# Conditional Resolving Parameters on Enhanced Hypercube Networks

*Bharati Rajan
Department of Mathematics
Loyola College
Chennai, India

Albert William
Department of Mathematics
Loyola College
Chennai, India

Indra Rajasingh
Department of Mathematics
Loyola College
Chennai, India

S. Prabhu
Department of Mathematics
Loyola College
Chennai, India

## Abstract

Given a graph $G = (V, E)$, a set $W \subset V$ is a resolving set if for each pair of distinct vertices $u, v \in V(G)$ there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for $G$. The cardinality of a minimum resolving set is called the *dimension* of $G$ and is denoted by $dim(G)$. A resolving set $W$ is said to be a *one size* resolving set if the size of the subgraph induced by $W$ is one, and a *one-factor* resolving set if $W$ induces isolated edges (one regular graph). The minimum cardinality of these sets denoted $or(G)$ and $onef(G)$ are called one size and one factor resolving numbers respectively. In this paper we investigate these resolving parameters for enhanced hypercube networks.

**Keywords**: Resolving set, basis, one size resolving set, one factor resolving set, and enhanced hypercube networks

## 1 INTRODUCTION

A query at a vertex $v$ discovers or verifies all edges and non-edges whose endpoints have different distance from $v$. In the network verification problem [1], the graph is known in advance and the goal is to compute a minimum number of queries that verify all edges and non-edges. This problem has previously been studied as the problem of placing landmarks in graphs or determining the metric dimension of a graph [8]. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [5, 21, 22].

For an ordered set $W = \{w_1, w_2...w_k\}$ of vertices and a vertex $v$ in a connected graph $G$, the *code* or *representation* of $v$ with respect to $W$ is the $k$-vector

$$C_W(v) = (d(v, w_1), d(v, w_2)...d(v, w_k))$$

where $d(x, y)$ is the distance between the vertices $x$ and $y$. The set $W$ is a *resolving set* for $G$ if distinct vertices of $G$ have distinct codes with respect to $W$. The minimum cardinality of a resolving set for $G$ is called the *resolving number* or *dimension* and is denoted by $dim(G)$.

## 2 AN OVERVIEW OF THE PAPER

The concept of resolvability in graphs has previously appeared in literature. Slater [21, 22] introduced this concept, under the name *locating sets*, motivated by its application to the placement of a minimum number of sonar detecting devices in a network so that the position of every vertex in the network can be uniquely determined in terms of its distance from the set of devices. He referred to a minimum resolving set as a reference set and called the cardinality of a minimum resolving set as the *location number*. Independently, Harary and Melter [5] discovered this concept, but used the term metric dimension, rather than location number. Later, Khuller et al. [8] also discovered these concepts independently and used the term metric dimension. These concepts were rediscovered by Chartrand et al. [2] and also by Johnson [7] while attempting to develop a capability of large data sets of chemical graphs.

It was noted in [4] that determining the metric dimension of a graph is *NP*-complete. It has been proved that the metric dimension problem is *NP*-hard [8] for general graphs. Manuel et al. [12] have shown that the problem remains *NP*-complete for bipartite graphs. There are many applications of resolving sets to problems of network discovery and verification [1], pattern recognition, image processing and robot navigation [8], geometrical routing protocols [10], connected joins in

*This research is supported by The Major Research Project-No. F. 38-120/2009(SR) of the University Grants Commission, New Delhi, India.

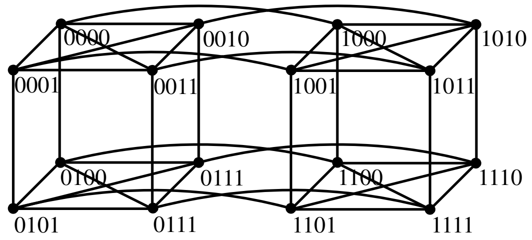


Figure 1: An enhanced hypercube $Q^{4,2}$ with binary representation

graphs [20], coin weighing problems [23]. This problem has been studied for trees multi-dimensional grids [8], Petersen graphs [14], torus networks [11], Benes networks [12], honeycomb networks [13], enhanced hypercubes [15] and Illiac networks [18].

