Stability of Functional Equations in Multi-Banach Spaces via Fixed Point Approach

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ABSTRACT

In this paper, using the fixed point approach, we proved the Hyers-Ulam-Rassias stability of a Jensen-type quadratic functional equations $f(ax \pm ay) - 2a^2[f(x) + f(y)]$ and $2f((x \pm y)/2) - f(x) - f(y)$ in Multi-Banach Spaces using the ideas from Dales and Polyakov [4].

Keywords

Fixed Point Alternative, Jensen-Type Quadratic functional equations, Multi-Banach spaces

1. INTRODUCTION

One of the interesting questions in the theory of non- linear functional analysis involved is the stability problem of functional equations as follows: Under what conditions is there a homomorphism near an approximately homomorphism between a group and a metric group, which was first given by S. M. Ulam [15]. In 1941, D. H. Hyers [2] gave the first affirmative answer to this question for approximately additive functions under the assumption of Banach spaces. Th. M. Rassias [17] gave the generalized version of Hyer's result for approximately linear mappings.

In1994, P. Gavruta [13] provided a further generalization of Th. M. Rassias [17] result in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function $\phi(x, y)$ for the existence of unique linear mapping. During last decades, Hyers-Ulam-Rassias stability of various functional equations have been extensively introduced by a number of mathematicians ([1], [16], [18-20]) on various spaces such as normed spaces, Banach space, Fuzzy normed space, RN-space, IRN-space, Non-Archimedean space etc.

In 1983, F. Skof [3] first proved the stability of the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for the mapping $f : X \to Y$, where X is a normed space and Y is a Banach space. P. Cholewa [12] again generalized the Skof's result for abelian groups. Lator on, Skof's [3] result was generalized by many mathematicians on various spaces. The functional equations

$$D(fx, fy) = f(ax \pm ay) - 2a^2[f(x) + f(y)] \quad (1.1)$$

and

$$D'(fx, fy) = 2f((x \pm y)/2) - f(x) - f(y) \quad (1.2)$$

for all x, y∈X are called Jensen- Type Quadratic functional equations. In 2009, S.Y.Jang, Rye Lee, Choonkil Park, and Dong Yun Shin [14] proved the Fuzzy stability of equation (1.1) and (1.2).

In the section 2, we adopt some usual terminology, notion and conventions of the theory of Multi-Banach spaces. In the last section, we prove the stability problem in the sense of Hyers-Ulam-Rassias for the functional equations (1.1) and (1.2) on Multi-Banach spaces by using fixed point approach. We also present some corollaries in reference to our results.

2. PRELIMINARIES

The multi-Banach space was first investigated by Dales and Polyakov [4]. Theory of multi-Banach spaces is similar to the operator sequence space and has some connections with operator spaces and Banach spaces. In 2007, H. G. Dales and M. S. Moslehian [5] first proved the stability of mappings on multi-normed spaces and also gave some examples on multi-normed spaces. The asymptotic aspects of the quadratic functional equations in multi-normed spaces was investigated by M. S. Moslehian, K. Nikodem, and D. Popa [9] in 2009. In last two decades, the stability of functional equations on multi-normed spaces was proved by many mathematicians ([7], [10], [21]).

Now, we adopt some usual terminology, notion and convention of the theory of multi-Banach spaces from [4] and [5].

Let $(E, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \ldots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinate-wise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by S_k the group of permutations on k symbols.

Definition 2.1(Multi - norm) A multi-norm on $\{E^k : k \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in E$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(N1) $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1, \ldots, x_k)\|_k$, for $\sigma \in S_k$, $x_1, \ldots, x_k \in E$;

(N2) $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|(x_1, \ldots, x_k)\|_k$, for $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, $x_1, \ldots, x_k \in E$;

(N3) $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$, for $x_1, \ldots, x_{k-1} \in E$;

(N4) $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$, for $x_1, \ldots, x_{k-1} \in E$

In this case, we say that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space (see [4], [5]).

