\ \mu_{(C(2^nx)/4^n)-(f(2^nx)/4^n)}(t/2),\ \mu_{(T(2^nx)/4^n)-(f(2^nx)/4^n)}(t/2)\ \}$

$\geq\ \mu^1_{\phi(0,2^nx)}(4^n(4-\alpha)t)\ \geq \mu^1_{\phi(0,x)}\left(\frac{4^n(4-\alpha)t}{\alpha^n}\right)$

Since $Lim_{n\to\infty}\,(4^n(4-\alpha)t)/\alpha^n = \infty$, we get $Lim_{n\to\infty}$ $\mu^1_{\phi(0,x)}(4^n(4-\alpha)t)/\alpha^n = 1$. Therefore it follows that $\mu_{C(x)-T(x)}(t) = 1$ for all $t > 0$ and so $C(x) = T(x)$. This completes the proof.

Theorem 4.2:- Let X be a real linear space, (Z, μ^1, min) an RN-space. Let $\phi: X\times X\to Z$ be a function such that for some $0<\alpha<8$

$$\mu^1_{\phi(2x,2x)}(t)\ \geq\ \mu^1_{\alpha\phi(x,x)}(t) \tag{4.14}$$

and

$$\mu^1_{\phi(0,2x)}(t)\ \geq\ \mu^1_{\alpha\phi(0,x)}(t) \tag{4.15}$$

$f(0)=0$ and $Lim_{n\to\infty}\mu^1_{\phi(2^nx,2^ny)}8^n(t)\ = 1$, for all $x, y\in X$ and all $t > 0$. Let (Y, μ, min) be a complete RN- space. If $f: X\to Y$ an odd function satisfies the inequality

$$\mu_{6f(x+y)-6f(x-y)+4f(3y)-3f(x+2y)+3f(x-2y)-9f(2y)}(t)\ \geq\ \mu^1_{\phi(x,y)}(t) \tag{4.16}$$

for all $x, y\in X$ and all $t > 0$. Then there exists a unique cubic mapping $C: X\to Y$ such that

$\mu_{C(x)-f(x)}(t) \geq \mu^1_{4\phi(x,x)}3(8-\alpha)(t) + \mu^1_{\phi(0,x)}3(8-\alpha)(t)$, (4.17)

for all $x \in X$ and all $t > 0$

Proof:- By using the Lemma (2.5) equation (4.16) implies that

$E_{\lambda,\mu}$ $(6f(x+y)-6f(x-y)+4f(3y)-3f(x+2y)+3f(x-2y)-9f(2y))$

$= \inf\{t > 0: \mu_{6f(x+y)-6f(x-y)+4f(3y)-3f(x+2y)+3f(x-2y)-9f(2y)}(t) > 1-\lambda\}$

$\leq \inf\{t>0;\ \mu^1_{\phi(x,y)}(t) > 1-\lambda\}$

$= E_{\lambda,\mu^1}$ $(\phi(x,y))$ for all $x, y \in X$ and $\lambda \in (0, 1)$ (4.18)

Putting $x = 0$, and replace y by x, we get

$E_{\lambda,\mu}$ $(6f(x)-6f(-x)+4f(3x)-3f(2x)+3f(-2x)-9f(2x))$
$\leq E_{\lambda,\mu^1}$ $(\phi(0,x))$
$E_{\lambda,\mu}$ $(12f(x)+4f(3x)-15f(2x)) \leq E_{\lambda,\mu^1}$ $(\phi(0,x))$,
for all $x \in X$. (4.19)

Putting $y = x$ in (4.18) we get

$E_{\lambda,\mu}$ $(3f(x)-f(3x)+3f(2x)) \leq E_{\lambda,\mu^1}$ $(\phi(x,x))$, (4.20)

for all $x \in X$. Now from (4.19) and (4.20), we have

$E_{\lambda,\mu}(24f(x)-3f(2x))$
$\leq E_{\lambda,\mu}(12f(x)+4f(3x)-15f(2x)-12f(x)-4f(3x)+12f(2x))$
$\leq E_{\lambda,\mu}(12f(x)+4f(3x)-15f(2x)) + 4E_{\lambda,\mu}(3f(x)-f(3x)+3f(2x))$
$\leq 4\, E_{\lambda,\mu^1}\phi(x,x) + E_{\lambda,\mu^1}\phi(0,x)$, $E_{\lambda,\mu}\left(\frac{f(2x)}{8} - f(x)\right)$
$\leq \frac{1}{24}[4\, E_{\lambda,\mu^1}\phi(x,x) + E_{\lambda,\mu^1}\phi(0,x)]$ (4.21)

for all $x \in X$. Now replacing x with $2x$ in (4.21) and then dividing by 8, the resulting inequality with (4.21) gives

