On Regular Generalized Open Sets In Topological Space

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ABSTRACT
In this paper we introduce and study #regular generalized open (briefly, #rg-open) sets in topological space and obtain some of their properties. Also, we introduce #rg-neighbourhood (shortly #rg-nbhd) in topological spaces by using the notion of #rg-open sets. Applying #rg-closed sets we introduce #rg-closure and discuss some basic properties of this.

KEYWORDS
rg-open sets,, rg-nbhd, rg-closure.

AMS SUBJECT CLASSIFICATION (2000)

1. INTRODUCTION:

Regular open sets and rw-open sets have been introduced and investigated by Stone[16] and Benchalli and Wali[1] respectively. Levine[8,9], Biswas[3], Cameron[4], Sundaram and Sheik john[17], Bhattacharyya and Lahiri[2], Nagaveni[12], Pushpalatha[15], Gnanambal[6], Gnanambal and Balachandran[7], Palaniappan and Rao[13] and Maki, Devi and Balachandran[10] introduced and investigated semi open sets, generalized closed sets, regular semi open sets, weakly closed sets, semi generalized closed sets , weakly generalized closed sets, strongly generalized closed sets, generalized pre-regular closed sets, regular generalized closed sets, and generalized α-generalized closed sets respectively. We introduce a new class of sets called #regular generalized open sets which is properly placed in between the class of open sets and the class of rg-open sets.

Throughout this paper (X,τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. For a subset A of a topological space X, cl(A) and int(A) denote the closure of A and the interior of A respectively. X\A or A^c denotes the complement of A in X. We recall the following definitions and results.

1.1. Definition
A subset A of a space X is called

1) a preopen set[11] if $A \subseteq$ intcl (A) and a preclosed set if clint (A)$\subseteq$ A.
2) a semiopen set[8] if $A \subseteq$ clint (A) and a semiclosed set if intcl (A) $\subseteq$A.

3) a regular open set[16]if A = intcl (A) and a regular closed set if A = clint (A).
4) a π- open set[20] if A is a finite union of regular open sets.
5) regular semi open[4]if there is a regular open U such $U \subseteq A \subseteq cl(U)$.

1.2. Definition
A subset A of (X,τ) is called
1) generalized closed set (briefly, g-closed)[9] if cl (A)$\subseteq$ U whenever A$\subseteq$U and U is open in X.

2) regular generalized closed set (briefly, rg-closed)[13] if cl (A)$\subseteq$U whenever A$\subseteq$U and U is regular open in X.

3) generalized preregular closed set (briefly, gpr-closed)[6] if pcl (A)$\subseteq$U whenever A $\subseteq$ U and U is regular open in X.

4) weakely generalized closed set (briefly, wg-closed)[12] if clint (A) $\subseteq$U whenever A$\subseteq$ U and U is open in X
.
5) π-generalized closed set (briefly, πg-closed)[5] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is π-open in X.

6) weakely closed set (briefly, w-closed)[15] if cl(A) $\subseteq$U whenever A$\subseteq$U and U is semi open in X.

7) regular weakly generalized closed set (briefly, rwg-closed)[12] if clint(A)$\subseteq$U whenever A $\subseteq$U and U is regular open in X.

8) rw-closed [1] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is regular semi open.

9) *g-closed [19] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is w-open.

10)#rg-closed[18] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is rw-open.

2. #REGULAR GENERALIZED OPEN SETS AND #REGULAR GENERALIZED NEIGHBOURHOODS.

2.1. Definition
A subset A of a space X is called #regular generalized open (briefly #rg-open) set if its complement is #rg-closed. The family of all #rg– open sets in X is denoted by #RGO(X).

2.2. Remark
cl(X\A) = X\int(A).

2.3. Theorem
A subset A of X is #rg-open if and only if F$\subseteq$int(A) whenever F is rw-closed and F$\subseteq$A.

Proof
(*Necessity*). Let A be #rg-open . Let F be rw-closed and F⊆A then X\A⊆X\F, whenever X\F is rw-open. Since X\A is #rg-closed, cl(X\A)⊆X\F. By Remark 2.2, X\int(A)⊆ X\F. That is F⊆int(A).