Suppose that ((Ek, ||·||k) : k∈N) is a multi-normed space, and take k∈N. We need the following two properties of multi-norms. They can be found in [4].

||(x,…, x)||k = ||x||, for x∈E,

$\max_{i\in N_k}$ ||xi|| ≤ ||(x1,…, xk)||k ≤ $\sum_{i=1}^{k}\| x_i \|$ ≤ k $\max_{i\in N_k}$ ||xi||, for x1,…,xk ∈E.

It follows from (b) that if (E, ||·||) is a Banach space, then (Ek, ||·||k) is a Banach space for each k∈N; in this case, ((Ek, ||·||k) : k∈N) is a multi-Banach space.

Lemma 2.2. Suppose that k∈N and (x1,…, xk) ∈ Ek. For each j ∈ {1,…, k}, let $(x_n^j)_{n=1,2,\ldots}$ be a sequence in E such that limn–∞ x_n^j = xj. Then

limn→∞ $(x_n^1 - y_1, \ldots, x_n^k - y_k)$ = (x1 − y1,…, xk − yk)

holds for all (y1,…, yk) ∈Ek (see [4], [5]).

Definition 2.3. Let ((Ek, ||·||k) : k∈N) be a multi-normed space. A sequence {xn} in E is a multi-null sequence if for each ε > 0, there exists n0∈N such that

$\sup_{k\in N}$ ||(xn,…, xn+k–1)||k ≤ ε (n ≥ n0).

Let x ∈ E, we say that the sequence {xn} is multi-convergent to x in E and write limn→∞ xn = x if (xn − x) is a multi-null sequence (see [4, 5]).

Theorem 2.4. (Fixed Point Alternative) Let (X, d) be a complete generalized metric space and J : X→X be a strictly contractive mapping, that is,

d(Jx, Jy) ≤ Ld(x, y) ∀x, y∈X,

for some L ≤ 1. Then, for each fixed element x∈X, either

d(Jnx, Jn+1x) = ∞ ∀n ≥ 0,

or

d(Jnx, Jn+1x) < ∞ ∀n ≥ n0,

for some natural number n0≥0. Moreover, if the second alternative holds, then

the sequence {Jnx} is convergent to a fixed point y* of J;

y* is the unique fixed point of J in the set

V : = {y ∈ X| $d(J^{n_0}$ x, y) < ∞};

d(y, y*) ≤ 1/(1 – L) d(y, Jy), for all y∈V ([6], [11]).

Lemma 2.5. If a mapping f : X→Y satisfies the functional equation (1.1) and (1.2) then f is a Quadratic mapping.

3. MAIN RESULTS

In this section, we prove the Hyers – Ulam – Rassias stability of functional equations (1.1) and (1.2). Throughout this section, let E be a linear space and ((Fn, ||·||n): n∈N) be a multi-Banach space.

3.1 STABILITY OF THE FUNCTIONAL EQUATION (1.1) BY FIXED POINT METHOD

First, we prove a lemma, which gives a useful strictly contracting mapping.

Lemma 3.1. Let E be a linear space and (F^n, ||·||) be a Banach space for all n∈N. Let $0 < \alpha < a^2$ and a mapping ψ : $E^n \to [0, \infty)$ such that

$$\psi(ax_1, ax_2, \ldots, ax_k) \le \alpha\, \psi(x_1, x_2, \ldots, x_k)$$

for all $x_1, \ldots, x_k \in E$. Let S = {h : E→F : h(0) = 0}, and the generalized metric d defined on S by

$$d(g, h) = \inf\{v\in(0, \infty): \sup_{k\in N} \|(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k))\|_k \le v\psi(x_1, x_2, \ldots, x_k),\ \forall x_1, \ldots, x_k \in E\}$$

Then, it is easy to show that (S, d) is a complete generalized metric on S (see [8]). Define a mapping J_0 : S→S by

$$J_0g(x) = \frac{g(a^n x)}{a^{2n}}$$

for all g∈S is a strictly contractive mapping.