$E_{\lambda,\mu}\left(\frac{f(2^2x)}{8^2} - \frac{f(2x)}{8}\right) \leq \frac{1}{192}[4\, E_{\lambda,\mu^1}\phi(2x,2x) + E_{\lambda,\mu^1}\phi(0,2x)]$,

$E_{\lambda,\mu}\left(\frac{f(2^2x)}{8^2} - f(x)\right)$
$\leq \frac{1}{24}\left(\frac{E_{\lambda,\mu^1}4\phi(2x,2x) + E_{\lambda,\mu^1}\phi(0,2x)}{8}\right) + \frac{1}{24}\left[E_{\lambda,\mu^1}4\phi(x,x) + E_{\lambda,\mu^1}\phi(0,x)\right]$

for all $x \in X$. By induction on 'n' we can write

$E_{\lambda,\mu}\left(\frac{f(2^nx)}{8^n} - f(x)\right) \leq \frac{1}{24}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix) + E_{\lambda,\mu^1}\phi(0,2^ix)}{8^i}\right)$
for all $x \in X$. (4.22)

We divide (4.22) by 8^m and replacing x with $2^m x$, we get

$E_{\lambda,\mu}\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^mx)}{8^m}\right)$

$\leq \frac{1}{24}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^i2^mx,2^i2^mx) + E_{\lambda,\mu^1}\phi(0,2^i2^mx)}{8^{i+m}}\right)$
$\leq \frac{1}{24}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^{i+m}x,2^{i+m}x) + E_{\lambda,\mu^1}\phi(0,2^{i+m}x)}{8^{i+m}}\right)$ (4.23)

This implies that $\{f(2^nx)/8^n\}$ is a Cauchy sequence in X by taking the Lim $m\to\infty$ since (Y, μ, min) is a complete RN – Space it follows that the sequence $\{f(2^nx)/8^n\}$ converges in (Y, μ, min). Taking $m = 0$ in (4.23), we get

$E_{\lambda,\mu}\left(\frac{f(2^nx)}{8^n} - f(x)\right) \leq \frac{1}{24}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix) + E_{\lambda,\mu^1}\phi(0,2^ix)}{8^i}\right)$

Now taking
$E_{\lambda,\mu}$ $(C(x)-f(x))$
$\leq E_{\lambda,\mu}\left(C(x) - \frac{f(2^nx)}{8^n}\right) + E_{\lambda,\mu}\left(\frac{f(2^nx)}{8^n} - f(x)\right)$
$\leq E_{\lambda,\mu}\left(C(x) - \frac{f(2^nx)}{8^n}\right) + \frac{1}{24}\sum_{i=0}^{n-1}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix) + E_{\lambda,\mu^1}\phi(0,2^ix)}{8^i}\right)$
$E_{\lambda,\mu}$ $(C(x)-f(x)) \leq \frac{1}{24}\sum_{i=0}^{\infty}\left(\frac{E_{\lambda,\mu^1}4\phi(2^ix,2^ix) + E_{\lambda,\mu^1}\phi(0,2^ix)}{8^i}\right)$
$\leq \frac{1}{24}\sum_{i=0}^{\infty}\frac{4\alpha^i}{8^i}E_{\lambda,\mu^1}\phi(x,x) + \frac{1}{24}\sum_{i=0}^{\infty}\frac{\alpha^i}{8^i}E_{\lambda,\mu^1}\phi(0,x)$
$\leq E_{\lambda,\mu^1}\phi(x,x)\ \frac{4}{3(8-\alpha)} + E_{\lambda,\mu^1}\phi(0,x)\ \frac{1}{3(8-\alpha)}$ (4.24)
that is

