Inverse Independence Number of a Graph

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ABSTRACT
The concept of inverse domination was introduced by Kulli V.R. and Sigarakanti S.C. [9] . Let D be a γ - set of G. A dominating set $D_1 \subseteq$ V- D is called an inverse dominating set of G with respect to D. The inverse domination number γ' (G) is the order of a smallest inverse dominating set. Motivated by this definition we define another parameter as follows. Let D be a maximum independent set in G. An independent set $S \subseteq$ V- D is called an inverse independent set with respect to D. The inverse independence Number $\beta_0^{-1}(G)$ = max {|S| : S is an inverse independent set of G}.We find few bounds on inverse domination number and also initiate the study of the inverse independence number giving few bounds on inverse independence number of a graph.

Keywords
Inverse domination number, Independence number and Inverse Independence number.

1. INTRODUCTION
The terminologies and notations used here are as in Harary [5]. By order and size we mean the number of vertices and number of edges respectively in a graph. A set $D \subseteq V$ is a dominating set of a graph G = (V, E), if every $v \in V$- D is adjacent to some $u \in D$. The domination number $\gamma = \gamma(G)$ is the cardinality of a minimum dominating set of G. This concept is well studied in [6]. A set $D \subseteq V$ is said to be independent if no two vertices in D are adjacent. The independence number $\beta_0(G)$ is the maximum cardinality of an independent set of G. We say that an edge x and a vertex v cover each other if x is incident on v. A set $D \subseteq V$ is said to be a vertex cover if every edge in G is covered by some vertex in D. The vertex covering number $\alpha_o(G)$ is the cardinality of a minimum vertex cover of G. A minimum vertex cover (maximum independent set) is denoted as α_o– set (respectively β_0 – set).

Remark 1.1. A set $S \subseteq V$ is a vertex cover if and only if V- S is an independent set.

The independence number and vertex covering number are related by the classical Gallai's Theorem.

Theorem 1.1. [Gallai]**.** For any graph G,

$$\alpha_0(G) + \beta_0(G) = p$$

2. BOUNDS ON INVERSE DOMINATION NUMBER

The concept of inverse domination is introduced by Kulli V.R and Sigarakanti S.C [9]. Let D be a γ - set of G. If $D_1 \subseteq$V- D is a dominating set, then D_1 is called the inverse dominating set of G with respect to D. The inverse domination number γ' (G) is the order of a smallest inverse dominating set. If D is a minimal dominating set of an isolate free graph G, then V- D is also a dominating set of G. Therefore every isolate free graph has an inverse dominating set. Henceforth, by a graph G, we mean an isolate free and simple graph. It is observed that $\gamma(G) \leq \gamma'(G)$ and $\gamma(G) + \gamma'(G) \leq p$. Domke et.al [3] characterized the graphs for which $\gamma(G) + \gamma'(G) = p$. Tamiz Chelvam and Grace Prema G.S. [10], characterized the graphs for which domination and inverse domination numbers are equal. Ameenal Bibi K. and Selvakumar R. [1] studied the Split and nonsplit inverse domination. They also extended inverse domination to semi-total block graph in [2]. Another parameter called disjoint domination number $\gamma\gamma$ (G) defined by Hedetnimi et.al [7] as min{ $|S_1|+|S_2|$; S_1, S_2 are disjoint dominating sets of G}. They call G is $\gamma\gamma$-minimum if $\gamma\gamma(G) = 2\gamma(G)$ and G is $\gamma\gamma$-maximum if $\gamma\gamma(G)$ = p. The following theorem from [6] which gives a lower bound for domination number is used to obtain an upper bound for inverse domination number.

Theorem 2.1[6]**.** For any (p, q) graph G,

$$\max \{p - q, \quad \frac{p}{1+\Delta}\} \leq \gamma$$

Proposition 2.2. For any (p, q) graph G,

$$\gamma' \leq \min\{q, \frac{p\Delta}{1+\Delta}\}$$

Further, this bound is sharp.

Proof. Let D be any γ - set. Since V- D is an inverse dominating set, we have $\gamma' \leq |V - D| = p - \gamma \leq p - \frac{p}{1+\Delta} = \frac{p\Delta}{1+\Delta}$ (using Theorem 2.1). Also $\gamma' \leq p - \gamma \leq p - (p - q) = q$. The bound is sharp for any star graph.