Proof: - It is easy to show that d is a complete metric on X. (see [8]). Given g, h∈S, let v∈(0, ∞) be an arbitrary constant with d(g, h) ≤ v. Then from the definition of d, it follows for each $x_1, \ldots, x_k \in E$ that

$$\sup_{k\in N} \|(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k))\| \le v\, \psi(x_1, \ldots, x_k),$$

and so

$$\sup_{k\in N} \|(J_0g(x_1) - J_0h(x_1), \ldots, J_0g(x_k) - J_0h(x_k))\|$$

$$\le \left\| \left(\frac{g(a^n x_1)}{a^{2n}} - \frac{h(a^n x_1)}{a^{2n}}, \ldots, \frac{g(a^n x_k)}{a^{2n}} - \frac{h(a^n x_k)}{a^{2n}} \right) \right\|$$

$$\le \frac{1}{a^{2n}} \|(g(a^n x_1) - h(a^n x_1), \ldots, g(a^n x_k) - h(a^n x_k))\|$$

$$\le \frac{\alpha^n}{a^{2n}}\, v\, \psi(x_1, \ldots, x_k)$$

for all $x_1, \ldots, x_k \in E$. Hence, it holds that

$$d(J_0g, J_0h) \le \frac{\alpha^n}{a^{2n}} d(g, h)$$

for all g, h∈S. Hence J_0 is a strictly contractive mapping with Lipschitz constant α^n / a^{2n}.

Theorem 3.1. Let (E, ||·||) be a normed space and let ((F^n, $\|\cdot\|_n$): n∈N) be a multi-Banach space. Let ε≥0, and let f:E→F be a mapping satisfying f(0) = 0 such that

$$\sup_{k\in N} \|D f(x_1, y_1), \ldots, D f(x_k, y_k)\|_k \le \varepsilon \quad (3.1)$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exists a unique quadratic mapping Q : E→F such that

$$\sup_{k\in N} \|(f(x_1) - Q(x_1), \ldots, f(x_k) - Q(x_k))\|_k$$

$$\leq \frac{a^{2n}\varepsilon}{(a^{2n}-1)(2a^2-2)} \quad \text{for all } x_1,\ldots,x_k \in E.$$

(3.2)

Proof: - Let $y_1,\ldots,y_k = 0$ in (3.1), we get

$$\sup_{k\in N}\left\|\left(\frac{f(ax_1)}{a^2}-f(x_1),\ldots,\frac{f(ax_k)}{a^2}-f(x_k)\right)\right\|_k \leq \frac{\varepsilon}{2a^2}$$

(3.3)

Again replacing x with ax and dividing by a^2, in (3.3), we get

$$\sup_{k\in N}\left\|\left(\frac{f(a^2x_1)}{a^4}-f(x_1),\ldots,\frac{f(a^2x_k)}{a^4}-f(x_k)\right)\right\|_k \leq \frac{\varepsilon}{2a^4}+\frac{\varepsilon}{2a^2} \quad (3.4)$$

By using induction for a positive integer 'n', we get

$$\sup_{k\in N}\left\|\left(\frac{f(a^nx_1)}{a^{2n}}-f(x_1),\ldots,\frac{f(a^nx_k)}{a^{2n}}-f(x_k)\right)\right\|_k \leq \frac{\varepsilon}{2}\sum_{i=1}^{n}\frac{1}{a^{2i}}$$

$$\leq \frac{\varepsilon}{2}\sum_{i=1}^{\infty}\frac{1}{a^{2i}} \quad \text{for all } x_1,\ldots,x_k \in E.$$

(3.5)

Now, let $S = \{h : h : E\to F,\ h(0) = 0\}$, and introduce the generalized metric d on E defined by

$$d(g, h) = \inf\{v\in(0,\infty);\ \sup_{k\in N}\|(g(x_1)-h(x_1),\ldots, g(x_k)-h(x_k))\|_k \leq v;\ x_1,\ldots, x_k\in E\}$$

