Complex Dynamics of Superior Multibrots

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ABSTRACT
The Multibrot fractal is a modification of the classic Mandelbrot and Julia sets and it is given by the complex function $z_{n+1} = z_n^d + c; d \geq 2$ where n and c is a constant. This Fractal is particularly interesting, with beautiful shapes and lots of spirals. In this paper we have presented different characteristics of Multibrot function using superior iterates. Further, different properties like trajectories, fixed point, its complex dynamics and its behavior towards Julia set are also discussed in the paper.

Keywords
Complex dynamics, Multibrot.

1. INTRODUCTION
In mathematics the Mandelbrot set, named after Benoit Mandelbrot, is a set of points in the complex plane, the boundary of which forms a fractal. Mathematically the Mandelbrot set can be defined as the set of complex values of *c* for which the orbit of 0 under iteration of the complex quadratic polynomial $z_{n+1} = z_n^2 + c$ remains bounded. Julia sets [6] provide a most striking illustration of how an apparently simple process can lead to highly intricate sets. Function on the complex plain C as simple *as* $z_{n+1} = z_n^2 + c$ give rise of fractals of an exotic appearance. Function z_n for complex c have many fascinating mathematical properties and produce a wide range of interesting images [7]. The superior iterates introduced by Rani and Kumar [11] in the study of chaos and fractal were found to be very effective in generating the fractals beyond the traditional limits. The Multibrot function is given by complex function $z_{n+1} = z_n^d + c; d \geq 2$ where *n* and *c* is a constant. The complex dynamics of Multibrot function, generally known as Multibrot fractal, is a modification of the classic Mandelbrot and Julia sets. The Multibrot (Julia) type is particularly interesting, with beautiful shapes and lots of spirals. In this paper we have presented different characteristics of Multibrot function using superior iterates and generated the superior Multibrot fractal. Further, different properties like trajectories, fixed point, its complex dynamics and its behaviour towards Julia set are also discussed in the paper.

2. PRELIMINARIES
Definition 2.1: Julia sets
French mathematician Gaston Julia [12] investigated the iteration process of a complex function intensively, and attained the Julia set, a very important and useful concept. At present Julia set has been applied widely in computer graphics, biology, engineering and other branches of mathematical sciences.

Consider the complex-valued quadratic function $z_{n+1} = z_n^2 + c; c \in C$

Where c be the set of complex numbers and n is the iteration number. The Julia set for parameter c is defined as the boundary between those of z_0 that remain bounded after repeated iterations and those escape to infinity. The Julia set on the real axis are reflection symmetric, while those with complex parameter show rotation symmetry with an exception to $c = (0,0)$ see Rani and Kumar [11].

Definition 2.2: Multibrot set
The Multibrot function is given by complex function $z_{n+1} = z_n^d + c; d \geq 2$ where *n* and *c* is a constant. The exponent *d* may be further generalized to negative and fractional values. So, the collection of point satisfies the given relation is called Multibrot set. The complex dynamics of Multibrot function, generally known as Multibrot fractal, is a modification of the classic Mandelbrot and Julia sets.

Definition 2.3: Superior Orbit
Let A be a subset of real or complex numbers and $f : A \to A$. For $x_0 \in A$, construct a sequence $\{x_n\}$ in A in the following manner

$$
\begin{aligned}
x_1 &= s_1 f(x_0) + (1 - s_1) x_0 \\
x_2 &= s_2 f(x_1) + (1 - s_2) x_1 \\
&\vdots \\
x_n &= s_n f(x_{n-1}) + (1 - s_n) x_{n-1}
\end{aligned}
$$

Where $0 < s_n \le 1$ and $\{s_n\}$ is convergent to a non-zero number.

The sequence $\{x_n\}$ constructed above is called Mann sequence of iterates or superior sequence of iterates. Let z_0 be an arbitrarily element of C. Construct sequences $\{z_n\}$ of points of C in the following manner:

$$z_n = sf(z_{n-1}) + (1-s) z_{n-1}, n = 1, 2, 3....,$$

Where f is a function on a subset of C and the parameter s lies in the closed interval $[0,1]$.