$\inf\{\,t > 0\,;\ \mu_{C(x)-f(x)}(t) > 1-\lambda\}$
$\leq \inf\{t > 0\,;\ \mu^1_{4\phi(x,x)}3(8-\alpha)(t) + \mu^1_{\phi(0,x)}3(8-\alpha)(t) > 1-\lambda\}$ (4.25)

then, we have

$\mu_{C(x)-f(x)}(t) \geq \mu^1_{4\phi(x,x)}3(8-\alpha)(t) + \mu^1_{\phi(0,x)}3(8-\alpha)(t)$ (4.26)
Now, To prove uniqueness, let there exists another cubic mapping $T: X \to Y$ which satisfies the equation (4.16), then clearly $C(2^nx) = 2^nC(x)$ and $T(2^nx) = 2^nT(x)$ for all $n \in N$. Then it follows from (4.16)

$\mu_{(C(x)-T(x))}(t) = Lim_{\,n\to\infty}\ \mu_{(C(2^nx)/8^n)-(T(2^nx)/8^n)}(t)$
$\mu_{(C(2^nx)/8^n)-(T(2^nx)/8^n)}(t)$
$\geq \min\{\mu_{(C(2^nx)/8^n)-(f(2^nx)/8^n)}(t/2),\ \mu_{(T(2^nx)/8^n)-(f(2^nx)/8^n)}(t/2)\}$
$\geq \mu^1_{\phi(0,2^nx)}(8^n(8-\alpha)t) \geq \mu^1_{\phi(0,x)}\left(\frac{8^n(8-\alpha)t}{\alpha^n}\right)$

Since $Lim_{n\to\infty}$ $(8^n(8-\alpha)t)/\alpha^n = \infty$, we get $Lim_{n\to\infty}$ $\mu^1_{\phi(0,x)}(8^n(8-\alpha)t)/\alpha^n = 1$. Therefore it follows that $\mu_{C(x)-T(x)}(t) = 1$ for all $t > 0$ and so $C(x) = T(x)$.

Corollary 4.3:- Let X be a linear space, (Z, μ^1, min) be a RN space, and (Y, μ^1, min) be a complete RN-space. If $f: X \to Y$ be a mapping such that

$$\mu_{6f(x+y)-6f(x-y)+4f(3y)-3f(x+2y)+3f(x-2y)-9f(2y)}(t) \geq \mu^1_{(\|x\|^p+\|y\|^p)}(t), \quad (4.30)$$

for all $x, y \in X$, $\forall\, t > 0$, $f(0) = 0$, $\theta \geq 0$ and $p < 2$, then there exists a unique cubic mapping $C: X \to Y$ and a unique quadratic mapping $Q: X \to Y$ such that

$$\mu_{f(x)-C(x)-Q(x)}(t)$$

$$\geq \mu_{3\theta\|x\|^p}(4-2^p)(t) + \mu_{3\theta\|x\|^p}(8-3^p)(t), \quad (4.31)$$

for all $x \in X$ and all $t > 0$.

Proof: - Define $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ and applying Theorem (4.3) we get the desired result.

5.1 Stability of Jensen-Type Quadratic Functional Equations (1.3) and (1.4) in RN-space

We prove the Hyers-Ulam-Rassias stability of equation (1.3) in random normed space.

Theorem 5.1:- Let X be a linear space, (Z, μ^1, min) an RN-space, and let $\phi: X \times X \to Z$ be a function such that for some $0<\alpha<4$

$$\mu^1_{\phi(2x,0)}(t) \geq \mu^1_{\alpha\phi(x,0)}(t) \quad (5.1)$$

$f(0) = 0$ and $Lim_{n\to\infty}\mu^1_{\phi(2^n x, 2^n y)} 4^n(t) = 1$, for all $x, y \in X$ and all $t > 0$. Let (Y, μ, min) be a complete RN-space. If $f: X \to Y$ is a mapping such that

$$\mu_{(2(x+y)/2+2f(x-y)/2-f(x)-f(y))}(t) \geq \mu^1_{\phi(x,y)}(t),$$

for all $x, y \in X$ and all $t > 0$ (5.2)