(3.6)

Then, it is easy to show that d is a complete generalized metric on S [see [8]). Let us define a function $J_0 : S\to S$ by $J_0\, h(x) = h(a^nx)/a^{2n}$. We claim that J_0 is a strictly contractive mapping. Let g, h∈S and $v\in(0, \infty)$ be an arbitrary constant with $d(g, h) \leq v$. Now by using the definition of d, we get

$$\sup_{k\in N}\|(g(x_1) - h(x_1),\ldots, g(x_k) - h(x_k))\|_k \leq v$$

for all $x_1,\ldots, x_k \in E$.

Therefore,

$$\sup_{k\in N}\|(J_0\, g(x_1) - J_0\, h(x_1),\ldots, J_0\, g(x_k) - J_0\, h(x_k))\|_k =$$

$$\sup_{k\in N}\left\|\left(\frac{g(a^nx_1)}{a^{2n}}-\frac{h(a^nx_1)}{a^{2n}},\ldots,\frac{g(a^nx_1)}{a^{2n}}-\frac{h(a^nx_2)}{a^{2n}}\right)\right\|_k$$

$\leq \frac{v}{a^{2n}}$ for all $x_1,\ldots, x_k \in E$. Hence, we found that

$$d(J_0\, g, J_0\, h) \leq \frac{v}{a^{2n}} \leq \frac{1}{a^{2n}}\, d(g, h) \qquad (3.7)$$

for all g, h∈S. From (3.5), $d(J_0f, f) \leq \frac{\varepsilon}{2(a^2-1)}$. Using fixed point alternative, we show the existence of a fixed point of J_0, that is, the existence of a mapping $Q : E\to F$ satisfying the following:

(i) Q is a fixed point of J_0, that is $Q(a^nx) = a^{2n} Q(x)$ for all x∈E.

(ii) For any x∈E, we have $d(J_0^n f, Q) \to 0$, which implies that

$$Q(x) = \lim_{n\to\infty}\frac{f(a^nx)}{a^{2n}}, \text{ for all } x\in E \qquad (3.8)$$

(iii) Also, $d(f, g) \leq \frac{1}{1-L} d(J_0f, f)$ implies the inequality

$$d(f, g) \leq \frac{1}{1-\frac{1}{a^{2n}}}\, d(J_0f, f) \leq \frac{a^{2n}\varepsilon}{(a^{2n}-1)(2a^2-2)} \qquad (3.9)$$

Now, to prove that the mapping $Q : E \to F$ is quadratic, set $x_1 = x_k = a^nx$, $y_1 = y_l = a^ny$ in (3.1) and dividing both sides by a^{2n}, we have

$$\frac{1}{a^{2n}}\sup_{k\in N}\|D f(a^nx, a^ny),\ldots, D f(a^nx, a^ny)\|_k$$

$$\leq \lim_{n\to\infty}\frac{1}{a^{2n}}\|D f(a^nx, a^ny)\| \leq \lim_{n\to\infty}\frac{\varepsilon}{a^{2n}} = 0$$

for x, y∈E. Which shows that Q is a quadratic mapping satisfying (1.1) .

Since, Q is a unique fixed point of J_0, then if Q′ is another fixed point of J_0, thus Q = Q′ which completes the proof of theorem.