The sequence $\{z_n\}$ constructed above, denoted by $SO(f, z_0, s)$,is the superior orbit for the complex-valued function f with an initial choice z_0 and parameter s. We may denote it by $SO(f, x_0, s_n)$. Notice that $SO(f, x_0, s_n)$ with $s_n = 1$ is $O(f, x_0)$. We remark that the superior orbit reduces to the usual Picard orbit when $s_n = 1$.

Definition 2.4: Superior Multibrot set

The sequence $\{x_n\}$ constructed above is called Mann sequence of iteration or superior sequence of iterates. We may denoted by $SO(f, x_0, s_n)$. Now we define the Mandelbrot set for $Q_c(z) = z^n + c$, where $n = 2, 3, 4, ...$ with respect to Mann iterates. We call it Superior Mandelbrot set. The collection of points whose orbits are bounded under the superior iteration for the Multibrot function, described above, is called the filled superior Multibrot set.

3. Analysis of Superior Multibrot set

For $d = 2$, the multibrot set converts to Mandelbrot set. To further investigate the various characteristics of the superior multibrot, we divide our analysis in three categories on the basis of power ***d***.

Case 1. For the Positive power or $d \ge 2$

Case 2. For the Negative power or $d < 0$

Case 3. For the Non integer Power or d, i.e. fractional

Case 1. For $d \ge 2$:

For ***d*** = 2, we notice that the superior Multibrot fractal converts to usual superior Mandelbrot fractal. The case is also true for the corresponding superior Julia sets. The Multibrot set for higher values of ***d*** are given in Fig. 1-4. Further, the study for higher orders of d is given as follows:

Fig 1. Superior Multibrot set of third degree *d* = 3 and *s*= 0.5

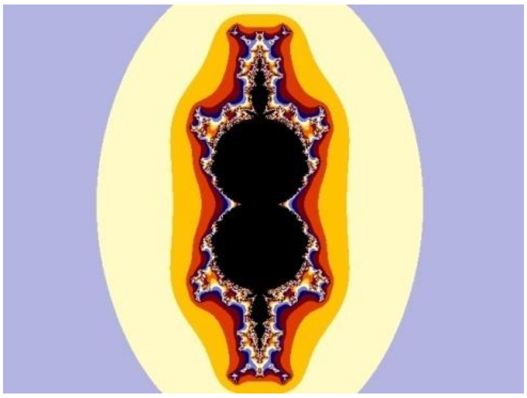


Fig 2. Superior Multibrort set of forth degree *d* = 4 and *s*= 0.5

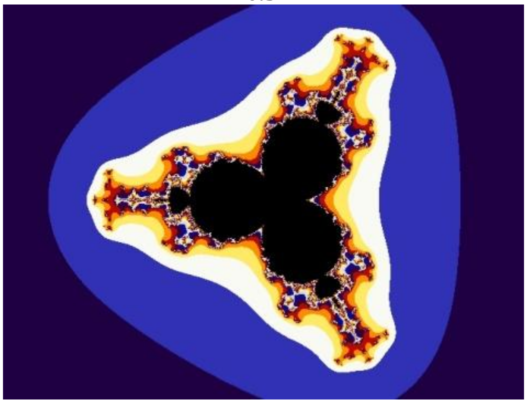


Fig 3. Superior Multibrot set of higher degree d = 5 and s= 0.5

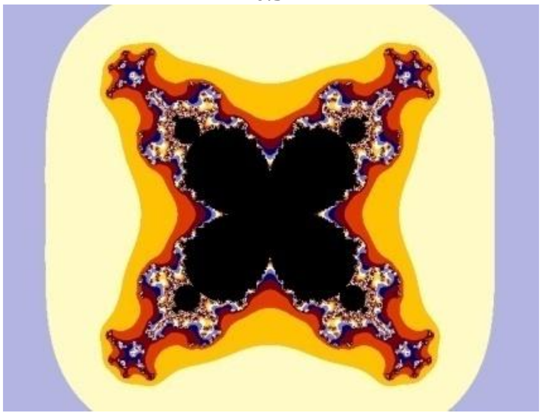
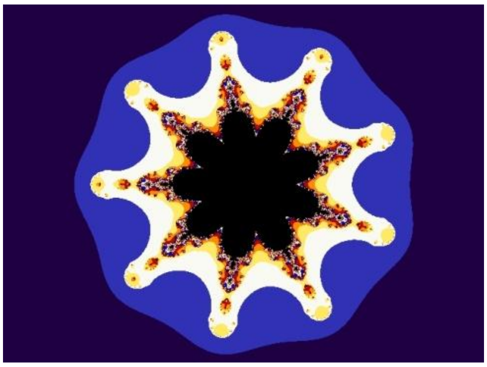


Fig 4. Superior Multibrot set of higher degree *d* = 10 and *s*= 0.5



The new set has some striking similarities to the superior Mandelbrot set. We found that for odd powers the superior Multibrot set is symmetric across both the real and imaginary axis, whereas for the even powers we find the symmetry to be along Real axis only. As we increase the power of superior multibrot function, we get *d-1* bulb in each image. In the next section we have analyzed Superior Julia sets for Multibrot fractals for $d \geq 2$ and calculated their fixed points.

Third degree superior Julia sets for Multibrot set:

Fig 5. Superior Julia set for *F* (*z*) = (-0.0189, 19.2277) at *d*=3 and *s* = 0.1

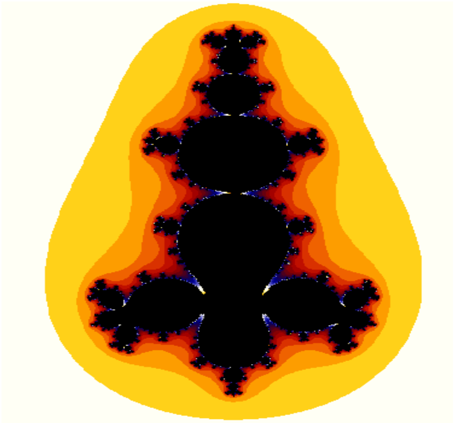


Table 1. *F*(*z*) for (z_0 = -0.0189, 19.277) at *d*=3 and *s* = 0.1

Number of iteration *i*	I *F*(*z*)I
63	2.1554
64	2.8613
65	2.1554
66	2.8613
67	2.1554
68	2.8613
69	2.1554
70	2.8613
71	2.1554

(We skipped 59 iterations and after 60 iterations value converges to two fixed point.)

Fig 6. Orbit of Superior Julia set for *F* (*z*) = (-0.0189, 19.2277) at *d*=3 and *s* = 0.1

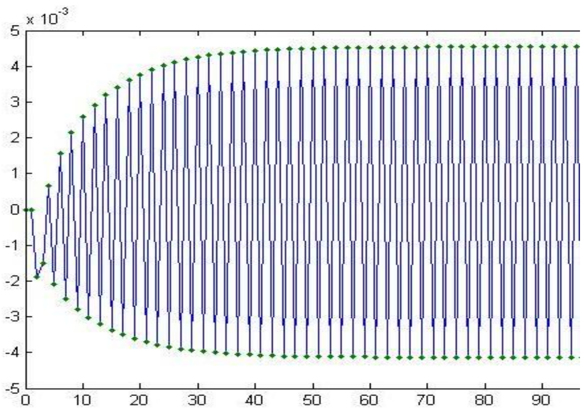
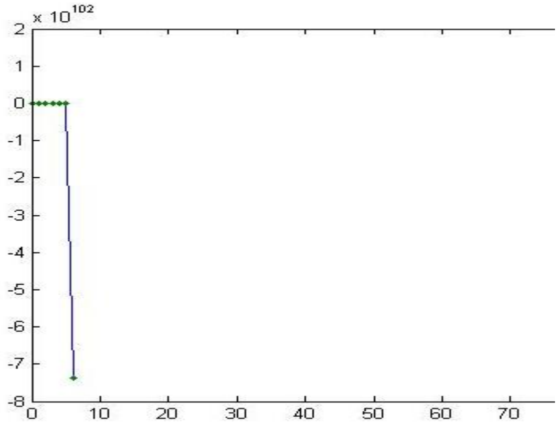