(*Sufficiency*). Suppose F is rw-closed and F⊆A implies F⊆int(A). Let X\A⊆U where U is rw-open. Then X\U⊆A, where X\U is rw-closed. By hypothesis X\U⊆int(A). That is X\int(A)⊆U. By remark2.2 cl(X\A)⊆U, implies, X\A is #rg-closed and A is #rg-open.

2.4. Theorem
If int(A) ⊆ B ⊆ A and A is #rg-open, then B is #rg-open.

Proof
Let A be #rg-open set and int(A) ⊆B⊆A. Now int(A)⊆B ⊆ A implies X\A ⊆ X\B ⊆ X\int(A). That is X\A ⊆ X\B ⊆ cl(X\A). Since X\A is #rg-closed, X\B is #rg-closed and B is #rg-open.

2.5. Remark
For any A ⊆ X, int(cl(A)\A) = ϕ.

2.6. Theorem
If A⊆X is #rg-closed then cl(A)\A is #rg-open.

Proof
Let A be #rg-closed. Let F be rw-closed set such that F⊆cl(A)\A. Then by theorem.2.7 [18], F=ϕ. So, F ⊆ int(cl(A)\A). This shows cl(A)\A is #rg-open.

2.7. Theorem
Every open set in X is #rg-open but not conversely.

Proof
Let A be an open set in a space X. Then X\A is closed set. By theorem 2.1[18], X\A is #rg-closed. Therefore A is #rg-open set in X.

The converse of the theorem need not be true, as seen from the following example.

2.8. Example
Let X={a,b,c,d} be with topology τ={X,ϕ, {a}, {b}, {a.b}, {a.b,c}} then the set A={b,c} is #rg-open but not open set in X.

2.9.Corollary
Every regular open set is #rg – open but not conversely.

Proof
Follows from Stone [16] and theorem 2.7.

2.10. Corollary
Every π-open set is #rg-open but not conversely.

Proof
Follows from Dontchev and Noiri [5] and theorem 2.7.

2.11.Theorem
Every #rg-open sets in X is rg-open set in X, but not conversely..

Proof
Let A be #rg-open set in space X. Then X\A is # rg-closed set in X. By theorem 2.2[18], X\A is rg-closed set in X. Therefore A is rg-open in X.

The converse of the above theorem need not be true as seen from the following example.

2.12. Example
Let X={a,b,c,d} be with topology τ={X,ϕ, {a}, {b}, {a.b}, {a.b,c}} then the set A={c,d} is rg-open but not #rg-open set in X.

2.13. Theorem
Every #rg-open set in X is *g –open set in X , but not conversely.

Proof
Let A be #rg-open set in space X. Then X\A is # rg-closed set in X. By theorem 2.3[18], X\A is *g-closed set in X. Therefore A is *g-open in X.

The converse of the above theorem need not be true as seen from the following example

.2.14. Example
Let X={a,b,c,d} be with topology τ={X,ϕ, {a}, {b}, {a.b}} then the set A={a,b,d} is *g-open but not #rg-open set in X.

2.15. Theorem
Every #rg-open set in X is g-open, but not conversely.

Proof.
Let A be #rg-open set in X. Then A^c is #rg-closed set in X. By theorem 2.5[18], A^c is g-closed set in X. Hence A is g-open in X.

The converse of the above theorem need not be true as seen from the following example.

2.16. Example.
Let X= {a,b,c,d} be with topology τ={X, ϕ, {a}, {b}, {a.b}} then the set A={a,b,c} is g-open but not open in X.

2.17. Theorem
If a subset A of a space X is #rg-open then it is πg-open set in X.

Proof
Let A be #rg-open set in space X. Then X\A is # rg-closed set in X. By theorem 2.4[18], X\A is πg-closed set in X. Therefore A is πg-open in X.

The converse of the above theorem need not be true as seen from the following example.

2.18. Example
Let X={a,b,c,d} be with topology τ={X,ϕ, {a}, {b}, {a.b}, {a,b,c}} then the set A={a,b,d} is πg-open but not #rg-open set in X.

