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Crossing Number of Join of Triangular Snake with mK_1, Path and Cycle

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ABSTRACT
The exact values of the crossing number is known only for a specific family of graphs. There are few results concerning crossing numbers of join of some graphs. We give the exact value of crossing number of the join of a certain graph $\boldsymbol{G}$ on with an empty graph, a path and a cycle respectively on m vertices.

General Terms
Computational Mathematics, Graph Theory.

Keywords
Drawing of a graph, crossing number, union and join of graphs, triangular snake, path, cycle

1. INTRODUCTION
Crossing number minimization is one of the fundamental optimization problems in the sense that it is related to various other widely used notions. Besides its mathematical interest, there are numerous applications, most notably those in VLSI design and in combinatorial geometry [1, 2, 3]. Researchers in computer science are also focussing their attention to this area of graph theory as the study of crossing numbers of graphs finds applications in network design and circuit layout. Minimizing the number of wire crossings in a circuit greatly reduces the chance of cross-talk in long crossing wires carrying the same signal and also allows for faster operation and less power dissipation. It is also an important measure of non-planarity of a graph.

Let $G = (V, E)$ be a simple connected undirected graph with vertex set V and edge set E. A drawing D of a graph G is a representation of G in the Euclidean plane R^2 where vertices are represented as distinct points and edges by simple polygonal arcs joining points that correspond to their end vertices. A drawing D is good or clean if no edge crosses itself, no pair of adjacent edges cross, two edges cross at most once and no more than two edges cross at one point.

The number of crossings of D is denoted by $cr(D)$ and is called the crossing number of the drawing D. The crossing number $cr(G)$ of a graph G is the minimum $cr(D)$ taken over all good or clean drawings D of G. If a graph G admits a drawing D with $cr(D) = 0$ then G is said to be planar; otherwise non-planar. It is well known that K_5, the complete graph on 5 vertices and $K_{3,3}$ the complete bipartite graph with 3 vertices in its classes are non-planar. According to Kuratowski's famous theorem, a graph is planar if and only if contains no subdivision of K_5 or $K_{3,3}$.

The study of crossing numbers began during the Second World War with Paul Turán. For an arbitrary graph computing $\boldsymbol{cr(G)}$ is NP-hard [4]. The bounds for $\boldsymbol{cr(K_n)}$ and $\boldsymbol{cr(K_{m,n})}$ are obtained by Guy [5]. The crossing number problem for generalized Petersen graphs has been investigated in [6] and [7]. For hypercubes and cube connected cycles the crossing number is given in [8]. Cimikowski [9] has obtained the bound for the crossing number of mesh of trees. We have obtained the bounds for the crossing number for two different representations of the standard butterfly and Benes networks [10]. We also have obtained the upper bounds for the crossing number for two different honeycomb tori namely, the honeycomb rectangular torus and the honeycomb rhombic torus [11].

The exact values of the crossing number are known only for a specific family of graphs. The cartesian product is one of the graph classes for which crossing number problem is investigated [12]. There are a few results concerning crossing numbers of join of some graphs [13, 14, 15]. In this paper, we obtain the exact value of crossing number of the join of a certain graph $\boldsymbol{G}$, with an empty graph, a path and a cycle respectively on $\boldsymbol{m}$ vertices.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Their union, denoted by $G_1 \cup G_2$, is the graph $(V_1 \cup V_2, E_1 \cup E_2)$. For any two vertex disjoint graphs G_1 and G_2 their join, denoted by $G_1 + G_2$, is obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2.When $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of G_1 and G_2 and that of the complete bipartite graph $K_{m,n}$. It has been long conjectured that the crossing number of the complete bipartite graph $K_{m,n}$ equals the Zarankiewicz's number $Z(m,n) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. This conjecture has been verified by Kleitman [16] for $min\{m,n\} \leq 6$. Marian Klesc [15] has given the exact values for the crossing number of join of two paths, join of two cycles and join of a path and a cycle. In addition, he has given the exact values for crossing numbers of $G + P_n$ and $G + C_n$ for all graphs G of order at most four [14]. He has also obtained this result for some of the graphs on five and six vertices [15]. In this paper we give the exact values of crossing numbers of $G + mK_1$, $G + P_m$ and $G + C_m$ where G is a triangular snake on n vertices.