A partition $V_1, V_2, \ldots, V_d$ of the vertex set V is called a domatic partition of G if each V_i, $1 \leq i \leq d$ is a dominating set of G. The domatic number d(G) is the maximum order of a domatic partition of G. In our next result we obtain a bound for γ' in terms of domatic number d(G).

Proposition 2.3. Let G be a graph of order p and domatic number d(G). Let $V_1, V_2, \ldots V_d$ be the domatic partition of G such that V_1 is a γ - set and V_2 is a γ' set of G.

Then $\gamma' \leq \frac{p-\gamma}{d-1}$.

Further, this bound is sharp.

Proof. Let $V_1, V_2, \ldots V_d$ be the domatic partition of G satisfying the condition stated in the proposition. Then $|V_1| + |V_2| + \ldots + |V_d| = p$. Since $\gamma' \leq |V_i|$, for each $2 \leq i \leq d$, we have $(d-1)\gamma' \leq |V_2| + \ldots + |V_d| = p - |V_1| = p - \gamma$ and the result follows. The bound is sharp for any star graph $K_{1,n}$, complete graph K_p and cycle C_{3n}.

We improve the bound obtained in Proposition 2.3 in the next corollary.

Corollary 2.3.1. Let G be a graph with p vertices, and domatic number d(G). Then

$$\gamma' \leq \frac{p\Delta}{(1+\Delta)(d-1)}.$$

Further, this bound is sharp.

Proof. From Proposition 2.2, $p - \gamma \leq p - \frac{p}{1+\Delta} = \frac{p\Delta}{1+\Delta}$. Using this in Proposition 2.3, we get the desired result. The bound is sharp for any star graph $K_{1,n}$ and complete graph K_p.

For any cycle C_{3n}, $\gamma'(C_{3n}) = n$ and $d(C_{3n}) = 3$. Therefore $\gamma'(C_{3n}) = n = \frac{3n\times 2}{3\times 2} = \frac{p\Delta}{(1+\Delta)(d-1)}$. Thus C_{3n} attains the bound in the Corollary. For the graphs with d(G) > 2, the bound given in Corollary 2.3.1 is an improved bound than that given in Proposition 2.2.

Domke et.al [3] proved that for any Tree T of order p, $\frac{p+1}{3} \leq \gamma'(T)$. Using the same proof technique we now obtain a lower bound for inverse domination number in terms of the size of the graph.

Proposition 2.4.

For any graph G of order p, and size q,

$$\frac{2p-q}{3} \leq \gamma'$$

Proof. Let D_1 be the γ- set and D_2 be the γ'-set of G. Then there are at least γ edges from D_1 to D_2 and $V-(D_1 \cup D_2)$ edges from D_1 to $V-D_1$ and $V-(D_1 \cup D_2)$ edges from D_2 to $V-D_2$. Therefore $q \geq 2(V-(D_1 \cup D_2)) + \gamma$
$= 2(p - (\gamma + \gamma')) + \gamma$
$= 2p - \gamma - 2\gamma' \geq 2p - 3\gamma'$ which yields the desired bound.

If G is a cycle with p vertices, then $\frac{(2p-q)}{3} = \frac{(2p-q)}{3} = \frac{p}{3} = \gamma'$. Hence the bound is sharp for any cycle Cp. One can easily observe that bound is also attained for the graph $K_4 - x$ where x is any edge of the complete graph K_4.

Since for any tree T, q = p -1, the next result, originally given by Domke et.al [3] follows as a Corollary to the above Proposition.

Corollary 2.4.1. For any Tree T of order p,

$$\frac{p+1}{3} \leq \gamma'(T)$$

Remark 2.1. If H is any spanning subgraph of G, then $\gamma'(G) \leq \gamma'(H)$. Based on this observation we have the next result.

Proposition 2.5. If G is Hamiltonian, then $\gamma'(G) \leq \left\lceil \frac{p}{3} \right\rceil$

Proof. Let G be a Hamiltonian graph. Then there exists a Hamiltonian cycle C_p which is a spanning subgraph of G. Then from the above remark we have $\gamma'(G) \leq \gamma'(C_p) \leq \left\lceil \frac{p}{3} \right\rceil$.