2.19. Remark
The following example shows that #rg-open sets are independent of rw-open sets, semi open sets, α-open sets, gα-open sets, sg-open sets, semi pre open sets, pre open sets and swg-open sets.

2.20. Example
Let X= {a,b,c,d} be with topology τ = { ϕ, X, {a}, {b}, {a,b}, {a,b,c}}. Then

(i) #rg-open sets in (X,τ) are ϕ, X, {a}, {b}, {c}, {a,b}, {b,c}, {a,c}, {a,b,c}.

(ii) rw-open sets in (X,τ) are ϕ, X, {a}, {b}, {c}, {d}, {a,b}, {c,d}, {a,b,c}.

(iii) semi open sets in (X,τ) are ϕ, X, {a}, {b}, {a,b}, {b,c}, {a,d}, {b,d}, {a,c}, {b,c,d}, {a,c,d}, {a,b,d}, {a,b,c}.

(iv) α-open sets in (X,τ) are ϕ, X, {a}, {b}, {a,b}, {a,b,d}, {a,b,c}

(v) gα-open sets in (X,τ) are ϕ, X, {a}, {b}, {a,b}, {a,b,d}, {a,b,c}.

(vi) sg- open sets in (X,τ) are ϕ, X, {a}, {b}, {a,b}, {a,d}, {b,c}, {a,c}, {b,d}, {a,c,d}, {a,b,d}, {a,b,c}, {b,c,d}.

(vii) semi pre open sets in (X,τ) are ϕ, X, {a}, {b}, {a,b}, {a,d}, {b,c}, {a,c}, {b,d}, {a,c,d}, {a,b,d}, {a,b,c}, {b,c,d}.

(viii) pre open sets in (X,τ) are ϕ, X, {a}, {b}, {a,b}, {a,b,d}, {a,b,c }.

(ix) swg-open sets in (X,τ) are ϕ, X, {a}, {b}, {a,b}, {a,b,c}, {a,b,d}

2.21. Remark.

From the above discussion and known results we have the following implications (Diagram 1).

A→ B means A implies B but not conversely,, A↮ means A and B are independent.

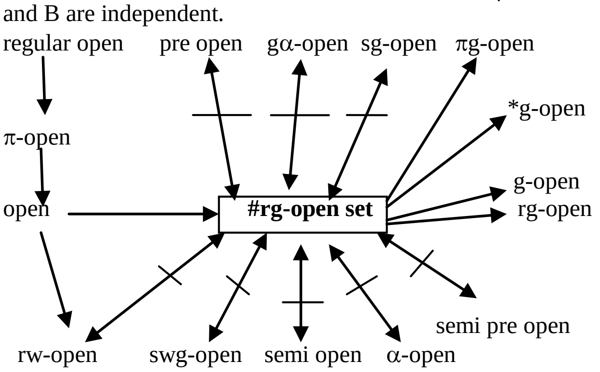

Diagram 1

2.22. Theorem

If A and B are #rg-open set in a space X. Then A∩B is also #rg-open set in X.

Proof

If A and B are #rg-open sets in a space X. Then X\A and X\A are #rg-closed sets in a space X. By theorem 2.6[18] (X\A)∪ (X\B) is also #rg-closed sets in X. Therefore A∩B is #rg-open set in X.

2.23. Remark

The union of two #rg-open sets in X is generally not a #rg-open set in X.

2.24. Example

Let X={a,b,c,d} be with topology τ={X,ϕ, {a}, {b}, {a.b}} then the set A={b,c} and B={b,d} are #rg-open set in X but A∪B = {b,c,d} is not #rg-open set in X.

2.25.Theorem

If a subset A of a topological space X is both rw-closed and #rg-open then it is open.

Proof.

Let A be rw-closed and #rg-open set in X. Now A ⊆A. By theorem 2.3 A⊆int(A). Hence A is open.

2.26.Theorem

If a set A is #rg-open in X, then G=X, whenever G is rw-open and (int(A)∪(X\A))⊆G.

