A Study of New Fractals Complex Dynamics for Inverse and Logarithmic Functions

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ABSTRACT
The object Mandelbrot set given by Mandelbrot in 1979 and its relative object Julia set have become a wide and elite area of research nowadays due to their beauty and complexity of their nature. Many researchers and authors have worked to study and reveal the new concepts unexplored in the complexities of these two most popular sets of fractal geometry. In this paper we review the recently done work on complex functions for producing beautiful fractal graphics, by few eminent researchers contributing a lot to the field of fractal geometry. The reviewed work mainly emphasizes on the complex functional dynamics of Ishikawa iterates for inverse and logarithmic function and existence of relative superior *Mandel-bar* set.

Keywords
Fractals, Complex dynamics, Relative Superior Mandelbrot Set, Relative Superior Julia Set, Ishikawa Iteration, Relative Superior Mandel-bar Set, Midgets.

1. INTRODUCTION
The object Mandelbrot set given by Mandelbrot in 1979 and its relative object Julia set have become a wide and elite area of research nowadays due to their beauty and complexity of their nature. Several programs and papers have used the escape-time methods to produce images of fractals based on the complex mapping $z \rightarrow (z^n + c)^{-1}$, where exponent *n* is a positive integer. The fractals generated from the self-squared function $z \rightarrow z^2 + c$, where *z* and *c* are the complex quantities, have been studied extensively in the literature[6,8]. Recently the generalized transformation function $z \rightarrow z^{-n} + c$ for positiveinteger value of has been considered by K.W.Shirriff [13].

The dynamics of anti-polynomial $z \rightarrow \bar{z}^d + c$ of complex polynomial $z^d + c$, where $d \geq 2$, leads to interesting Tricorns and Multicorns antifractals with respect to function iteration[5, 14]. Multicorns are symmetrical objects.

The study of connectedness locus for anti-holomorphic polynomials $\bar{z}^2 + c$ defined as Tricorns coined by Milnor, plays an intermediate role between quadratic and cubic polynomials. Crowe etal.[4] considered as in formal analogy with Mandelbrot set and named it as Mandel-bar set and also brought its features bifurcations along axis rather than at points.

For the transcendental function, like logarithmic function, Julia set may be defined as closure of the set of the points whose orbits may escape to infinity under the iteration of Q_c. Equivalently, the Julia set is also closure of the set of the repelling periodic points. For a quadratic family, the only singular value is critical value $c = Q_c(0)$, since *0* is the only critical point. Further infinity is the super attracting fixed point for Q_c. The Mandelbrot set on the other hand is the set of values of c for which the orbit of 0 under Q_c does not tends to infinity. Equivalently, Mandelbrot set takes those values of *c*, for which Julia set of, Q_c is connected.

We investigate the dynamics of the Mandel-bar set for the transformation of the function $z \rightarrow (z^n + c)^{-1}$, for $n \geq 2$, and analyze the *z* plane fractal images generated from the iteration of this function using Ishikawa iteration procedure and analyze the drastic changes that occur in the visual characteristics of the images from *n*=2,3,4,…

2. ELABORATION OF CONCEPTS INVOLVED

2.1 Mandelbrot Set
Definition 1. [10] The Mandelbrot set M for the quadratic $Q_C(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of point 0 is bounded, that is, $M = \{c \in C : \{Q_c^n(0)\}; n = 0, 1, 2, 3\ldots \text{ is bounded}\}$
An equivalent formulation is

$$M = \{c \in C : \{Q_c^n(0) \text{ does not tends to } \infty \text{ as } n \rightarrow \infty\}\}$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

2.2 Julia Set
Definition 2. [10] The set of points *K* whose orbits are bounded under the iteration function of $Q_c(z)$ is called the Julia set. We choose the initial point 0, as 0 is the only critical point of $Q_c(z)$.

2.3 Ishikawa Iteration

Definition 3. Ishikawa Iterates[9]**:** Let *X* be a subset of real or complex number and $f : X \to X$ for all $x_0 \in X$, we have the sequence $\{x_n\}$ and $\{y_n\}$ in *X* in the following manner:

$$y_n = S'_n\, f(x_n) + (1 - S'_n)x_n$$
$$x_{n+1} = S_n f(y_n) + (1 - S_n)x_n$$

where $0 \le S'_n \le 1,\ 0 \le S_n \le 1$ and S'_n & S_n are both convergent to non-zero number.