Theorem 3.2. Let E be a linear space, and let $((F^n, \|\cdot\|_n) : n\in N)$ be a multi-Banach space. Suppose $\psi : E^{2k} \to [0, \infty)$ for some $0 < \alpha < a^2$, k∈N.

$$\psi(ax_1, ay_1,\ldots, ax_k, ay_k) \leq \alpha\, \psi(x_1, y_1,\ldots, x_k, y_k) \qquad (3.10)$$

for all $x_1,\ldots, x_k, y_1,\ldots, y_k \in E$. If $f : E\to F$ is a mapping satisfying f(0) = 0 such that

$$\|D f(x_1, y_1),\ldots, D f(x_k, y_k)\|_k \leq \psi(x_1, y_1,\ldots, x_k, y_k) \qquad (3.11)$$

for all $x_1,\ldots, x_k, y_1,\ldots, y_k\in E$. Then, there exists a unique quadratic mapping $Q : E\to F$ such that

$$\|(f(x_1)-Q(x_1),\ldots, f(x_k)-Q(x_k))\|_k$$

$$\leq \frac{a^{2n}}{\alpha(a^{2n}-\alpha^n)(a^2-\alpha)}\, \psi(x_1, 0,\ldots, x_k, 0) \qquad (3.12)$$

for all $x_1,\ldots, x_k \in E$.

Proof :- Let $y_1, \ldots, y_k = 0$ in (3.1), we get

$\|(f(ax_1) - a^2 f(x_1), \ldots, f(ax_k) - a^2 f(x_k))\|_k$

$$\leq \frac{1}{2}\ \psi(x_1, 0, \ldots, x_k, 0) \tag{3.13}$$

again replacing x with ax, in (3.13) we obtain

$\|(f(a^2x) - a^4 f(x), \ldots, f(a^2x_k) - a^2 f(x_k))\|_k$

$$\leq \frac{a^2}{2}\psi(ax_1, 0, \ldots, ax_k, 0) + \frac{1}{2}\psi(x_1, 0, \ldots, x_k, 0)$$

By using induction for a positive integer 'n', we get

$$\left\| \frac{f(a^n x)}{a^{2n}} - f(x), \ldots, \frac{f(a^n x_k)}{a^{2n}} - f(x_k) \right\|_k \leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{a^{2(i+1)}}$$

$\psi(a^i x_1, 0, \ldots, a^i x_k, 0)$

$$\leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{\alpha^i}{a^{2(i+1)}}\psi(x_1, 0, \ldots, x_k, 0) \qquad \text{(using (3.10))}$$

$$\leq \frac{1}{2(a^2-\alpha)}\ \psi(x_1, 0, \ldots, x_k, 0) \tag{3.14}$$

Let $S = \{h : E \to F : h(0) = 0\}$ and introduce the generalized metric d as in Lemma (3.1). Define a function $J_0 : S \to S$ by

$$J_0\, h(x) = \frac{h(a^n x)}{a^{2n}}$$

for all $x \in E$, which is a strictly contractive mapping (see lemma 3.1). Given $g, h \in S$, let $v \in (0, \infty)$ be an arbitrary constant with $d(g, h) \leq v$. From the definition of d, it follows that

$$\sup_{k\in N} \|(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k))\|_k \leq v\, \psi(x_1, \ldots, x_k)$$

By using (3.14), we obtain

$$d(J_0 f, f) \leq \frac{1}{2(a^2-\alpha)}\ \psi(x_1, 0, \ldots, x_k, 0)$$

Using fixed point alternative, we deduce the existence of a unique fixed point of J_0, that is, the existence of mapping $Q : E \to F$ such that $Q(a^n x) = a^{2n}Q(x)$ for all $x \in E$. Moreover, we have $d(J_0^n f, Q) \to 0$, which implies that

$$Q(x) = \lim_{n\to\infty}\frac{f(a^n x)}{a^{2n}}, \text{ for all } x \in E$$

Hence,

$$d(f, Q) \leq \frac{1}{1-L}\ d(J_0 f, f) \text{ implies the inequality}$$

$$d(f, Q) \leq \frac{a^{2n}}{2(a^{2n}-\alpha^n)(a^2-\alpha)}\quad \psi(x_1, 0, \ldots, x_k, 0)$$