Proof

Suppose that A is #rg-open in X. Let G is rw-open and (int(A)∪(X\A))⊆G. Thus $G^c \subseteq (int(A) \cup A^c)^c = (int(A))^c \cap A$. That is $G^c \subseteq (int(A))^c \backslash A^c$. Since $(int(A))^c = cl(A^c)$, $G^c \subseteq cl(A^c) \backslash A^c$. Now, G^c is rw-closed and A^c is #rg-closed, by theorem 2.7 [18] , $G^c = \phi$. Hence G=X.

2.27. Theorem

Every singleton point set in a space is either #rg-open (or) rw-closed.

Proof

It is follows from theorem 2.8[18].

2.28. Definition

Let X be a topological space and let x ∈ X. A subset N of X is said to be a #rg-nbhd of x iff there exists a #rg-open set U such that x ∈ U⊆N.

2.29. Definition

A subset N of space X, is called a #rg-nbhd of A ⊂X iff there exists a #rg-open set U such that A⊆U⊆ N.

2.30. Theorem

Every nbhd N of x ∈ X is a #rg-nbhd of X , but not conversely.

Proof

Let N be a nbhd of point x ∈ X. Then there exists an open set U such that x ∈ U ⊆ N. Since every open set is #rg-open set, U is a #rg-open set such that x ∈U ⊆ N. This implies N is #rg-nbhd of x.

The converse of the above theorem need not be true as seen from the following example.

2.31. Example

Let X = {a, b, c, d} with topology τ = {X, ϕ, {a}, {b}, {a, b} , {a, b, c}}. Then #RGO (X) = {X,ϕ,{a},{b},{c}, {a,b}, {a,c}, {b,c}, {a,b,c}}. The set {c,d} is #rg-nbhd of the point c, since the #rg-open sets {c} is such that c ∈{c}⊂ {c,d}. However, the set {c,d} is not a nbhd of the point c, since no open set U exists such that c∈ U ⊆{c,d}.

2.32. Theorem

Every #rg-open set is #rg-nbhd of each of its points, but not conversely.

Proof

Suppose N is #rg-open. Let x ∈ N. For N is a #rg-open set such that x ∈ N⊆N. Since x is an arbitrary point of N, it follows that N is a #rg-nbhd of each of its points.

The converse of the above theorem is not true in general as seen from the following example.

2.33. Example

Let X = {a,b, c, d} with topology τ = {X, ϕ, {a}, {b}, {a, b} }. Then #RGO (X) = {X,ϕ,{a},{b},{c},{d}, {a,b}, {a,c}, {b,c}, {a,d}, {a,c}}. The set {b, d} is a #rg-nbhd of the point b, since the #rg-open set {b} is such that c∈ {b}⊆ {b,d}. Also the set {b, d} is a #rg-nbhd of the point {d}, Since the #rg-open set {d} is such that d ∈ {d}⊆ {c, d}. Hence {b, d} is a #rg-nbhd of each of its points, but the set { b, d} is not a #rg-open set in X.

2.34. Remark
The #rg-nbhd N of x ∈ X need not be a #rg-open in X. It is seen from the following example.

2.35. Example
Let X = {a, b, c, d} with topology τ = {X, ϕ, {a}, {b}, {c}, {a,b},{a,c},{b,c},{a,b,c}}.Then #RGO(X)= {X,ϕ,{a},{b},{c}, {a,b}, {a,c}, {b,c}, {a,b,c}}. Here {a,b,d} is a #rg-nbhd of {a}, since {a,b} is a #rg-open set such that a∈{a,b}⊆{a,b,d}, but it is not a #rg-open set.

2.36. Theorem
If F is a #rg-closed subset of X, and x∈ F^c then there exists a #rg-nbhd N of x such that N∩F =ϕ.

Proof
Let F be #rg-closed subset of X and x∈F^c. Then F^c is #rg-open set of X. So by theorem 2.32. F^c contains a #rg-nbhd of each of its points. Hence there exists a #rg-nbhd N of x such that N⊆ F^c. Hence N ∩ F =ϕ.

3.#RG-CLOSURE AND THEIR PROPERTIES.