2.4 Relative Superior Orbit

Definition 4.[12] The sequence x_n and y_n constructed above is called Ishikawa sequence of iteration or relative superior sequence of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$.

Notice that $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1$ is $SO(x_0, s_n, t)$ i.e. mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduce to $O(x_0, t)$.

We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is relative superior orbit.

2.5 Relative Superior Mandelbrot Set

Now we define Mandelbrot set for the function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

Definition 5.[10] Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where n = 1,2,3,… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded i.e. $RSM = \{c \in C : Q_c^k(0) : k = 0,1,2,3...\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number.

2.6 Relative Superior Julia Set

Definition 6.[2] The set of points RSK whose orbits are bounded under relative superior iteration of function Q(z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK.

2.7 Mandel-bar Set

Definition 7. [3] The Mandel-bar set A_c, for the quadratic $A_c(z) = z'^n + c$ is defined as the collection of all $c \in C$ for which the orbit of point 0 is bounded, that is, $A_c = \{c \in C : A_c(0)_{n=0,1,2,3,...} \text{ is bounded}\}$.

An equivalent formulation is

$A_c = \{c \in C : A_c(0) \text{ not tends to } \infty \text{ as } n \to \infty\}$.

2.8 Relative Superior Mandel-bar Set

Definition 8.[11] Relative superior Mandel-bar set RSMB for the function of the form $Q_c(z) = z^n + c$, where n=1,2,3,4,… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded i.e.

$RSMB = \{c \in C : Q_c^{\,k}(0) : k = 0,1,2,3,\ldots\}$ is bounded.

3. GENERATING PROCESS

The basic principle of generating fractals employs the iterative formula: $z_{n+1} \leftarrow f(z_n)$ where z_0 = the initial valueof z, and z_i = the value of complex quantity z at the i^{th} iteration [7][8]. For example, the Mandelbrot's self-squared function for generating fractal is: $f(z) = z^2+c$, where z and c are both complex quantities. We propose the use of transformation function $z \to z^n + c, n \ge 2$ and $z \to (z^n + c)^{-1}$ for generating fractal images with respect to Ishikawa iterates, where z and c are the complex quantities and n is a real number. Each of these fractal images is constructed as two-dimensional array of pixel. Each pixel is represented by a pair of (x,y) coordinates. The complex quantities z and c can be represented as:

$$z = z_x + iz_y$$
$$c = c_x + ic_y$$

where $i = \sqrt{(-1)}$ and z_x , c_x are the real parts and z_y , c_y are the imaginary parts of z and c respectively. The pixel coordinates (x,y) may be associated with (c_x,c_y) or (z_x,z_y) .

Based on this concept, the fractal images can be classified as follows:

(a) **z-Plane**fractals, wherein (x,y) is a function of (z_x,z_y).
(b) **c-Plane** fractals, wherein (x,y) is a function of (c_x,c_y).

In the literature, the fractals for n=2 in z plane are termed as the Mandelbrot set while the fractals for n=2 in c plane are known as Julia sets [13]

4. ESCAPE CRITERION FOR RELATIVE SUPERIOR JULIA AND MANDELBROT SETS

4.1 Escape Criterion for Quadratics

[14] Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z| \to \infty$ as $n \to \infty$. So, $|z| \ge |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

4.2 Escape Criterion for Cubics

[14] Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. This gives the escape criterion for cubic polynomials.

4.3 General Escape Criterion

[14] Consider $|z| > \max\{|c|, (2/s)^{1/2}, (2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$ is the escape criterion.

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z \to (z^n + c)^{-1}$. However, instead of starting with z_0 = infinity, it is simpler to start with $z_1 = c$, which yields the same result. A critical point of $z \to F(z) + c$ is a point where $F'(z) = 0$.