3. INVERSE INDEPENDENCE NUMBER

The concept of inverse dominating sets motivated us to define another parameter as follows. Let D be a maximum independent set in G. An independent set S in V- D is called an inverse independent set with respect to D. The inverse independence Number $\boldsymbol{\beta_0^{-1}(G)}$ is the order of the largest inverse independent set of G. We initiate the study of the new parameter and obtain few bounds on inverse independence number of a graph.

Remark 2.1. If $S \subseteq V$, then the subgraph induced by the set S is denoted as $\langle S \rangle$. Infact, any inverse independent set of G is a maximal independent set of $\langle V - D \rangle$, the subgraph induced by the set V- D where D is a β_0-set of G. This fact confirms the existence of an inverse independent set for any graph G. Therefore $\beta_0^{-1}(G) = \beta_0 \langle V - D \rangle$.

Another variety of independence number called strong independence number introduced by S.S.Kamth and R.S.Bhat [8]. A vertex v is said to be strong if $d(v) \geq d(u)$ for every u adjacent to v. An independent set in which every vertex is strong is called a strong independent set. Several properties of strong independent sets are studied in [8]

3.1 Inverse Covering Number

Let D be a minimum vertex covering of G. A set $S \subseteq V- D$ which is a covering of G is called an inverse covering of G with respect to the covering D. Then the Inverse Covering number $\alpha_0^{-1}(G)$ is the order of smallest inverse covering of G. Note that $\alpha_0^{-1}(G)$ need not exist for every graph, for if D is a covering of G then V- D need not be a covering of G. For example the complete graph K_p does not have any inverse covering. A graph G is said to be Invertible if G admits an inverse vertex covering.

Example 3.1.

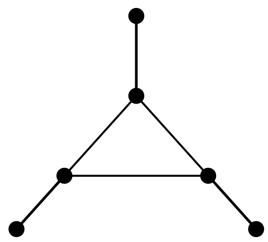

Fig.1.The Corona $K_3 \cdot K_1$

The corona of two graphs G_1 and G_2 (as defined by Harary [4]) is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 . For example, the Corona $K_3 \cdot K_1$ is shown in the Fig. 1.

It is well known that every maximal independent set is a minimal dominating set in any graph. But β_0^{-1} - set need not be a dominating set. For example for the Corona, $\beta_0(K_3 \cdot K_1) = 3$ and $\beta_0^{-1}(K_3 \cdot K_1) = 2$ and any one pendent vertex together with a nonadjacent nonpendent vertex forms a β_0^{-1} – set which is not a dominating set. Further α_0^{-1} does not exists for this graph. Therefore $K_3 \cdot K_1$ is not an invertible graph, whereas every complete bipartite graph $K_{m,n}$ is invertible.

At the outset we give inverse independence number and inverse covering numbers of some standard graphs without proof.

Proposition 3.1.

(i) For any Path P_n and Cycle C_n with n vertices

$$\beta_0^{-1}(P_n) = \left\lfloor\frac{n}{2}\right\rfloor = \beta_0^{-1}(C_n)$$

and $$\alpha_0^{-1}(P_n) = \left\lceil\frac{n}{2}\right\rceil$$

$$\alpha_0^{-1}(C_n) = \frac{n}{2} \quad \text{if n is even}$$

and C_n is not invertible if n is odd.

(ii) For any complete graph K_n,

$$\beta_0^{-1}(K_n) = 1$$

and K_n is not invertible for any n.

(iii) For any complete bipartite graph $K_{m,n}$

$$\beta_0^{-1}(K_{m,n}) = \min(m, n)$$

$$\alpha_0^{-1}(K_{m,n}) = \max(m, n)$$

(iv) For any wheel graph W_n

$$\beta_0^{-1}(W_n) = \beta_0^{-1}(C_{n-1}) = \left\lfloor\frac{n-1}{2}\right\rfloor$$

and W_n is not invertible for any n.

We now give a condition for G to be invertible.

Proposition 3.2. Any graph G is invertible if and only if there exists an α_o-set of G which is independent.