3.1. Definition
For a subset A of X, #rg-cl(A)=∩{F :A⊆F , F is #rg closed in X}.

3.2. Definition
Let (X,τ) be a topological space and $\tau_{\#rg}$ = {V⊆X: #rg-cl(X\V)=X\V}.

3.3. Definition
For any A⊆X, #rg-int(A) is defined as the union of all #rg-open set contained in A.

3.4. Remark
If A⊆X is # rg-closed then #rg-cl(A) = A, but the converse is not true.

3.5. Example
Let X={a,b,c,d} be with topology τ={X,ϕ, {a}, {b}, {a.b}}. Let A= {a} then #rg-cl(A)=A, but A is not #rg-closed.

3.6. Theorem
Suppose $\tau_{\#rg}$ is a topology. If A is #rg-closed in (X,τ), then A is closed in (X,$\tau_{\#rg}$).

Proof
Since A is #rg-closed in (X,τ), #rg-cl(A)=A. This implies X\A∈$\tau_{\#rg}$. That is X\A is open in (X,$\tau_{\#rg}$). Hence A is closed in (X, $\tau_{\#rg}$).

3.7. Remark
(i) #rg-cl(ϕ) =ϕ and #rg-cl(X)=X
(ii) A ⊆ #rg-cl(A).

3.8. Theorem
For any x∈X, x∈ #rg-cl(A) if and only if V∩A ≠ ϕ for every #rg-open set V containing x.

Proof
(Necessity). Suppose there exists a #rg-open set V containing x such that V∩A=ϕ. Since A⊆ X\V , #rg-cl(A) ⊆X\V implies x∉ #rg-cl(A) a contradiction.

(Suffiency).Suppose x∉ #rg-cl(A), then there exists a #rg-closed subset F containing A such that x∉F. Then x∈ X\F and X\F is #rg-open. Also (X\F)∩A =ϕ, a contradiction.

3.9. Remark
Let A and B be subsets of X , if A⊆B then #rg-cl(A)⊆#rg-cl(B).

3.10. Theorem
Let A and B be subsets of X, then #rg-cl (A∩B) ⊆
#rg-cl(A) ∩#rg-cl(B).

Proof
Since A∩B⊆A and B, by remark 3.9,#rg-cl(A∩B)⊆#rg-cl(A) and,#rg-cl(A∩B)⊆#rg-cl(B).Thus, #rg-cl(A∩B)⊆#rg-cl(A) ∩#rg-cl(B). In general, #rg-cl(A)∩#rg-cl(B) ⊄ #rg-cl(A∩B).

3.11. Example
Let X={a,b,c,d} be with topology τ={X,ϕ, {a}, {b}, {a.b}, {a.b,c}}. Let A={a} and B={b} and #rg-cl(A)={a,d}, #rg-cl(B)={b,d}. Then #rg-cl(A)∩#rg-cl(B)={d} ⊄ #rg-cl(A∩B).

3.12. Theorem
If A and B are #rg-closed sets then # rg-cl(A∪B) =
#rg-cl(A)∪#rg-cl(B).

Proof
Let A and B be # rg-closed in X. Then A∪B is also #rg-closed. Then #rg-cl(A∪B) = A∪B = #rg-cl(A)∪#rg-cl(B).

3.13. Theorem
(X\#rg-int(A)) = #rg-cl(X\A).

Proof
Let x∈ X\#rg-int(A), then x ∉ #rg-int(A). Thus every #rg-open set B containing x is such that B⊄A. This implies every #rg-open set B containing x intersects X\A. This means x∈#rg-cl(X\A). Hence (X\#rg-int(A))⊆#rg-cl(X\A). Conversely, let x∈#rg-cl(X\A). Then every #rg-open set U containing x intersects X\A. That is every #rg-open set U containing x is such that U⊄A, implies x ∉ #rg-int(A). Hence #rg-cl(X\A)⊆ (X\#rg-int(A)). Thus (X\#rg-int(A))=#rg-cl(X\A).

4. CONCLUSION
In this paper, we have introduced the weak and generalized form of open sets namely #rg-open set and established their relationships with some generalized sets in topological space.

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