Fig 7. Orbit of Superior Julia set for *F* (*z*) = (-0.0189, 19.2277) at *d*=3 and *s* = 1.0



Higher degree superior Julia sets for Multibrot set:

Fig 8. Superior Julia set for *F* (*z*) = (4.3271, 3.1202) at *d*=6 and *s* = 0.1

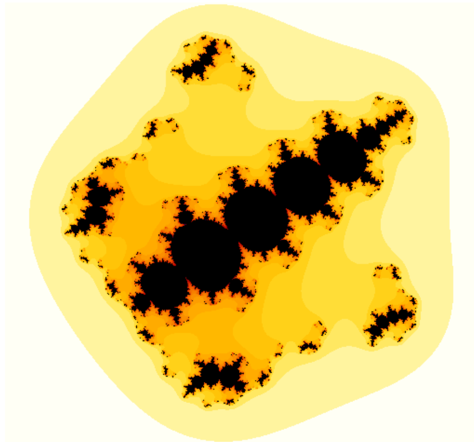


Table 2: *F* (*z*) for (z_0 = 4.3271, 3.1202) at *d*=6 and *s* = 0.1

Number of iteration i	I F(z)I
26	1.1512
27	1.3579
28	1.1512
29	1.3579
30	1.1512
31	1.3579
32	1.1512
33	1.3579
34	1.1512
35	1.3579
36	1.1512
37	1.3579

(We skipped 25 iterations and after 26 iterations value converges to two fixed point.)

Fig 9. Orbit of Superior Julia set for *F* (*z*) = (4.3271, 3.1202) at *d*=6 and *s* = 0.1

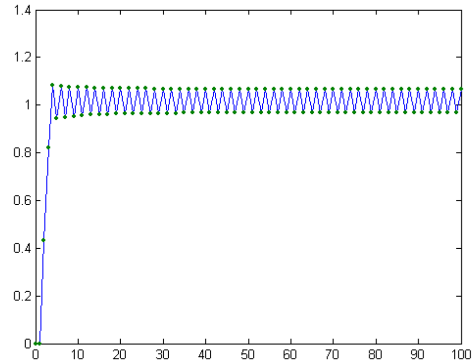
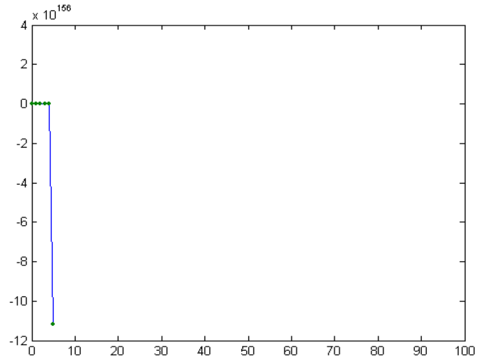


Fig 10. Orbit of Superior Julia set for *F* (*z*)= (4.3271, 3.1202) at *d*=6 and *s* = 1



Third degree superior Julia sets for Multibrot set is symmetric through *y* axis and after 60 iterations value converges to two fixed point. Fourth degree superior Julia sets for Multibrot set is symmetric through *x* axis with lots of spiral and after 14 iterations value converges to infinity. In Higher degree superior Julia sets for Multibrot set, we also get symmetry with bulb Sequences as they are superimposed on one another and after 26, 28 iterations value converges to two fixed point respectively.