Then there exists a unique cubic mapping $C: X \to Y$ such that $\mu_{f(x)-C(x)}(t) \geq \mu^1_{\alpha\phi(x,0)}(4-\alpha)(t)$ for all $x \in X$ and all $t > 0$. (5.3)

Proof: - By using the Lemma (2.5) and inequality (5.2), implies that

$$E_{\lambda,\mu}\ (2f((x+y)/2) + 2f((x-y)/2) - f(x) - f(y))$$

$= \inf\{t > 0$: $\mu_{(2f(x+y)/2+2f(x-y)/2-f(x)-f(y))}(t) > 1-\lambda\}$

$\leq \inf\{t > 0;\ \mu^1_{\phi(x,y)}(t) > 1-\lambda\}$

$= E_{\lambda,\mu^1}\ (\phi(x,y))$ for all $x, y \in X$, $\lambda \in (0, 1)$ (5.4)

Putting $y = 0$ and replacing x with $2x$, we get

$$E_{\lambda,\mu}\ (2f((2x)/2) + 2f((2x)/2) - f(2x)) \leq E_{\lambda,\mu^1}\ (\phi(2x,0))$$

$$E_{\lambda,\mu}\ (2f(x) + 2f(x) - f(2x)) \leq E_{\lambda,\mu^1}\ (\phi(2x,0))$$

$$E_{\lambda,\mu}\ (4f(x) - f(2x)) \leq E_{\lambda,\mu^1}\ (\phi(2x,0))$$

$$E_{\lambda,\mu}\left(\frac{f(2x)}{4} - f(x)\right) \leq \frac{1}{4} E_{\lambda,\mu^1}\ (\phi(2x,0))$$

Replacing x with $2x$ and dividing by 4, we have

$$E_{\lambda,\mu}\left(\frac{f(2^2x)}{4^2} - \frac{f(2x)}{4}\right) \leq \frac{1}{4^2} E_{\lambda,\mu^1}\ (\phi(2^2x,0))$$

$$E_{\lambda,\mu}\left(\frac{f(2^2x)}{4^2} - f(x)\right)$$

$$\leq \frac{1}{4^2} E_{\lambda,\mu^1}\ (\phi(2^2x,0)) + \frac{1}{4} E_{\lambda,\mu^1}\ (\phi(2x,0))$$

for all $x \in X$. By induction on 'n' we can write

$$E_{\lambda,\mu}\left(\frac{f(2^nx)}{4^n} - f(x)\right) \leq \sum_{i=1}^{n} \frac{1}{4^i} E_{\lambda,\mu^1}\ (\phi(2^ix,0)) \quad (5.5)$$

Again replacing x with $2^m x$ and dividing by 4^m in (5.5), we obtain

$$E_{\lambda,\mu}\left(\frac{f(2^{n+m}x)}{4^{n+m}} - \frac{f(2^mx)}{4^m}\right) \leq \sum_{i=1}^{n} \frac{1}{4^{i+m}} E_{\lambda,\mu^1}\ (\phi(2^{i+m}x,0))$$

This implies that the sequence $\{f(2^nx)/4^n\}$ is a Cauchy sequence as $m\to\infty$, since $\{Y, \mu, min\}$ is complete random normed space, thus the sequence $\{f(2^nx)/4^n\}$ is convergent in $\{Y, \mu, min\}$. Taking $m = 0$, we get

$$E_{\lambda,\mu}\left(\frac{f(2^nx)}{4^n} - f(x)\right) \leq \sum_{i=1}^{n} \frac{1}{4^i} E_{\lambda,\mu^1}\ (\phi(2^ix,0)) \quad (5.6)$$

and so,

$$E_{\lambda,\mu}\ (C(x) - f(x))$$

$$\leq E_{\lambda,\mu}\left(C(x) - \frac{f(2^nx)}{4^n}\right) + E_{\lambda,\mu}\left(\frac{f(2^nx)}{4^n} - f(x)\right)$$

Using above equation (5.6), we get

$$E_{\lambda,\mu}\ (C(x) - f(x))$$

$$\leq E_{\lambda,\mu}\left(C(x) - \frac{f(2^nx)}{4^n}\right) + \sum_{i=1}^{n} \frac{1}{4^i} E_{\lambda,\mu^1}\ (\phi(2^ix,0))$$