Many resolving parameters are formed by combining resolving property with another common graph-theoretic property such as being connected, independent, or acyclic. The generic nature of conditional resolvability in graphs provides various ways of defining new resolving parameters by considering different conditions. A resolving set $W$ of $G$ is connected if the subgraph induced by $W$ is a nontrivial connected subgraph of $G$. The minimum cardinality of a connected resolving set is called *connected resolving number* and it is denoted by $cr(G)$ [19]. A resolving set $W$ is said to be a one size resolving set [9] if the size of $W$ is one, a one factor resolving set [17] if $W$ induces isolated edges, and a path resolving set [17] if $W$ induces a path. In this paper we show the existence of a *one size* and a *one factor* resolving set in an enhanced hypercube network.

# 3 TOPOLOGICAL PROPERTIES OF ENHANCED HYPERCUBE NETWORKS

The hypercube has received considerable attention mainly due to its regular structure, small diameter, and good connection with a relatively small node degree [25]. The hypercube is a very popular, versatile and vertex-transitive interconnection network [3]. When the dimension of hypercube increases, the cardinality of its vertex set increases exponentially. Many variations of the hypercube have been suggested to improve the performance. One of the variations is the enhancement [24] of the hypercube with same number of vertices. The enhanced hypercubes are much more attractive than the normal hypercubes due to their potentially nice topological properties.

Let $Q^r$ denote the graph of the $r$-dimensional hypercube, $r \geq 1$. The vertex set is given by $V(Q^r) = \{(x_0x_1...x_{r-1}) : x_i = 0 \text{ or } 1\}$. Two vertices $(x_0x_1...x_{r-1})$ and $(y_0y_1...y_{r-1})$ of $Q^r$ are adjacent if and only if they differ exactly in one position. $Q^r$ is

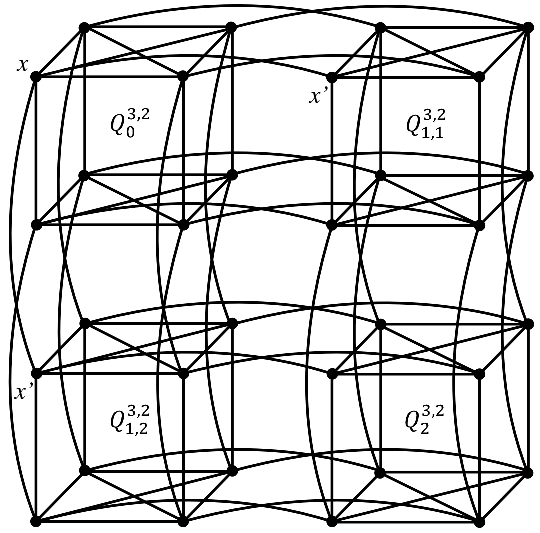


Figure 2: Four copies of $Q^{3,2}$ in $Q^{5,2}$

$r$-regular, bipartite, has $2^r$ vertices and $r2^{r-1}$edges and diameter $r$. It is hamiltonian if $r \geq 2$ and Eulerian if $r$ is even [25].

The enhanced hypercube $Q^{r,k}, 0 \leq k \leq r-1$, is a graph with vertex set $V(Q^{r,k}) = V(Q^r)$ and edge set $E(Q^{r,k}) = E(Q^r) \cup \{x_0x_1...x_{k-2}x_{k-1}x_k...x_{r-1}, x_0x_1...x_{k-2}\overline{x}_{k-1}\overline{x}_k...\overline{x}_{r-1})\}$. The edges of $Q^r$ in $Q^{r,k}$ are called the *hypercube edges* and the remaining edges are called *complementary edges* or *skips* [24]. See Figure 1. The enhanced hypercube, $Q^{r,k}, 0 \leq k \leq r-1$ is $(r+1)$-regular with $2^r$ vertices and $(r+1)2^{r-1}$edges. It is bipartite if and only if $r$ and $k$ have the same parity [6, 24].