For fix $x \in E$, let us replace $x_1, x_2, \ldots, x_k$ by $a^n x$ and $y_1, \ldots, y_k$ by $a^n y$ in (3.11) and dividing by a^{2n}. Then, using property (a), we get

$$\left\| \frac{f(a^n(ax+ay))}{a^{2n}} + \frac{f(a^n(ax-ay))}{a^{2n}} - \frac{2a^2 f(a^n x)}{a^{2n}} - \frac{2a^2 f(a^n y)}{a^{2n}} \right\|_k$$

$$\leq \frac{1}{a^{2n}}\ \psi(a^n x, a^n y, \ldots, a^n x, a^n y).$$

as limit $n \to \infty$, we obtain

$$T(ax + ay) + T(ax - ay) = 2a^2\, T(ax) + 2a^2 T(a\, y)$$

for all $x, y \in E$.

Thus, the uniqueness of Q follows from the fact that Q is the unique fixed point of J_0. Which completes the proof of the theorem.

Corollary 3.1. Let E be a linear space, and let $((F^n, \|\cdot\|_n) : n \in N)$ be a multi-Banach space. Let $\theta \geq 0$ and $f : E \to F$ be a mapping satisfying $f(0) = 0$ such that

$$\|D f(x_1, y_1), \ldots, D f(x_k, y_k)\|_k \leq \theta(\|x_1\| + \|y_1\|, \ldots, \|x_k\| + \|y_k\|) \tag{3.15}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then, there exists a unique quadratic mapping $Q : E \to F$ such that

$\|(f(x_1) - Q(x_1), \ldots, f(x_k) - Q(x_k))\|_k$

$$\leq \frac{a^n\theta}{2(a^{2n}-1)(a^2-a)}(\|x_1\|, \ldots, \|x_k\|) \tag{3.16}$$

for all $x_1, \ldots, x_k \in E$.

3.2 STABILITY OF THE FUNCTIONAL EQUATION (1.2) BY FIXED POINT METHOD

Theorem 3.3. Let E be a linear space and $((F^n, \|\cdot\|_n) n \in N)$ be a multi-Banach space and $f : E \to F$ satisfies $f(0) = 0$ such that

$$\sup_{k\in N} \|D' f(x_1, y_1), \ldots, D' f(x_k, y_k)\|_k \leq \varepsilon \tag{3.17}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then, there exists a unique mapping $C : E \to F$ such that

$$\sup_{k\in N} \|(f(x) - C(x_1), \ldots, f(x_k) - C(x_k))\|_k \leq \frac{\varepsilon}{3} \tag{3.18}$$

for all $x_1, \ldots, x_k \in E$.

Proof. Let $y_1, \ldots, y_k = 0$ and replacing $x_1, \ldots, x_k$ with $2x_1, \ldots, 2x_k$ in (3.17), we get

$$\sup_{k\in N} \|(f(2x_1) - 4 f(x_1), \ldots, f(2x_k) - 4(f(x_k))\|_k \leq \varepsilon \tag{3.19}$$

for all $x_1, \ldots, x_k \in E$. Now, let $S = \{g \,;\, g : E \to F \,;\, g(0) = 0\}$, and introduce the generalized metric d defined on S by

$$d(g, h) = \inf\{v \in (0,\infty);\ \sup_{k\in N} \|(g(x_1)-h(x_1),\ldots,g(x_k)-h(x_k))\|_k \le v : x_1,\ldots,x_k \in E\} \quad (3.20)$$