Taking $n \to \infty$, we get

$$E_{\lambda,\mu}\ (C(x) - f(x)) \leq \sum_{i=1}^{\infty} \frac{1}{4^i} E_{\lambda,\mu^1}\ (\phi(2^ix,0)) \leq \sum_{i=1}^{\infty} \frac{\alpha^i}{4^i} E_{\lambda,\mu^1}\ (\phi(x,0)) \leq \sum_{i=1}^{\infty} \left(\frac{\alpha}{4}\right)^i E_{\lambda,\mu^1}\ (\phi(x,0)) \leq$$

$$E_{\lambda,\mu^1}\ (\phi(x,0))\ \frac{\alpha}{4-\alpha} \quad (5.7)$$

that is, $\inf\{t > 0$: $\mu_{(2(x+y)/2+2f(x-y)/2-2f(x)-2f(y))}(t) > 1-\lambda\}$

$\leq \inf\{t > 0;\ \mu^1_{\phi(x,y)}(t) > 1-\lambda\}$ (5.8)

then, we get

$$\mu_{C(x)-f(x)}(t) \geq \mu^1_{\alpha\phi(x,0)}(4-\alpha)(t) \quad (5.9)$$

Now, to prove uniqueness of the quadratic mapping C, let us consider another quadratic equation $T: X \to Y$ which satisfies (5.2). Fix $x \in X$ then we have $C(2^n x) = 4^n C(x)$ and $T(2^n x) = 4^n T(x)$ for all $n \in N$. It follows from (5.2) that

$\mu_{C(x)-T(x)}(t) = \mathrm{Lim}_{n\to\infty}\ \mu_{(C(2^nx)/4^n)-(T(2^nx)/4^n)}(t)$

$\mu_{(C(2^nx)/4^n)-(T(2^nx)/4^n)}(t) \geq \min\{\mu_{(C(2^nx)/4^n)-(f(2^nx)/4^n)}(t/2),$

$\mu_{(T(2^nx)/4^n)-(f(2^nx)/4^n)}(t/2)\}$

$\geq \mu^1_{\phi(2^n x,0)}(4^n(4-\alpha)t) \geq \mu^1_{\phi(x,0)}\left(\frac{4^n(4-\alpha)t}{\alpha^n}\right)$

Since $\text{Lim}_{n\to\infty}$ $(4^n(4-\alpha)t)/\alpha^n) = \infty$, we get $\text{Lim}_{n\to\infty}$ $\mu^1_{\phi(x,0)}(4^n(4-\alpha)t)/\alpha^n = 1$. Therefore, it follows that $\mu_{C(x)-T(x)}(t) = 1$ for all $t > 0$ and so $C(x) = T(x)$. This completes the proof.

Corollary 5.2:- Let X be a linear space, (Z, μ^1, min) an RN-space and (Y, μ, min) a complete RN-space. Let $\phi: X\times X \to Z$ a function such that some $0<\alpha<9$

$$\mu^1_{\phi(3x,0)}(t) \geq \mu^1_{\alpha\phi(x,0)}(t) \quad (5.10)$$

$f(0)=0$ and $Lim_{n\to\infty}\mu^1_{\phi(3^n x, 3^n y)} 9^n(t) = 1$, for all $x, y \in X$ and all $t > 0$. If $f: X \to Y$ is a mapping such that

$$\mu_{(2(x+y)/2+2f(x-y)/2-f(x)-f(y))}(t) \geq \mu^1_{\phi(x,y)}(t), \quad (5.11)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique cubic mapping $C: X\to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu^1_{\alpha\phi(x,0)}(9-\alpha)(t), \text{ for all } x \in X \text{ and all } t > 0 \quad (5.12)$$

Proof:-Taking the value of α from 0 to 9 and using the Theorem (5.1) we get the required result.

Now, we prove the Hyers-Ulam stability of equation (1.4) in random normed space.