2. TRIANGULAR SNAKE
Let $G = (V, E)$ be a simple graph. A block of the graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block. The block-cutpoint graph of G is a bipartite graph H in which one partite set consists of the cut-vertices of G, and the other

has a vertex b_i for each block B_i of G. Rosa [18] has defined a *triangular snake* (Δ-*snake*) as a connected graph in which all blocks are triangles and the block-cut-point graph is a path. See Figure 1(a).

In other words, Δ-*snake* or TS_n is a path with vertices $t_1, t_2, ..., t_n$, n is odd, along with the edges $t_{2i-1}t_{2i+1}, 1 \le i \le \frac{n-1}{2}$.

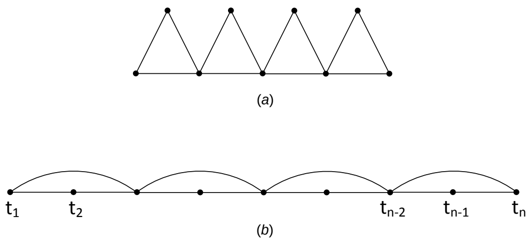

Figure 1: A Triangular Snake and its different drawing

3. THE GRAPH $TS_n + mK_1$

The graph TS_n in figure 1(b) consists of 3-cycles $t_{2i-1}t_{2i}t_{2i+1}, 1 \le i \le \frac{n-1}{2}$ and there are $\left\lfloor \frac{n}{2} \right\rfloor$ such cycles. The graph $TS_n + mK_1$ consists of one copy of TS_n and m vertices, $m = 1, 2, ..., 6$ which are adjacent to every vertex of TS_n. We consider a drawing of $TS_n + mK_1$, where $\left\lfloor \frac{n}{2} \right\rfloor$ vertices of TS_n are placed in the negative positions of the x-axis, $\left\lceil \frac{n}{2} \right\rceil$ vertices of TS_n in the positive positions of the x-axis, $\left\lfloor \frac{m}{2} \right\rfloor$ vertices of mK_1 in the positive positions of the y-axis, $\left\lceil \frac{m}{2} \right\rceil$ vertices of mK_1 in the negative positions of the y-axis and draw nm edges by straight line segments. The motivation for the drawing of $TS_n + mK_1$ is obtained from the paper of Székely [4]. From Figure 2 one can easily see that $TS_n + mK_1 = K_{n,m} \cup P_n \cup S$. where P_n is a path on n vertices and S is the sub graph induced by the edges $\left\{ t_{2i-1}t_{2i+1} \;/ 1 \le i \le \frac{n-1}{2} \right\}$. It is clear that $TS_n + K_1$ is planar.

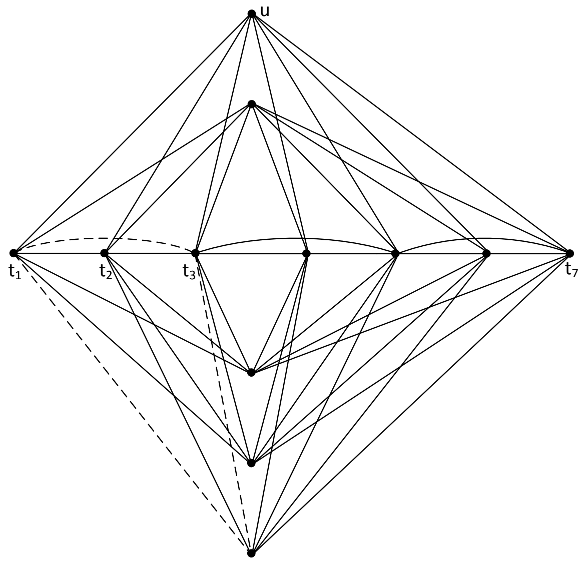

Figure 2: $TS_7 + 5K_1$

Lemma 1. Let A, B, C be mutually disjoint subsets of E. Then

$$cr_D(A \cup B) = cr_D(A) + cr_D(B) + cr_D(A, B)$$

$$cr_D(A, B \cup C) = cr_D(A, B) + cr_D(A, C)$$

where D is a good drawing of G.