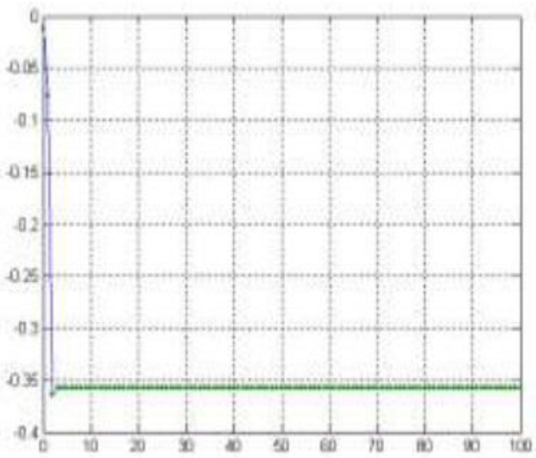
5. SIMULATION AND RESULTS

Fixed points of Quadratic Polynomial [12] :

Table 1: Orbit of F(z) at s=0.5 and s'=0.1 for (z0=-0.01192288639+0.01042379668i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.015837	6	0.85943
2	0.98458	7	0.85943
3	0.86429	8	0.85942
4	0.85883	9	0.85942
5	0.85933	10	0.85942

Fig.1:Observation : the value converges to a fixed point after 08 iterations



Fixed points of Cubic polynomial [12] :

Table 2 : Orbit of F(z) at s=0.5 and s'=0.1 for (z0=-0.00888346751+0.01650347336i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.018742	6	0.86749
2	0.97928	7	0.86747
3	0.85738	8	0.86747
4	0.86871	9	0.86747
5	0.86732	10	0.86747

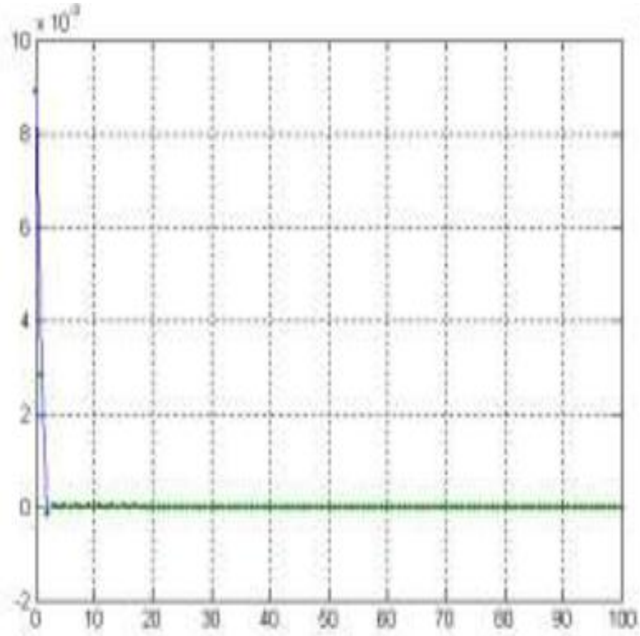


Fig.2:Observation : the value converges to a fixed point after 07 iterations

Fixed points of Bi-quadratic polynomial [12] :

Table 3 : Orbit of F(z) at s=0.5 and s'=0.1 for (z0=-0.01573769494+0.03678871897i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.040014	8	0.8968
2	0.98215	9	0.89704
3	0.90556	10	0.89699
4	0.88426	11	0.89699
5	0.90308	12	0.89699
6	0.89476	13	0.89699
7	0.8977	14	0.89699

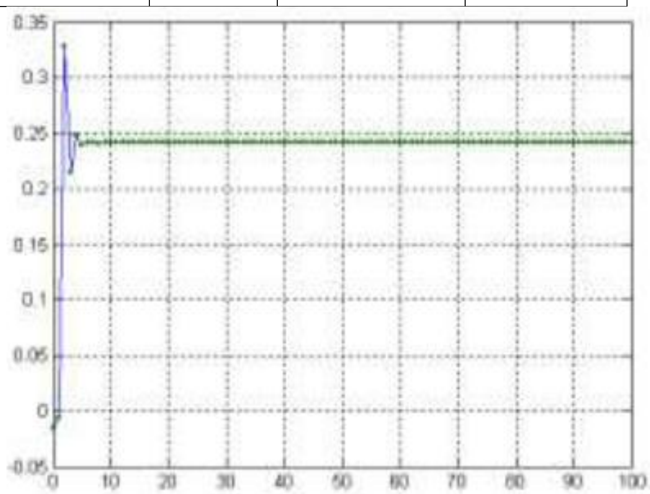


Fig.3: Observation : the value converges to a fixed point after 10 iterations

Generation of Relative Superior Mandelbrot Set:

Fig.4: For quadratic function:
s=0.8, s’=0.3

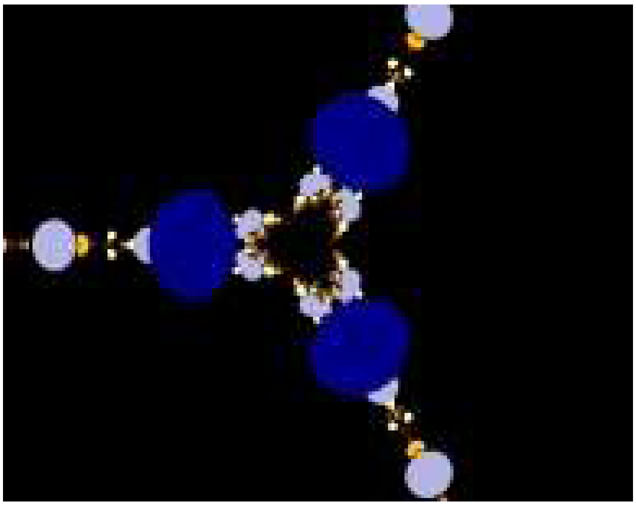
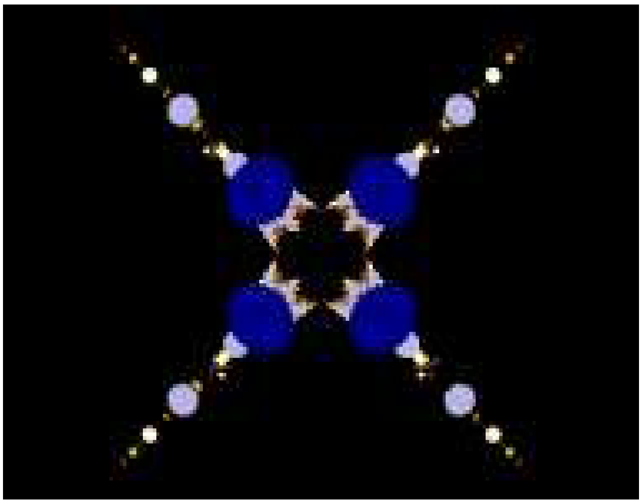


Fig.5: For cubic function:
s=0.8, s’=0.3



Generation of Relative Superior Julia Sets:

Fig.6: For quadratic function: s=0.5, s’=0.4
c=0.002169194079+0.465750756i

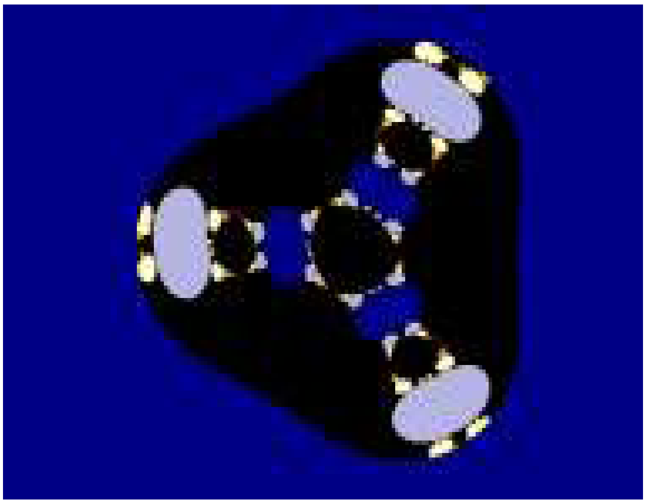
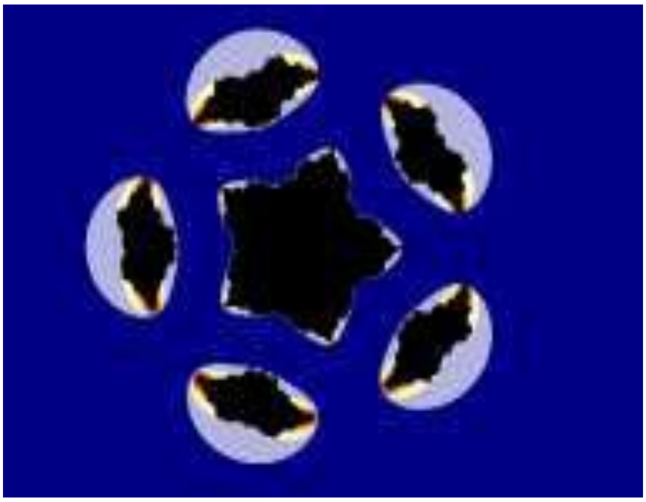


