An Algebraic Approach for stability Analysis of Linear Systems with Complex Coefficients

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ABSTRACT
In this paper employing Routh's table, a geometrical stability criterion for the analysis of linear time-invariant system is formulated. The proposed stability criterion is applied for the system, whose characteristic equation having complex coefficients. For this Routh like table is presented with complex terms and the signs pair-wise elements with the first column of the table are observed. The proof for the criterion is also given which is based on the Hurwitz's matrix and its determinants. It is found that the proposed method is termed as ''SIGN PAIR CRITERION'' and is illustrated with suitable examples.

General Terms
Stability Analysis, Complex Coefficients et. al.

Keywords
Hurwitz's matrix, Routh's array, Routh's table, Sign pair criterion.

1. INTRODUCTION
In the case of linear time invariant systems the roots can be analyzed with the help of characteristic polynomial F(s) of the system. If all the roots of F(s) =0 lie in the left half of s plane then the given linear system is said to be stable. If any of the roots exists with positive real part, then it represents the unstable nature of the given linear system. Many algebraic schemes are available for knowing the distribution of roots of F(s) =0, each having its own applications and merits [1,2,3]. To have computational simplicity, In this approach a Routh like table including real as well as complex coefficients is presented and the stability is observed from the first column with the help of the suggested sign pair criterion.

2. PROPOSED APPROACH
Let F(S)=0 be the nth degree characteristic equation of a linear time invariant system and written as

$$f(s) = s^n + (a_1 + jb_2)s^{n-1} + (a_2 + jb_2)s^{n-2} + \dots + (a_n + jb_n) = 0 \qquad (1)$$

Where ' ai +j bi 'are the complex coefficients,while a_i and b_i are real and 's' is the Laplace variable .The pure real coefficient of equation (1) are $a_1, a_2, a_3, \dots\dots, a_n$. and the corresponding imaginary terms of equation (1) are

$jb_1, jb_2, jb_3, \dots\dots, jb_n$.

The elements in the odd column of the first two rows of Routh like table are written as

$$\left.\begin{matrix} 1 & a_2 & a_4 & \cdots\cdots \\ a & a_3 & a_5 & \cdots\cdots \end{matrix}\right\} \qquad (2)$$

The even column elements in the first two rows are filled with imaginary terms as before and the complete first two rows are shown in below.

$$\left.\begin{matrix} 1 & jb_1 & a_2 & jb_3 & a_4 & \cdots\cdots\cdots \\ a_1 & jb_2 & a_3 & jb_4 & a_5 & \cdots\cdots\cdots \end{matrix}\right\} \qquad (3)$$

For the rows formed by the above elements, by applying Routh multiplication rule [4,5] the table is formed as shown below.

$$\begin{matrix} : & \begin{pmatrix} 1 \\ a_1 \end{pmatrix} & \begin{matrix} jb_1 & a_2 & jb_3 & a_4 & jb_5 & a_6 & \cdots\cdots \\ jb_2 & a_3 & jb_4 & a_5 & jb_6 & a_7 & \cdots\cdots \end{matrix} \\ : & \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} & \begin{matrix} c_2 & c_3 & c_4 & . & . & . \\ d_2 & d_3 & d_4 & . & . & . \end{matrix} \\ : & \begin{pmatrix} e_1 \\ f_1 \end{pmatrix} & \begin{matrix} e_2 & e_3 & e_4 & . & . & . \\ f_2 & f_3 & f_4 & . & . & . \end{matrix} \\ : & \begin{pmatrix} g_1 \\ h_1 \end{pmatrix} & \begin{matrix} g_2 & g_3 & g_4 & . & . & . \\ h_2 & h_3 & h_4 & . & . & . \end{matrix} \\ & . & . \quad . \quad . \end{matrix} \qquad \dots. (4)$$

From the elements of first column of the above table the following pair may be grouped respectively.

$P_1 = (1, a_1)$: $\quad P_2 = (c_1, d_1)$:
$P_3 = (e_1, f_1)$: $\quad P_4 = (g_1, h_1)$: …. (5)

For k= 0,1,2,3,… the odd pairs formed ($P_{(2k+1)}$) will be having pure real elements, while the even pairs (P_{2k}) , for k= 1,2,3,……..will contain pure imaginary elements.

3. PROPOSED CRITERION

3.1 Sign Pair Criterion
For F(S)=0 to have all roots in LHS of 's' plane the sign of each pair obtained from the respective $P_{(2k+1)}$ and P_{2k} will be the same , hence it is named as 'Sign Pair Criteion'.