Theorem 5.3:- Let X be a linear space, (Z, μ^1, min) an RN-space and (Y, μ, min) be a complete RN-space. Let $\phi: X\times X \to Z$ a function such that some $0<\alpha<4a^2$, $a \neq (\pm 1/2)$

$$\mu^1_{\phi(ax,0)}(t) \geq \mu^1_{\alpha\phi(x,0)}(t) \quad (5.13)$$

$f(0)=0$ and $Lim_{n\to\infty}\mu^1_{\phi(a^n x, a^n y)} a^{2n}(t) = 1$, for all $x, y \in X$ and all $t > 0$. If $f: X \to Y$ is a mapping such that

$$\mu_{(f(ax+ay)+f(ax-ay)-2a^2 f(x)-2a^2 f(y))}(t) \geq \mu^1_{\phi(x,y)}(t), \quad (5.14)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $C: X\to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu^1_{\phi(x,0)} 2(a^2-\alpha)(t), \text{ for all } x \in X \text{ and all } t > 0 \quad (5.15)$$

Proof: - By using Lemma (2.5) and inequality (5.14), implies that

$E_{\lambda,\mu}\ (f(ax+ay)+f(ax-ay)-2a^2 f(x)-2a^2 f(y))$

$= \inf\{t > 0: \mu_{(f(ax+ay)+f(ax-ay)-2a^2 f(x)-2a^2 f(y))}(t) > 1-\lambda\}$

$\leq \inf\{t > 0: \mu^1_{\phi(x,y)}(t) > 1-\lambda\} = E_{\lambda,\mu^1}\ (\phi(x,y)) \quad (5.16)$

for all $x, y \in X$, $\lambda \in (0, 1)$.

Putting $y = 0$, we get

$$E_{\lambda,\mu}\ (f(ax)+f(ax)-2a^2 f(x)-2a^2 f(0)) \leq E_{\lambda,\mu^1}\ (\phi(x,0))$$

$$E_{\lambda,\mu}\ (2f(ax)-2a^2 f(x)) \leq E_{\lambda,\mu^1}\ (\phi(x,0))$$

$$E_{\lambda,\mu}\left(\frac{f(ax)}{a^2}-f(x)\right) \leq \frac{1}{2a^2}E_{\lambda,\mu^1}\ (\phi(x,0))$$

Replacing x with $a\,x$ and dividing by a^2, we get

$$E_{\lambda,\mu}\left(\frac{f(a^2x)}{a^4}-\frac{f(ax)}{a^2}\right) \leq \frac{1}{2a^4}E_{\lambda,\mu^1}\ (\phi(ax,0))$$

$$E_{\lambda,\mu}\left(\frac{f(a^2x)}{a^4}-f(x)\right)$$

$$\leq \frac{1}{2a^4}E_{\lambda,\mu^1}\ (\phi(ax,0)) + \frac{1}{2a^2}E_{\lambda,\mu^1}\ (\phi(x,0))$$

for all $x \in X$. By induction we can write

$$E_{\lambda,\mu}\left(\frac{f(a^n x)}{a^{2n}}-f(x)\right) \leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{a^{2i+2}}E_{\lambda,\mu^1}\ (\phi(a^i x,0)) \quad (5.17)$$

Again replacing x by $a^m x$ and dividing by a^{2m}, we have

$$E_{\lambda,\mu}\left(\frac{f(a^{n+m}x)}{a^{2n+2m}}-\frac{f(a^m x)}{a^{2m}}\right) \leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{a^{2i+2m+2}}E_{\lambda,\mu^1}\ (\phi(a^{i+m}x,0)) \quad (5.18)$$

This implies that the sequence $\{f(a^n x)/a^{2n}\}$ is a Cauchy sequence as $m\to\infty$, since $\{Y, \mu, min\}$ is complete random normed space, thus the sequence $\{f(a^n x)/a^{2n}\}$ is convergent in $\{Y, \mu, min\}$. Taking $m = 0$, we get

$$E_{\lambda,\mu}\left(\frac{f(a^n x)}{a^{2n}}-f(x)\right) \leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{a^{2i+2}}E_{\lambda,\mu^1}\ (\phi(a^i x,0)) \quad (5.19)$$

and so,

$E_{\lambda,\mu}\ (C(x)-f(x))$

$$\leq E_{\lambda,\mu}\left(C(x)-\frac{f(a^n x)}{a^{2n}}\right) + E_{\lambda,\mu}\left(\frac{f(a^n x)}{a^{2n}}-f(x)\right)$$