Case2. For $d < 0$:

The study for Negative orders of ***d*** is given as follows:

Fig 11. Superior Multibrot set of third degree for *d* = -3 and *s*= 0.5

Fig 12. Superior Multibrort set of forth degree for *d* =- 4 and *s*= 0.5

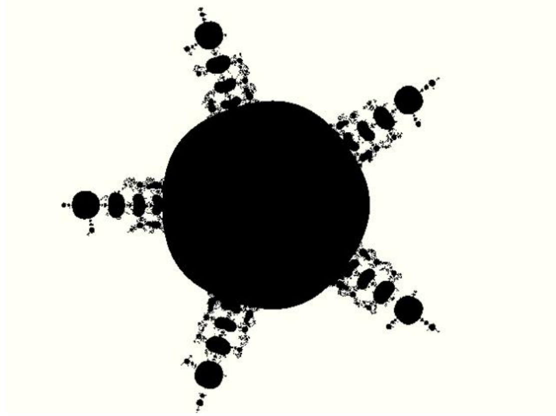


Fig 13. Superior Multibrot set of higher degree for *d* = -5 and *s*= 0.5

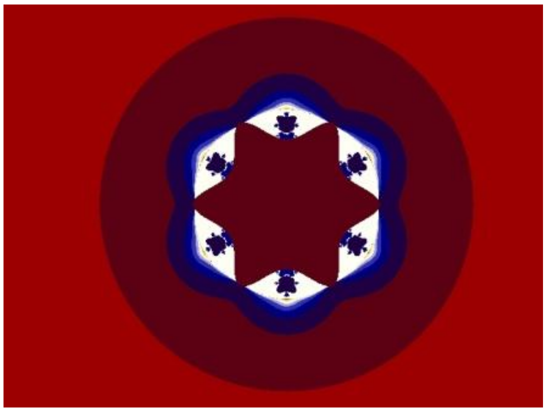
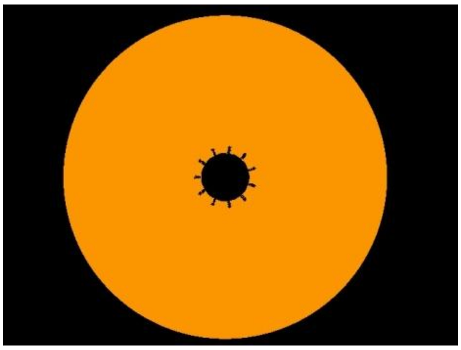


Fig 14. Superior Multibrot set of higher degree for *d* =-10 and *s*= 0.5



Further we have analysis the generation of Superior Julia sets for Multibrot fractals for $d \geq 2$.

Fig 15. Third degree superior Julia sets for Multibrot set For $F(z) = (-0.0507, 0.1672)$ at d=-3 and s = 0.1

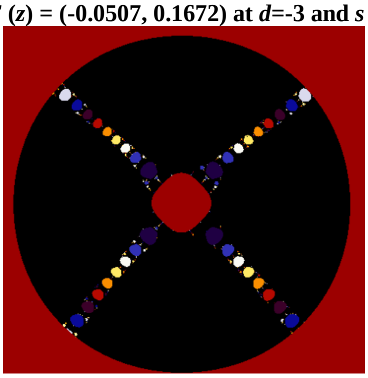


Fig 16. Fourth degree superior Julia sets for Multibrot set For $F(z) = (-0.0507, 0.1672)$ at d=-4 and s = 0.1

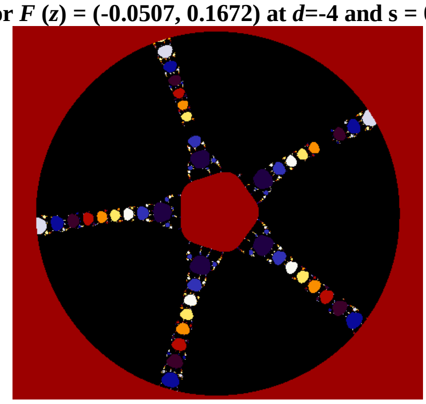


Fig 17. Higher degree superior Julia sets for Multibrot set For $F(z) = (-0.0507, 0.1672)$ at d=-5 and s = 0.1