Fig.7: For bi-quadratic function: s=0.8, s’=0.3,
c=-0.0227144337+0.04376545773i

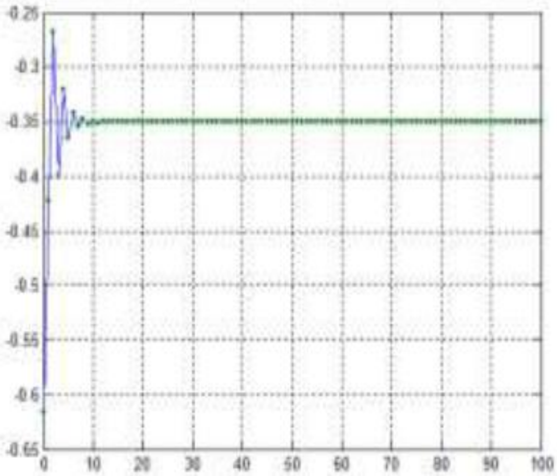


Fixed points for quadratic polynomial [3]:

Table 4 : Orbit of F(z) at s=0.5 and s’=0.7 for (z0=-0.6160374839+0.0135629073i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.61619	14	0.35866
2	0.5189	15	0.35835
3	0.288	16	0.35852
4	0.43079	17	0.35842
5	0.32218	18	0.35848
6	0.37886	19	0.35845
7	0.34703	20	0.35846
8	0.36492	21	0.35845
9	0.35484	22	0.35846
10	0.36049	23	0.35846
11	0.35732	24	0.35846
12	0.3591	25	0.35846
13	0.3581	26	0.35846

Fig.8: Observation : the value converges to a fixed point after 22
iterations

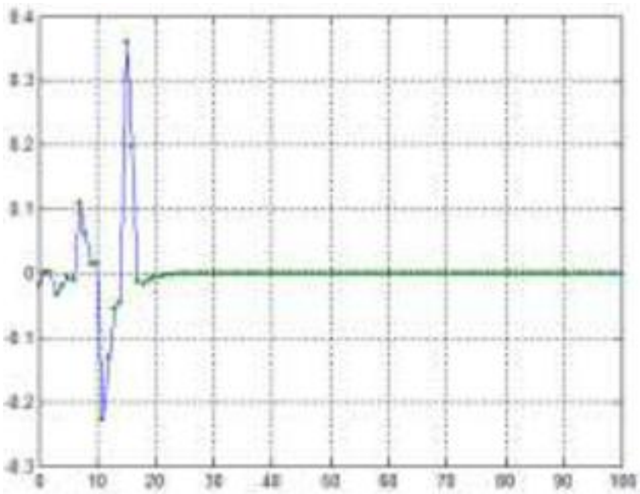


Fixed points for cubic polynomial [3]:

Table 5: Orbit of F(z) at s=0.4 and s'=0.2 for (z0=-0.0189704705+0.02867852789i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
18	0.6098	28	0.50274
19	0.45643	29	0.50223
20	0.52977	30	0.50252
21	0.4871	31	0.50236
22	0.51134	32	0.50245
23	0.49733	33	0.5024
24	0.50536	34	0.50243
25	0.50073	35	0.50241
26	0.50339	36	0.50242
27	0.50186	37	0.50242

Fig.9: Observation : we skipped 17 iteration and the value converges to a fixed point after 36 iterations



Generation of Relative Superior Mandel-bar Set:

Fig.10: For quadratic function:
s=0.6, s'=0.2

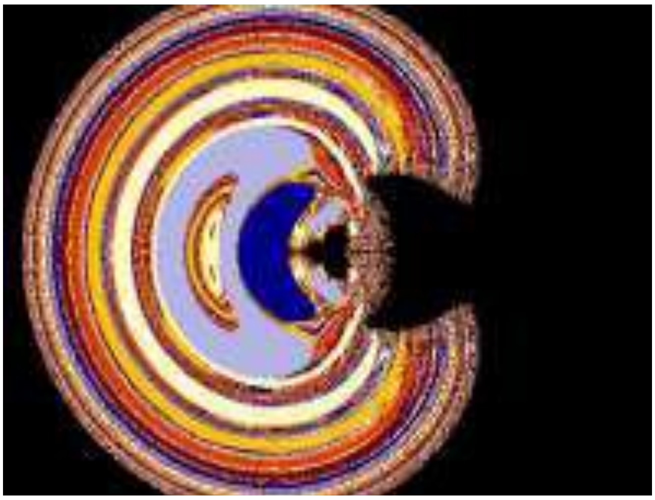


Fig.11: For cubic function:
s=0.4, s'=0.2

Fig.12: For bi-quadratic function:
s=0.6, s'=0.2

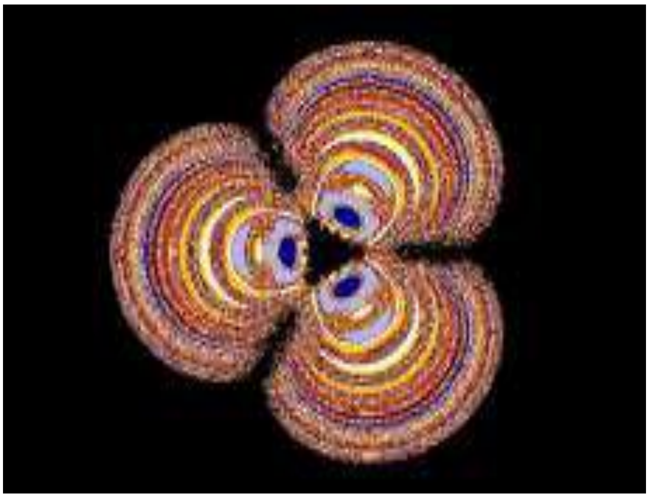


Fig.13: Generalization of RSMB :
s=0.5, s'=0.2, n=19

Generation of Relative Superior Julia Sets for Mandel-bar set:

Fig.14: For quadratic function:
s=0.6, s'=0.2, c=0.08166620257+0.00739899807i

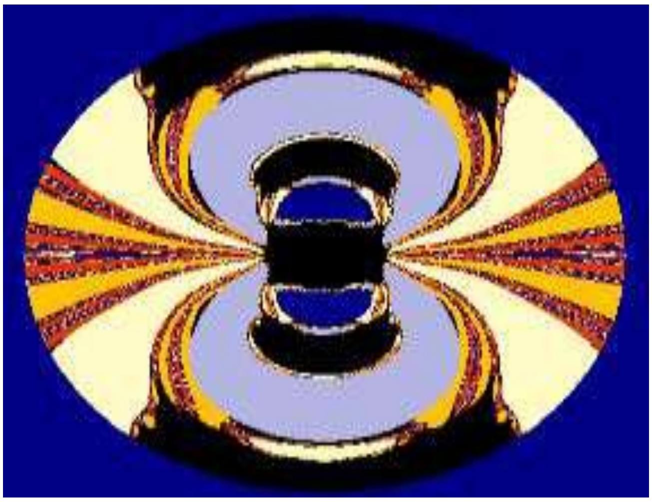
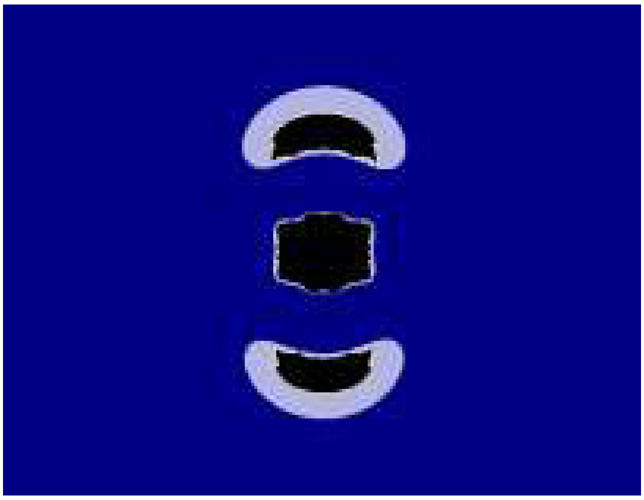