Proof. Suppose there exists an α_o-set D of G which is independent. Then from Remark 1.1, V-D is an inverse vertex cover. Hence $\alpha_0^{-1}(G)$ exists which implies G is invertible. Conversely, suppose G is invertible. Then $\alpha_0^{-1}(G)$ exists and S be the α_0^{-1}-set. Let S_1 be the corresponding α_o-set of G. We claim that S_1 is independent. Suppose S_1 is not independent. Then there exists atleast one edge x in $\langle S_1 \rangle$. But then the edge x is not covered by any vertex in S and hence S is not a vertex cover of G - a contradiction. Hence our claim. Thus S_1 is an α_o-set which is independent.

In the next theorem we characterize the invertible graphs.

Theorem 3.3. A graph G is invertible if and only if $\alpha_0^{-1}(G) + \beta_0^{-1}(G) = p$

Proof. Suppose G is invertible. Then by Proposition 3.2, there exists an α_o-set D which is independent. Then by Remark 1.1, V- D is an inverse vertex cover of G. Since no proper subset of V- D is a vertex cover, we have $\alpha_0^{-1} = |V - D| = p - \alpha_0$. Again, since D is independent α_o-set we have V-D is a β_0-set of G. Hence D is an inverse independent set. Therefore $|D| = \alpha_0 = \beta_0^{-1}$. Thus $\alpha_0^{-1} = p - \alpha_0 = p - \beta_0^{-1}$ which yields $\alpha_0^{-1}(G) + \beta_0^{-1}(G) = p$.

Corollary 3.3.1. If G is invertible, then

$$\alpha_0(G) = \beta_0^{-1}(G) \text{ and } \beta_0(G) = \alpha_0^{-1}(G).$$

4. BOUNDS ON INVERSE INDEPENDENCE NUMBER

Proposition 4.1. For any graph G with p vertices

(i) $\beta_0^{-1} \leq \alpha_0$.
(ii) $\beta_0^{-1} \leq \beta_0$
(iii) $\beta_0^{-1} \leq \frac{p}{2}$

Proof. Let D be any β_0^{-1}set of G. If S is any β_0 set of G then S is a maximum independent set of G. Since D is independent, we have $\beta_0^{-1} = |D| \leq |S| = \beta_0$. By definition of inverse independence number $S \cap D = \emptyset$. Also V- S is α_0-set and $S \cap D = \emptyset$ together imply that $D \subseteq$ V-S. Thus we have $\beta_0^{-1} = |D| \leq |V - S| = \alpha_0$. Now adding (i) and (ii) and using Theorem 1.1 we get (iii).

Proposition 4.2. Let G be a (p, q) graph with minimum degree δ then $\beta_0^{-1} \leq p - \delta$.

Proof. It is well known that $\delta \leq \alpha_0$. Then $\beta_0^{-1} \leq \beta_0 = p - \alpha_0 \leq p - \delta$ and the result follows. The bound is sharp for K_n, complete graph with n vertices and $K_{n,n}$, n-regular complete bipartite graphs with 2n vertices.

We improve the above bound under certain conditions.

Proposition 4.3. Let G be a (p, q) graph with maximum degree Δ and G has a β_0^{-1} set containing atleast one maximum degree vertex. Then $\beta_0^{-1} \leq p - \Delta$.

Proof. Let D be a β_0^{-1} set containing atleast one maximum degree vertex and S be the set of all vertices with maximum degree. Then $S \cap D \neq \emptyset$. Let $v \in S \cap D$. Then v is a maximum degree vertex in D and D is independent together imply that $D \subseteq V - N(v)$. Therefore $\beta_0^{-1} = |D| \leq |V - N(v)| = p - \Delta$

Example 4.1

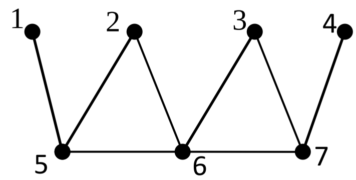

Fig.2

For example for graph, $K_{m,n}$, $\beta_0^{-1}(K_{m,n}) = n = p - m = p - \Delta$, if $n \leq m$. Note that in this case β_0^{-1}-set contains all the maximum degree vertices of $K_{m,n}$. For the graph shown in the Fig 2, S ={1, 2, 3, 4} is a β_0 set. D ={5, 7} is a β_0^{-1}-set. But D does not contain any vertex of maximum degree vertex. $\beta_0^{-1} = 2 < 3 = p - \Delta$. Therefore it seems that $\beta_0^{-1} \leq p - \Delta$ for any graph but we could not either prove the result or give a counter example. Hence we put it as an open problem.