The following result is due to Kleitman on the crossing number of complete bipartite graph $K_{n,m}$.

Lemma 2. For $min\{m, n\} \le 6$, $cr(K_{n,m}) = Z(n, m)$ where $Z(n, m) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$.

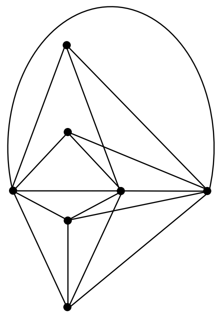

Figure 3: $TS_3 + 4K_1$

Theorem 1. $cr(TS_3 + mK_1) = Z(3, m)$

Proof. TS_3 is a 3-cycle. Figure 3 gives a drawing of $TS_3 + 4K_1$ from which it is clear that $cr(TS_3 + mK_1) \le Z(3, m)$. Also $TS_3 + mK_1$ contains $K_{3,m}$ as a subgraph. Hence $cr(TS_3 + mK_1) \ge Z(3, m)$.

Theorem 2. For a triangular snake TS_n on n vertices,

$$cr(TS_n + mK_1) = Z(n, m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor, n \ge 5, m \le 6.$$

Proof. It is clear that an optimal drawing of $TS_n + mK_1$ contains $K_{n,m}$. Further the number of 3-cycles in TS_n is $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lfloor \frac{m}{2} \right\rfloor$ vertices of mK_1 are placed in the positive positions of the y-axis. Hence $cr(TS_n + mK_1) \le Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor$. We now claim that $cr(TS_n + mK_1) \ge Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor$.

Since $TS_n + mK_1 = K_{n,m} \cup P_n \cup S$. we have by Lemma 1

$$cr(TS_n + mK_1) = cr(K_{n,m} \cup P_n \cup S)$$
$$= cr(K_{n,m} \cup P_n) + cr(S) + cr(K_{n,m} \cup P_n, S)$$

$$cr(K_{n,m} \cup P_n, S) = cr(K_{n,m}, S) + cr(P_n, S).$$

Also, $cr(K_{n,m} \cup P_m) = Z(n,m), cr(S) = 0 \; and \; cr(P_n, S) = 0.$

Claim: $cr(K_{n,m}, S) \ge \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor$.

Consider the 3-cycle $t_1t_2t_3$ and a vertex u of mK_1 which is in the positive y-axis. This vertex is adjacent with all the vertices of TS_n , in particular with t_2. Also t_2 is inside the bounded region denoted by dotted lines as in Figure 2. Hence the edge ut_2 must cross the edges in the boundary at least once. Without loss of generality we say that the edge ut_2 is crossed by the edge t_1t_3. In general, all the edges ut_{2i} must have at least one crossing with $t_{2i-1}t_{2i+1}$.

Since the number of 3-cycles in TS_n is $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lfloor \frac{m}{2} \right\rfloor$ vertices of mK_1 are placed in the negative positions of the y-axis, $cr(K_{n,m}, S) \ge \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor$.

Therefore,

$$cr(TS_n + mK_1) = Z(n,m) + cr(K_{n,m}, S)$$
$$\ge Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor.$$

Following results are the easy consequences of Theorem 2.

Theorem 3. For a path on m vertices,

$$cr(TS_n + P_m) = Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor, \quad n \ge 5, m \le 6.$$

Theorem 4. For a cycle on m vertices,

$$cr(TS_n + C_m) = Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + 2, \quad n \ge 5, m \le 6.$$

4. CONCLUSION

In this paper we have obtained the exact values of crossing numbers of $G + mK_1$, $G + P_m$ and $G + C_m$ where G is a triangular snake on n vertices. It would be interesting to investigate the exact value of the crossing number of some other graphs with different graph operations.

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