Using above equation (5.19), we get

$E_{\lambda,\mu}\ (C(x)-f(x))$

$$\leq E_{\lambda,\mu}\left(C(x)-\frac{f(a^n x)}{a^{2n}}\right) + \frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{a^{2i+2}}E_{\lambda,\mu^1}\ (\phi(a^i x,0))$$

Taking $n \to \infty$, we get

$$E_{\lambda,\mu}\ (C(x)-f(x)) \leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{a^{2i+2}}E_{\lambda,\mu^1}\ (\phi(a^i x,0))$$

$$\leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{\alpha^i}{a^{2i+2}}E_{\lambda,\mu^1}\ (\phi(x,0))$$

$$\leq E_{\lambda,\mu^1}\ (\phi(x,0))\ \frac{1}{2(a^2-\alpha)} \quad (5.20)$$

that is,

$\inf\{t > 0: \mu_{(f(ax+ay)+f(ax-ay)-2a^2 f(x)-2a^2 f(y))}(t) > 1-\lambda\}$

$\leq \inf\{t > 0;\ \mu^1_{\phi(x,y)}(t) > 1-\lambda\}$ (5.21)

then, we get $\mu_{C(x)-f(x)}(t) \geq \mu^1_{\phi(x,0)} 2(a^2-\alpha)t$ (5.22)

Now, to prove uniqueness of the quadratic mapping C, let us consider another quadratic mapping $T: X \to Y$ which satisfies (5.14). Fix $x \in X$ then we have $C(a^n x) = a^{2n} C(x)$ and $T(a^n x) = a^{2n} T(x)$ for all $n \in N$. It follows from (5.14) that

$\mu_{C(x)-T(x)}(t) = \mathrm{Lim}_{n\to\infty}\ \mu_{(C(a^n x)/a^{2n})-(T(a^n x)/a^{2n})}(t)$

$\mu_{(C(a^n x)/a^{2n})-(T(a^n x)/a^{2n})}(t) \geq \min\{\mu_{(C(a^n x)/a^{2n})-(f(a^n x)/a^{2n})}(t/2),$
$\mu_{(T(a^n x)/a^{2n})-(f(a^n x)/a^{2n})}(t/2)\}$

$\geq \mu^1_{\phi(a^n x,0)}(a^{2n}(a^2-\alpha)t) \geq \mu^1_{\phi(x,0)}\left(\frac{a^{2n}(a^2-\alpha)t}{\alpha^n}\right)$

Since $\mathrm{Lim}_{n\to\infty}\ (a^{2n}(a^2-\alpha)t)/\alpha^n = \infty$, we get $\mathrm{Lim}_{n\to\infty}$ $\mu^1_{\phi(x,0)}(a^{2n}(a^2-\alpha)t)/\alpha^n = 1$. Therefore it follows that $\mu_{C(x)-T(x)}(t) = 1$ for all $t > 0$ and so $C(x) = T(x)$.
This completes the proof.

Corollary 5.4:- Let X be a linear space, (Z, μ^1, min) be a RN-space and (Y, μ, min) be a complete RN-space. Let $\phi: X \times X \to Z$ a function such that some $\alpha > 4a^2$, $a \neq (\pm 1/2)$

$$\mu^1_{\varphi(ax,0)}(t) \geq \mu^1_{\alpha\phi(x,0)}(t) \quad (5.23)$$

$f(0) = 0$ and $Lim_{n\to\infty}\mu^1_{\phi(a^n x, a^n y)} a^{2n}(t) = 1$, for all $x, y \in X$ and all $t > 0$. If $f: X \to Y$ is a mapping such that

$$\mu_{(f(ax+ay)+f(ax-ay)-2a^2 f(x)-2a^2 f(y))}(t) \geq \mu^1_{\phi(x,y)}(t), \quad (5.24)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $C: X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu^1_{\phi(x,0)} 2(a^2-\alpha)(t), \text{ for all } x \in X \text{ and all } t > 0 \quad (5.25)$$

Proof: - Applying Theorem (5.3) we get the desired result.