# 4 ONE SIZE RESOLVING NUMBER

In this section we determine a bound for the one size resolving number of enhanced hypercube networks.

**Definition 4.1.** *A set $W$ of $G$ is a one size resolving set if the size of subgraph induced by $W$ is one and distinct vertices of $G$ have distinct codes with respect to $W$. The minimum cardinality of a one size resolving set in $G$ is the one size resolving number and is denoted by $or(G)$.*

A one size resolving set of cardinality $or(G)$ is called an or-set of $G$. If $G$ is a connected graph of order $n$ containing an or-set, then it is clear that $2 \leq or(G) \leq n-1$.

We now proceed to identify a one size resolving set in an enhanced hypercube network $Q^{r,2}$. It is clear that there are four copies of $Q^{r-2,2}$ in $Q^{r,2}$. We denote them as $Q_0^{r-2,2}$, $Q_{1,1}^{r-2,2}$, $Q_{1,2}^{r-2,2}$ and $Q_2^{r-2,2}$. Figure 2 exhibits the four copies of $Q^{3,2}$ in $Q^{5,2}$. Let $x \in V(Q_0^{r-2,2})$. A vertex $x' \in V(Q_{1,1}^{r-2,2})$ or $V(Q_{1,2}^{r-2,2})$ is called an *image* of $x$ if $d(x, x') = 1$. Note that vertices in $Q_0^{r-2,2}$, at

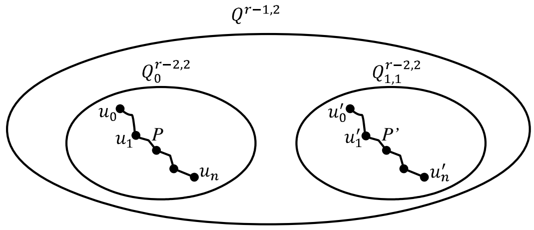


Figure 3: Image of a path $P$

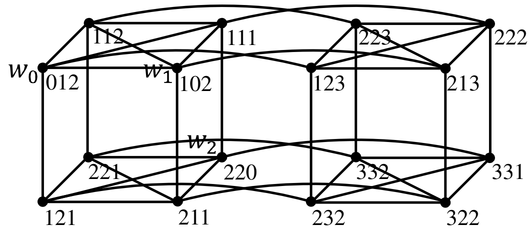


Figure 4: One size resolving set in $Q^{4,2}$

distance 1 from $x$ are not considered as images of $x$. If $x'$ is the image of $x$ in $Q_{1,1}^{r-2,2}$ then $x$ is called the *pre-image* of $x'$. Let $P = u_0u_1...u_n$ be a path in $Q_0^{r-2,2}$. Then the path $P' = u'_0u'_1...u'_n$ where $u'_i$ is the image of $u_i$ in $Q_{1,1}^{r-2,2}(Q_{1,2}^{r-2,2})$ is called the image of $P$ in $Q_{1,1}^{r-2,2}(Q_{1,2}^{r-2,2})$ and $P$ is called the pre-image of $P'$. See Figure 3. We use the following result of B. Rajan et al. [15].

**Lemma 4.1.** [15] Let $x \in V(Q_0^{r-2,2})$ and let $x' \in V(Q_{1,1}^{r-2,2})$ be the image of $x$. Let $w$ be any vertex in $Q_0^{r-2,2}$. Then $d(x', w) = 1 + d(x, w)$.