3.2 Proof for the Sign Pair Criterion
Interchanging the first two rows as given in (3), the generalized Hurwitz matrix of order 2n is formed [4,5,6,7] as given below.

$$H_{2n}=\begin{bmatrix} a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & a_7 & . & . & . \\ 1 & jb_1 & a_2 & jb_3 & a_4 & jb_5 & a_6 & . & . & . \\ 0 & a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & a_7 & . & . \\ 0 & 1 & jb_1 & a_2 & jb_3 & a_4 & jb_5 & a_6 & . & . \\ 0 & 0 & a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & . & . \\ 0 & 0 & 1 & jb_1 & a_2 & jb_3 & a_4 & jb_5 & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{bmatrix} \quad (6)$$

From the equation (6) the Hurwitz sub determinant can be formed; in this, the necessary conditions for a system to be stable are as shown below.

$(i)\ \Delta_0 = 1 > 0$ &

$(ii)\ \Delta_1 = a_1 > 0$

The sufficient conditions are obtained as given below

$$(iii)\ \Delta_2 = j(b_1 a_1 - b_2)$$

$$(iv)\ \Delta_3 \begin{vmatrix} a_1 & jb_2 & a_3 \\ 1 & jb_1 & a_2 \\ 0 & a_1 & jb_2 \end{vmatrix}$$

$$= a_1(-b_1 b_2 - a_1 a_2) - (-b^2 - a_1 a_3)$$

$$(v)\ \Delta_4 \begin{vmatrix} a & jb_2 & a_3 & jb_4 \\ 1 & jb_1 & a_2 & jb_3 \\ 0 & a_1 & jb_2 & a_3 \\ 0 & 1 & jb_1 & a_2 \end{vmatrix}$$

$$= \begin{vmatrix} jb_1 & a_2 & jb_3 \\ a_1 & jb_2 & a_3 \\ 1 & jb_1 & a_2 \end{vmatrix} - \begin{vmatrix} jb_2 & a_3 & jb_4 \\ a_1 & jb_2 & a_3 \\ 1 & jb_1 & a_2 \end{vmatrix}$$

In such manner other sub determinants will be obtained. It can be easily observed from Δ_0, Δ_1, Δ_2, Δ_3 and Δ_4 that the pair P(1, a_1)will form same sign.

Since, Δ_1 is real Δ_3 will be real and Δ_2 is pure complex along with Δ_4 is real. This procedure is extended and is observed for the sign of each pair.

$\Delta_0 = 1, \Delta_1 = a_1, \Delta_2 = j\alpha;$

$\Delta_3 = \beta, \Delta_4 = \gamma, \Delta_5 = j\sigma; \ldots\ldots$

In general the elements in the first column of Routh like table are formed using the values of sub determinants:

$$R_1 = 1;\ R_2 = a_1;\ R_3 = \frac{\Delta_2}{\Delta_1} = \frac{\Delta_2}{R_1}$$

$$R_4 = \frac{\Delta_3}{\Delta_2} = pure\ compex$$

$$R_5 = \frac{\Delta_4}{\Delta_3}$$

Without loss of generality and for simplicity the first column of Routh like table may be constructed with the following.

(R1=real positive: R2 = real positive)

(R3=pure imaginary: R4= pure imaginary)

(R5= real positive: R6= real positive)

(R7 =pure imaginary: R_8= pure imaginary)

From the above sign pair criterion is obtained

P1 =(R1,R2): P2=(R3,R4):

P3=(R5,R6): P4= (R7,R8)…..

It is ascertained that each pair has to maintain the sign for the roots of F(S)=0 to lie in left of s plane. Thus the necessary and sufficient conditions are established. The proposed sign pair criterion is (SPC) is applied for the following illustrations[8,9].

4. ILLUSTRATIONS

4.1 Example 1

Consider

$$f(s) = s^3 + (2 + j1)s^2 + (z + j1)s + (2 + j2) = 0$$

The Routh like table is formed as shown.

$$\begin{array}{llllll} : & \begin{pmatrix} 1 \\ 2 \end{pmatrix} & j1 & 3 & j2 \\ : & & j1 & 2 & 0 \\ : & \begin{pmatrix} j0.5 \\ j9.0 \end{pmatrix} & 2 & j2 & \\ : & & -6 & 0 & \\ : & \begin{pmatrix} \frac{7}{3} \\ \frac{12}{7} \end{pmatrix} & j2 & & \\ : & & & & \end{array}$$

Since n=3, the three pairs obtained from the table are

P_1 = (1,2)

P_2 = (j0.5,j9.0)

P_3 = (7/3,12/7)

Since each pair obeys SPC, all the three roots lie in the LHS of ‘s’ plane.