Example 5.1:- Let $(X, \|.\|)$ be a Banach Algebra and

$$\mu_x(t) = \begin{cases} \max\left\{1 - \frac{\|x\|}{t}, 0\right\} & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases}$$

for every $x, y \in X$ and $a \in R$, let

$$\mu^1_{\phi(x,y)}(t) = \begin{cases} \max\left\{1 - \frac{(2a+2a^2)\|x\| + (2a+2a^2)\|y\|}{t}, 0\right\} if & t > 0 \\ 0 & if \quad t \leq 0 \end{cases}$$

We know that norm is a distribution function and $Lim_{n\to\infty}\mu^1_{\phi(a^n x, a^n y)} a^{2n}(t) = 1$ for every $x, y \in X$ and $t > 0$. As we know that (X, μ, T) is a RN-space. In fact, $\mu_x(t) = 1$ for all $t > 0 \Rightarrow \frac{\|x\|}{t} = 0$ for all $t > 0 \Rightarrow x = 0$ and certainly (PN2) $\mu_{\lambda x}(t) = \mu_x\left(\frac{t}{\lambda}\right)$ for all $n \in X$ and $t > 0$. Next for every $x, y \in X$ and $t, s > 0$ we obtain

$$\mu_{x+y}(t+s) = \max\left\{1 - \frac{\|x+y\|}{t+s}, 0\right\} = \max\left\{1 - \left\|\frac{x+y}{t+s}\right\|, 0\right\}$$
$$= \max\left\{1 - \left\|\frac{x}{t+s} + \frac{y}{t+s}\right\|, 0\right\}$$
$$\geq \max\left\{1 - \left\|\frac{x}{t}\right\| - \left\|\frac{y}{t}\right\|, 0\right\} \geq T(\mu_x(t), \mu_y(s))$$

Also RN-space (X, μ, T) is complete for

$$\mu_{x-y}(t) \geq 1 - \frac{\|x-y\|}{t} \qquad (x, y \in X, t > 0)$$

and hence $(X, \|.\|)$ is complete.

Let us define a mapping $f: X \to X$, $f(x) = x^2 + \|x\|x_0$, where x_0 is a unit vector in X. Now by using a simple calculation, we get

$\left\|f(ax+ay) + f(ax-ay) - 2a^2 f(x) - 2a^2 f(y)\right\|$

$= \left\|\|ax+ay\| + \|ax-ay\| - 2a^2\|x\| - 2a^2\|y\|\right\|\|x_0\|$

$\leq \left|(2a-2a^2)\|x\| + (2a+2a^2)\|y\|\right|\|x_0\|$

$\leq \left|(2a+2a^2)\|x\| + (2a+2a^2)\|y\|\right|\|x_0\|$

for all $x, y \in X$. Hence $\mu_{f(ax+ay)+f(ax-ay)-2a^2 f(x)-2a^2 f(y)}(t) \geq \mu^1_{\phi(x,y)}(t)$ for all $x, y \in X$ and $t > 0$

Now let

$$\mu^1_{\phi(a^n x,0)} a^{2n}(a^2-\alpha)t = \max\left\{1 - \frac{(2a+2a^2)\|a^n x\|}{a^{2n}(a^2-\alpha)t}, 0\right\}$$

where $0 < \alpha < 4a^2$ $a \neq (\pm 1/2)$

$\Rightarrow Lim_{n\to\infty}\mu^1_{\phi(a^n x,0)} a^{2n}(a^2-\alpha)t = 1$

which shows that all the conditions of Theorem (5.3) hold.
Since

$$\mu^1_{\phi(x,0)} 2(a^2-\alpha)t = \max\left\{1 - \frac{(2a+2a^2)\|x\|}{2(a^2-\alpha)t}, 0\right\},$$

We deduce that $C(x) = x^2$ is the unique cubic mapping $C: X \to X$ such that

$$\mu_{f(x)-C(x)}(t) \geq \max\left\{1 - \frac{(a+a^2)\|y\|}{(a^2-\alpha)t}, 0\right\},$$

for all $y \in X$ and $t > 0$.

4. CONCLUSION

Throughout this paper we introduced the following results:

(i) In the subsection 3.1, using the Hyers-Ulam approach we proved the stability of functioan equation (1.1) in random normed space.

(ii) In the subsection 4.1, we proved the stability of functional equation (1.2).

(iii) Further, in subsection 5.1 we proved the stability of equation (1.3) and (1.4) using an example and also introduced some corollaries for different conditions.

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