**Lemma 4.2.** *Let $x \in V(Q_0^{r-2,2})$. Let $x'_1 \in V(Q_{1,1}^{r-2,2})$ and $x'_2 \in V(Q_{1,2}^{r-2,2})$ be the images of $x$. Then $x'_1$ and $x'_2$ are equidistant from every vertex of $Q_0^{r-2,2}$.*

*Proof.* Since the shortest paths from $x'_1$ and $x'_2$ to any vertex of $Q_0^{r-2,2}$ pass through $x$, the conclusion follows. □

**Theorem 4.1.** *Let $G = Q^{r,2}$, then $or(G) \leq r - 1$, $r > 3$.*

*Proof.* We prove the theorem by induction on $r$.

**Base Case:**

Let $G = Q^{4,2}$ and $W_1 = \{w_0, w_1, w_2\}$, where $w_0 = 0001, w_1 = 0011$ and $w_2 = 0110$. It follows from the definition of hypercube edges that $w_0$ is adjacent to $w_1$ and that $w_2$ is non-adjacent to $w_0$ and $w_1$. It is easy to check that $W_1$ is a resolving set for $Q^{4,2}$. Figure 4 shows the distinct codes of vertices in $Q^{4,2}$, with respect to $W_1 = \{w_0, w_1, w_2\}$. Thus $W_1$ is a one size resolving set for $G$. Now assume that the result is true for the enhanced hypercube $Q^{r-1,2}$. Let $W_1 = \{w_0, w_1...w_{r-3}\}$ where $w_0 = \underbrace{0000...01}_{(r-1)-bit}$ and $w_i = x_0x_1...\overline{x}_{r-i-2}\overline{x}_{r-i-1}...x_{r-2}$, $1 \leq i \leq r-3$, $x_s = 0$, $0 \leq s \leq r-2$ be a one size resolving set for $Q^{r-1,2}$. Here $w_0w_1 \in E$. Since and $w_{k+1}$ and $w_j$, $0 \leq j \leq k, 1 \leq k \leq r-4$ differ in two bits they are not adjacent in $Q^{r-1,2}$. This justifies the fact that the size of $W_1$ is one. Moreover $W_1 \subset V(Q_0^{r-2,2})$. Divide $Q^{r,2}$ into four copies of $Q_0^{r-2,2}, Q_{1,1}^{r-2,2}, Q_{1,2}^{r-2,2}$ and $Q_2^{r-2,2}$. There exist vertices $x \in Q_{1,1}^{r-2,2}$ and $y \in Q_{1,2}^{r-2,2}$ having the same code with respect to every vertex of $Q_0^{r-2,2}$ and in particular with respect to every vertex of $W_1$. Hence $W_1$ cannot resolve $x$ and $y$. We exhibit a resolving set for $Q^{r,2}$. We claim that $W$ is a resolving set for $Q^{r,2}$. Define $W = \{u_0, u_i : 1 \leq i \leq r-3\} \cup \{\underbrace{x_0\overline{x}_1\overline{x}_2x_3...00}_{r-bit}\}$ where $u_0 = 0w_0$ and $u_i = 0w_i, 1 \leq i \leq r-3$. Now define $W = \{u_i : 0 \leq i \leq r-2\}$ where $u_0 = 0w_0 = \underbrace{x_0x_1...x_{r-2}\overline{x}_{r-1}}_{r-bit}$ and $u_i = 0w_i, 1 \leq i \leq r-3$ and $u_{r-2} = \underbrace{x_0\overline{x}_1\overline{x}_2x_3...x_{r-1}}_{r-bit}$ set is obtained by appending a 0 to each element of $W_1$ and including the additional vertex $\underbrace{x_0\overline{x}_1\overline{x}_2x_3...00}_{r-bit}$. Hence $W = \{w_0, w_1...w_{r-2}\}$ where $w_0 = \underbrace{x_0x_1...x_{r-2}\overline{x}_{r-1}}_{r-bit}$ and $w_i = x_0x_1...\overline{x}_{r-i-1}\overline{x}_{r-i}...x_{r-1}$, where $1 \leq i \leq r-2$, $x_s = 0$, $0 \leq s \leq r-1$. Clearly the size of $W$ is one. We claim that $W$ is a resolving set of $Q^{r,2}$.