For the sake of clarity Hurwitz matrix as well as the sub determinants are provided below

$$H_6 = \begin{bmatrix} 2 & j1 & 2 & 0 & 0 & 0 \\ 1 & j1 & 3 & j2 & 0 & 0 \\ 0 & 2 & j1 & 2 & 0 & 0 \\ 0 & 1 & j1 & 3 & j2 & 0 \\ 0 & 0 & 2 & j1 & 2 & 0 \\ 0 & 0 & 1 & j1 & 3 & j2 \end{bmatrix}$$

$$\Delta_0 = 1;\ \Delta_1 = 2$$

$$\Delta_2 = j1; \frac{\Delta_2}{\Delta_1} = j0.5$$

$$\Delta_3 = -9; \left(\frac{\Delta_3}{\Delta_2}\right) = j9$$

$$\Delta_4 = -21; \left(\frac{\Delta_4}{\Delta_3}\right) = \frac{7}{3}$$

In a similar way Δ_5 is evaluated and is found that

$$\frac{\Delta_5}{\Delta_4} = \frac{12}{7}$$

Thus we get the pairs

$$P_1 = [\Delta_0, \Delta_1];\ P_2 = \left[\frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_2}\right];\ P_3 = \left[\frac{\Delta_4}{\Delta_3}, \frac{\Delta_5}{\Delta_4}\right]$$

Since each pair obeys SPC, all the three roots lie in the LHS of 's' plane.

4.2 Example 2

Let $f(s) = s^3 + s^2(1 + j6) + s(-13 + j15) + (-7 - j10) = 0$

The Routh like table is formed as shown.

$$\begin{array}{lcccc} : & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{matrix} j6 \\ j5 \end{matrix} & \begin{matrix} -13 \\ -7 \end{matrix} & \begin{matrix} -j10 \\ 0 \end{matrix} \\ : & \begin{pmatrix} j1 \\ -j1 \end{pmatrix} & \begin{matrix} -6 \\ 3 \end{matrix} & \begin{matrix} -j10 \\ 0 \end{matrix} & \\ : & \begin{pmatrix} -3 \\ (\frac{-1}{3}) \end{pmatrix} & -j10 & & \end{array}$$

From the first column the pairs are formed as

$P_1 = (1,1)$: $P_2=(j1,-j1)$:

$P_3=(-3,-1/3)$

The pairs P_1 and P_3 obey the proposed SPC while P_2 fails. Thus F(s)=0 has two roots in the LHS and one root in the RHS of S plane.

4.3 Example 3

Consider

$$f(s) = s^4(2 - j6) + (9 - j13)s^3 + (12 - j9)s^2 + (6 - j2)s + 1 = 0$$

The Routh like table is formed

$$\begin{array}{lccccc} : & \begin{pmatrix} 2 \\ 9 \end{pmatrix} & \begin{matrix} -j6 \\ -j13 \end{matrix} & \begin{matrix} 12 \\ 6 \end{matrix} & \begin{matrix} -j9 \\ -j2 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} \\ : & \begin{pmatrix} -j\frac{28}{9} \\ -j160 \end{pmatrix} & \begin{matrix} \frac{106}{9} \\ -18.75 \end{matrix} & \begin{matrix} -j\frac{77}{9} \\ -j4.89 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} & \\ : & \begin{pmatrix} 12.142 \\ 92.653 \end{pmatrix} & \begin{matrix} -j8.456 \\ j100 \end{matrix} & & & \\ : & \begin{pmatrix} -j21.4 \\ -j2.87 \end{pmatrix} & 1 & & & \end{array}$$

From the first column of the table we formulated the four pairs (n=4):

P1=(2,9) P2 = (-j28/9, -j160)

P3 = (12.142, 92.653)

P_4 = (-j21.4, -j2.87)

All the pairs obey SPC. Thus all the four roots lie in the LHS of 's' plane and hence the system is stable.

5. CONCLUSION

In this paper, extending the Routh criterion, a stability criterion is formulated which is directly applicable to handle the characteristic equation having complex coefficients. For inferring the stability, the computations remain as that of original Routh array. Further the usage of Sign Pair Criterion (SPC) to find the distributions of roots either in LHS or RHS of 's' plane are also presented in this work. Thus the proposed approach shows the unique extension of the results deduced from the Routh Table for stability investigation of Linear Time Invariant Systems.

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