Conjecture 4.4. Let G be a (p, q) graph with maximum degree Δ , then $\beta_0^{-1} \leq p - \Delta$.

Proposition 4.5. Let G be a (p, q) graph with maximum degree Δ, then $\beta_0^{-1} \leq \frac{p\Delta - q}{\Delta}$.

Proof. Since a vertex can cover at most Δ edges and we need at least $\frac{q}{\Delta}$ edges to cover all the edges of G. Therefore $\frac{q}{\Delta} \leq \alpha_0$. Then $\beta_0^{-1} \leq \beta_0 = p - \alpha_0 \leq p - \frac{q}{\Delta}$ and the result follows. The bound is sharp for any regular complete bipartite graph.

The chromatic number $\chi(G)$ is the minimum number of colours required to colour the vertices of a graph G such that no two adjacent vertices have same colour. If $\chi(G) = k$, then the vertex set of G is partitioned in to k independent sets. Next proposition gives a bound for inverse independence number in terms of vertex covering and chromatic number.

Proposition 4.6. Let G be a (p, q) graph with chromatic number $\chi(G)$ and vertex covering number $\alpha_0(G)$. Let $V_1, V_2, V_3, \dots, V_k$ be the partition of the vertex set V in to k independent sets such that V_1 is a β_0 set and V_2 is a β_0^{-1} set of G. Then $\frac{\alpha_0}{\chi - 1} \leq \beta_0^{-1}$. Further this bound is sharp.

Proof. Let $\chi(G) = k$. Suppose that $V_1, V_2, V_3, \dots, V_k$ be the partition of the vertex set V in to k independent sets satisfying the condition of the stated in the proposition. Then
$|V_2| + |V_3| + \cdots + |V_k| = p - |V_1|$. Therefore
$p - \beta_0 = \alpha_0 = |V_2| + |V_3| + \cdots + |V_k| \leq (k-1)\beta_0^{-1}$ and hence the result follows. The bound is sharp for complete graphs and complete bipartite graphs.

The next proposition gives an upper bound for the number of edges when independence number and inverse independence numbers are known.

Proposition 4.7. Let G be a (p, q) graph with $\beta_0 = k$ and $\beta_0^{-1} = s$. Then

$$q \leq \frac{1}{2}\left(p^2 - k^2 - s^2 - (p - k - s)\right).$$

Proof. For any (p, q) graph G, $q \leq \binom{p}{2}$. Further, since $\beta_0 = k$ and $\beta_0^{-1} = s$ there cannot exist $\binom{k}{2} + \binom{s}{2}$ edges in G. This implies $q \leq \binom{p}{2} - \binom{k}{2} - \binom{s}{2}$

$$= \frac{1}{2}\left(p^2 - k^2 - s^2 - (p - k - s)\right).$$

Corollary 4.7.1. Let G be a (p, q) graph with $\beta_0 = k$ and and $\beta_0^{-1} = s$. Then

$$\beta_0^{-1} \leq \frac{1}{2} + \sqrt{\frac{1}{4} + [p(p-1) - k(k-1) - 2q]}$$

Proof. Suppose that $\beta_0^{-1} = s$. Then from Proposition 4.7, we have $q \leq \binom{p}{2} - \binom{k}{2} - \binom{s}{2}$.
Hence $2q \leq p(p-1) - k(k-1) - s(s-1)$. This yields
$s^2 - s - [p(p-1) - k(k-1) - 2q] \leq 0$. Solving this quadratic equation we get

$$s \leq \frac{1 + \sqrt{1 + 4[p(p-1) - k(k-1) - 2q]}}{2}$$

$$= \frac{1 + 2\sqrt{\frac{1}{4} + [p(p-1) - k(k-1) - 2q]}}{2}$$

$$= \frac{1}{2} + \sqrt{\frac{1}{4} + [p(p-1) - k(k-1) - 2q]}.$$

On the similar lines in the next corollary we get a bound for independence number of a graph when the inverse independence number is known.