**Case 1:** $x, y \in V(Q_0^{r-2,2})$ or $V(Q_{1,1}^{r-2,2})$ or $V(Q_{1,2}^{r-2,2})$

Since $W_1 \subset V(Q_0^{r-2,2})$ and since $Q_0^{r-2,2} \cup Q_{1,1}^{r-2,2}$ and $Q_0^{r-2,2} \cup Q_{1,2}^{r-2,2}$ are isomorphic to $Q^{r-1,2}$, by induction hypothesis $W_1$ resolves $x$ and $y$. The same argument applies to the following cases.

**a)** $x \in V(Q_0^{r-2,2})$ and $y \in V(Q_{1,1}^{r-2,2})$

**b)** $x \in V(Q_0^{r-2,2})$ and $y \in V(Q_{1,2}^{r-2,2})$

**Case 2:** $x \in V(Q_{1,1}^{r-2,2})$ and $y \in V(Q_{1,2}^{r-2,2})$

We need to prove that $d(x, w) \neq d(y, w)$ for some $w$ in $W = \{w_0, w_1, w_2...w_{r-3}\} \cup \{w_{r-2}\}$. Let $x', y' \in V(Q_0^{r-2,2})$ be the images of $x$ and $y$ respectively.

**Case 2.1:** $x' = y'$

In this case $d(y, w_{r-2}) = d(y, y') + d(y', w_{r-2}) = 1 + d(x', w_{r-2}) = 1 + 1 + d(x, w_{r-2}) \neq d(x, w_{r-2})$.

**Case 2.2:** $x' \neq y'$ and $x', y' \in V(Q_0^{r-2,2})$

Now $x'$ and $y'$ are resolved by some $w$ in $W_1$. Hence $d(x', w) \neq d(y', w)$ and consequently $d(x, w) \neq d(y, w)$.

**Case 3:** $x \in V(Q_0^{r-2,2})$ and $y \in V(Q_2^{r-2,2})$

The proof is similar to Case 2.

**Case 4:** $x, y \in V(Q_2^{r-2,2})$

As before let $x'$ and $y'$ be the images of $x$ and $y$ respectively. There are two possibilities $x', y' \in V(Q_{1,1}^{r-2,2})$ or $V(Q_{1,2}^{r-2,2})$. Then $d(x', w) \neq d(y', w)$ for some $w \in W_1$ by Case 1.

**Case 5:** $x \in V(Q_{1,1}^{r-2,2})$ and $y \in V(Q_2^{r-2,2})$

Let $x' \in V(Q_0^{r-2,2})$ and $y' \in V(Q_{1,2}^{r-2,2})$ be the images of $x$ and $y$ respectively. Since $Q_0^{r-2,2} \cup Q_1, 2^{r-2,2}$ is resolved by $W_1$, there exist a $w \in W_1$ such that $d(x', w) \neq d(y', w)$. This implies that $d(x, w) \neq d(y, w)$. □

# 5 ONE FACTOR RESOLVING NUMBER

In this section we determine a bound for the one factor resolving number of enhanced hypercube networks.