Fig.15: For cubic function:
s=0.8, s'=0.3, c=-0.003854849909+0.01666833389i



Generation of Relative Superior Midget of the Logarithmic Function :

Fig.16 : For quadratic function:
s=0.8, s'=0.8

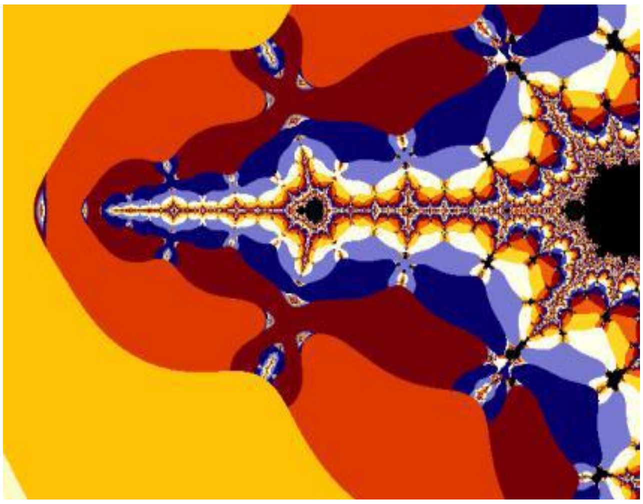
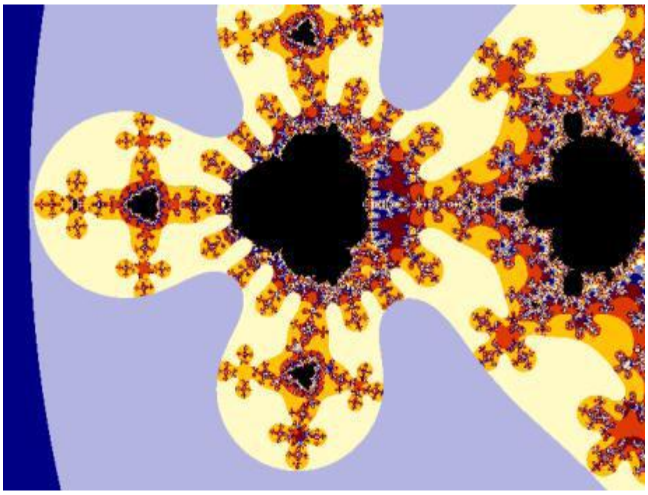


Fig.17: For bi-quadratic function:
s=0.8, s'=0.9



6. CONCLUSION

We have considered the function $z \rightarrow (z_n + c)^{-1}$, for $n \geq 2$, and mathematically analyzed the visual characteristics of the fractal images in the complex c and z planes respectively.Relative Superior Mandelbrot of inverse function showed lace like structure with multicolored small circles. Geometrical analysis of the Relative Superior Julia sets of inverse function shows that the boundary of the fixed point region forms a (n+1) hypocycloid. The geometry of Relative Superior Mandelbrot and Relative Superior Julia sets of inverse function showed their rotational as well as reflection symmetry.

In the dynamics of anti-polynomial of complex polynomial $z'^{n} + c$, where $n \geq 2$, there exist many Mandelbar sets for a value of n with respect to Relative Superior orbit.

Further, for odd values of n, all the Relative Superior Mandelbar sets are symmetrical objects. And for even values of n, all the Relative Superior Mandelbar sets are symmetrical about x-axis.

In the dynamics of complex logarithmic polynomial $z \to \log(z^n + c)$, where $n \geq 2$, the fractal generated with exponent n are found as (n+1) way rotationally symmetric. There are several ovoids or bulbs attached with the main body. The number of major secondary lobe is (n-1). The midgets observed for the logarithmic function are derived for even polynomials while for the odd function, bulbs gets disconnected.

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