Corollary 4.7.2. Let G be a (p, q) graph with $\beta_0^{-1} = s$. Then

$$\beta_0 \leq \frac{1}{2} + \sqrt{\frac{1}{4} + [p(p-1) - s(s-1) - 2q]}$$

4.1 Tripartite Split Graph.
A graph G is said to be tripartite split graph if the vertex set of G can be partitioned in to three subsets V_1, V_2, V_3 such that V_1 and V_2 are independent and $\langle V_3 \rangle$ is complete. For example, the tripartite split graph shown in the Fig.3, attains the bound in Proposition 4.7. For this graph, $\beta_0 = 3$, $\beta_0^{-1} = 2$, $p = 7$. Therefore $q = 21 - 3 - 1 = 17 = \binom{7}{2} - \binom{3}{2} - \binom{2}{2} = \binom{p}{2} - \binom{3}{2} - \binom{2}{2}$.

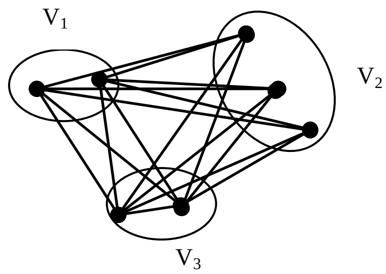

Fig 3. A Tripartite split graph

Similarly, the bound in Corollary 4.7.1 and 4.7.2 are attained for any tripartite split graph.
Again for the graph shown in Fig.3, we have

$$\beta_0 = 3 = \frac{1}{2} + \sqrt{\frac{1}{4} + [7(7-1) - 2(2-1) - 2 \times 17]}$$

$$= \frac{1}{2} + \sqrt{\frac{1}{4} + [p(p-1) - s(s-1) - 2q]}.$$

And $\beta_0^{-1} = 2 = \frac{1}{2} + \sqrt{\frac{1}{4} + [7(7-1) - 3(3-1) - 2 \times 17]}$

$$= \frac{1}{2} + \sqrt{\frac{1}{4} + [p(p-1) - k(k-1) - 2q]}$$

4.2 Characterization Of Graphs Attaining The Bound In Proposition 4.7.

One would ask which graphs will attain the bound in Proposition 4.7? This question is answered in our next result.
Proposition 4.8. Let G be a connected (p, q) graph with $\beta_0 = k$ and $\beta_0^{-1} = s$. Then
$q = \frac{1}{2}\left(p^2 - k^2 - s^2 - (p - k - s)\right)$ if and only if G is a tripartite split graph.

Proof. If G is a tripartite split graph with $\beta_0 = k$ and $\beta_0^{-1} = s$ it is not hard to verify that $q = \frac{1}{2}\left(p^2 - k^2 - s^2 - (p - k - s)\right)$. Conversely, suppose G is a graph with $\beta_0 = k$

and $\beta_0^{-1} = s$ and $q = \frac{1}{2}\left(p^2 - k^2 - s^2 - (p - k - s)\right)$. Let V_1 be the β_0 –set and V_2 be the β_0^{-1} –set and V_3 = V-V_1-V_2. Since V_1 and V_2 are independent and $q = \frac{1}{2}\left(p^2 - k^2 - s^2 - (p - k - s)\right)$,G has maximum number of edges. Therefore $\langle V_3 \rangle$ must be complete. Hence G is a tripartite split graph.

5. CONCLUSION

Kulli V.R and Sigarakanthi S.C [9] and Ameenal Bibi K and Salvakumar R [1] quoted some application of inverse domination. Graph theory serves as a model for any binary relation. In domination, both dominating sets and their inverses have important roles to play. Whenever, D is a dominating set, V-D is also a dominating set whenever G is an isolate free graph. In an information retrieval system, we always have a set of primary nodes to pass on the information. In case, the system fails, we have another set of secondary nodes, to do the job in the complement. When the complement set is connected, then there will be flow of information among the members of the complement. Thus, the dominating sets and the elements in the inverse dominating sets can stand together to facilitate the communication process. Similarly if we want to get an independent set in the compliment set we look for inverse independent set of G. They play very vital role in coding theory, computer science, operations research, switching circuits, electrical networks etc.

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