**Definition 5.1.** *A set $W$ of $G$ is a one factor resolving set for $G$ if $G[W] \approxeq tK_2$, for some integer $t$. The minimum $t$ for which $G[W] \approxeq tK_2$ is called the one factor resolving number of $G$ and it is denoted by $onef(G)$. This parameter $onef(G)$ reduces to the one size resolving number, $or(G)$ when the size of $G[W]$is one.*

**Theorem 5.1.** *Let $G = Q^{r,2}$,then $onef(G) \leq \left(\frac{r-1}{2}\right)$, where $r$ odd and $r > 3$.*

*Proof.* We prove the theorem by induction on $r$. Now assume that the result is true for the hypercube $Q^{r-2,2}$. Let $W_1 = \{\{(x_0x_1...\overline{x}_{r-2i-3}...x_{r-3}), (x_0x_1...\overline{x}_{r-2i-4}\overline{x}_{r-2i-3}...x_{r-3})\}, 0 \leq i \leq \frac{r-5}{2}\}$ be a one factor resolving set where $w_j$ and $w_{j+1}$ are adjacent, $0 \leq j \leq r-5, j$ even. Clearly $W_1 \subset V(Q_0^{r-2,2})$ Divide $Q^{r,2}$ into eight copies of $Q^{r-3,2}$, namely $Q_0^{r-3,2}, Q_{1,1}^{r-3,2}, Q_{1,2}^{r-3,2}, Q_{1,3}^{r-3,2}, Q_{2,1}^{r-3,2}, Q_{2,2}^{r-3,2}, Q_{2,3}^{r-3,2}$ and $Q_3^{r-3,2}$.

Now each of $Q_0^{r-3,2} \cup Q_{1,1}^{r-3,2}, Q_0^{r-3,2} \cup Q_{1,2}^{r-3,2}$ and $Q_0^{r-3,2} \cup Q_{1,3}^{r-3,2}$ is isomorphic to $Q^{r-2,2}$. Since $W_1 \subset V(Q_0^{r-3,2})$, $W_1$ resolves the above subcubes by assumption. Now there exist vertices $x \in Q_{1,1}^{r-3,2}, y \in Q_{1,2}^{r-3,2}$ and $z \in Q_{1,3}^{r-3,2}$ having the same code with respect to every vertex of $Q_0^{r-3,2}$ and in particular with respect to every vertex of $W_1$. Similarly there exist vertices one each in $Q_{2,1}^{r-3,2}, Q_{2,2}^{r-3,2}$ and $Q_{2,3}^{r-3,2}$ having the same code with respect to $W_1$. So we need to augment $W_1$. If a cube is resolved by some $W_1 \subset V(Q_0^{r-3,2})$ then the $s$-neighbor cube is also resolved by the same resolving set $W_1$ where $1 \leq s \leq 3$. Therefore it is enough to resolve $Q_{1,1}^{r-3,2}$, $Q_{1,2}^{r-3,2}$, $Q_{1,3}^{r-3,2}$. Let $w_{r-3} \in V(Q_{1,1}^{r-3,2})$. Now $W_1 \cup \{w_{r-3}\} \subset V(Q_0^{r-3,2} \cup Q_{1,1}^{r-3,2})$. This means that there are vertices one in each of $Q_{1,2}^{r-3,2} \cup Q_{2,1}^{r-3,2}$ and $Q_{1,3}^{r-3,2} \cup Q_{2,2}^{r-3,2}$ having same code as they are at distance 1 from $Q_0^{r-3,2} \cup Q_{1,1}^{r-3,2}$. Now choose $w_{r-2} \in V(Q_{1,2}^{r-3,2} \cup Q_{2,1}^{r-3,2})$ preferably $w_{r-2} \in V(Q_{2,1}^{r-3,2})$ so that $w_{r-2}$ and $w_{r-3}$ are adjacent. Therefore the augmented $W = \{w_0, w_1...w_{r-2}\}$, more precisely $W = \{\{(x_0x_1...\overline{x}_{r-2i-1}...x_{r-1}), (x_0x_1...\overline{x}_{r-2i-2}\overline{x}_{r-2i-1}...x_{r-1})\}, 0 \leq i \leq \frac{r-3}{2}\}$ is a one factor resolving set. □

# 6 CONCLUSION

In this paper we have discussed two different resolving parameters namely the one size resolving number and one factor resolving number for enhanced hypercube networks. These resolving parameters for Benes and Butterflies are under investigation.

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