Then, it is easy to show that d is a complete generalized metric on S (see [8]). Let us define a function $J_0 : S \to S$ by

$$J_0h(x) = \frac{1}{4} h(2x),\ \forall\ x\in E. \quad (3.21)$$

where J_0 is strictly contractive mapping with Lipschitz constant $1/4$. Given g, h∈S, let v∈(0, ∞) be an arbitrary constant with d(g, h) ≤ v. It follows from d that

$$\sup_{k\in N} \|(g(x_1) - h(x_1),\ldots, g(x_k) - h(x_k))\|_k \le v \quad (3.22)$$

for all $x_1, \ldots, x_k \in E$. Therefore,

$$\sup_{k\in N} \|(J_0g(x_1) - J_0h(x_1),\ldots, J_0g(x_k) - J_0h(x_k))\|_k =$$

$$\sup_{k\in N} \left\|\left(\frac{1}{4}g(2x_1) - \frac{1}{4}h(2x_1),\ldots,\frac{1}{4}g(2x_k) - \frac{1}{4}h(2x_k)\right)\right\|_k \le \frac{v}{4} \quad (3.23)$$

for all $x_1, \ldots, x_k \in E$. Hence, it shows that $d(J_0g, J_0h) \le \frac{v}{4}$

that is

$d(J_0g, J_0h) \le \frac{1}{4} d(g, h)$ for all g, h∈S. Now by using (3.19),

it holds that

$$d(J_0f, f) \le \frac{\varepsilon}{4}$$

Using fixed point alternative, there exists a fixed point of J_0, that is, the mapping $C : E \to F$ satisfying the following :

(i) C is a fixed point of J_0, that is

$C(2x) = 4C(x)$, for all x∈E.

Moreover the mapping C is unique fixed point of J_0 in the set $\Omega = \{h\in S : d(g, h) < \infty\}$.

(ii) We have $d(J_0^n f, C) \to 0$, which implies that

$$C(x) = \lim_{n\to\infty}(J_0^n f)(x) = \lim_{n\to\infty}\frac{f(2^n x)}{4^n},\ \forall\ x\in E.$$

(iii) Also, $d(f, C) \le \frac{d(J_0f, f)}{(1-L)}$ with $f\in\Omega$ implies that inequality

$$d(f, C) \le \frac{1}{1-\frac{1}{4}}\, d(J_0 f, f) \le \frac{\varepsilon}{3}$$

for all x∈E. Which implies that the inequality (3.18) holds. Now taking $x_1 = \ldots = x_k = 2^n x$, $y_1 = \ldots = y_k = 2^n y$ in (3.17), and dividing both sides by 4^n, we get,

$$\sup_{k\in N} \left\|\left(\frac{Df(2^n x, 2^n y)}{4^n},\ldots,\frac{Df(2^n x, 2^n y)}{4^n}\right)\right\|_k \le \frac{\varepsilon}{4^n}$$

taking limit as n→∞, we obtain.

$$\sup_{k\in N} \|D' f(x\ y)\| = 0$$

for all x, y∈E. It shows that C is a mapping satisfying the functional equation (1.2).The uniqueness of C follows from the fact that C is the unique fixed point of J_0. Hence completes the proof of the theorem.

Corollary 3.2. Let E be a linear space, and let $((F^n, \|\cdot\|_n) : n\in N)$ be a multi-Banach space. $f : E\to F$ satisfies f(0) = 0 such that

$$\sup_{k\in N} \|(D'f(x_1, y_1),\ldots, D'f(x_k, y_k))\| \le \psi(x_1, y_1,\ldots, x_k, y_k)$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ and $\psi : E^{2k} \to [0, \infty)$. Then there exists a unique mapping $C : E\to F$ such that

$$\sup_{k\in N} \|(f(x_1) - C(x_1),\ldots, f(x_k) - C(x_k))\|_k \le \frac{1}{3}\, \psi(x_1, 0, \ldots, x_k, 0)\ \forall x_1, \ldots, x_k \in E.$$

Proof :- Proof is similar to that of Theorem 3.3 by using the general condition instead of ε.

4. CONCLUSION

Throughout the paper we concluded the following results:

(i) In section 3 using the ideas of multi normed spaces from H. G. Dales and M. E. Polyakov [4], we proved the Hyers-Ulam-Rassias stability of the Jensen Type Quadratic functional equations (1.1) and (1.2) in Multi Banach spaces.

(ii) We also present some corollaries related to our results by using the general conditions.

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