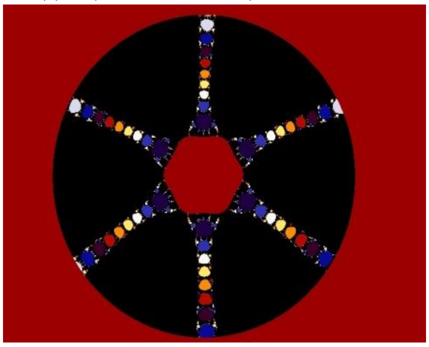


Fig 18. Higher degree superior Julia sets for Multibrot set For $F(z) = (-0.0507, 0.1672)$ at d=-10 and s = 0.1

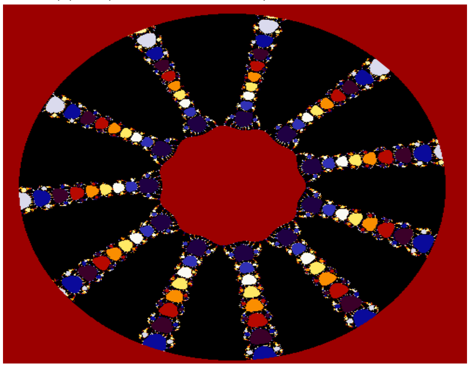


Fig 19. Third degree superior Julia sets for Multibrot set For $F(z) = (0.0015,-4.4522)$ at d=-3 and s = 0.1

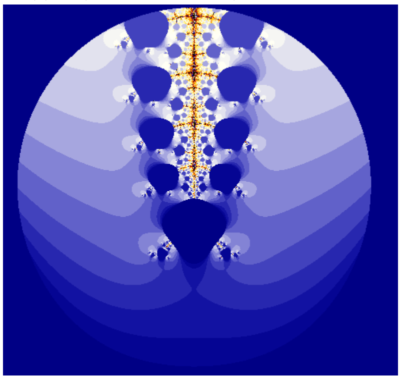


Fig 20. Third degree superior Julia sets for Multibrot set For $F(z) = (2.0527,-0.0498)$ at d=-3 and s = 0.1

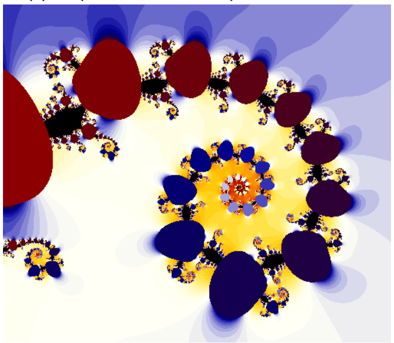


Fig 21. Higher degree superior Julia sets for Multibrot set For $F(z) = (-0.0507, 0.1672)$ at d=-10 and s = 0.5

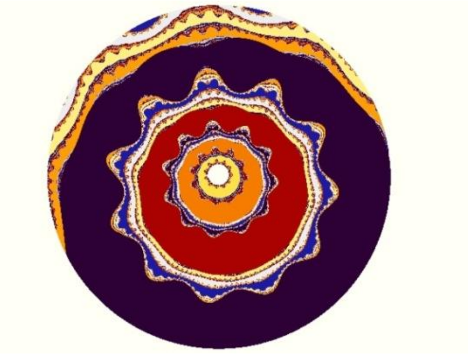
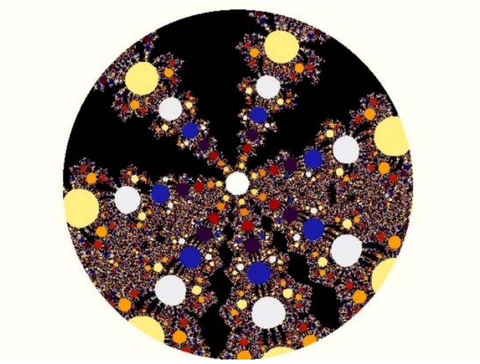


Fig 22. Higher degree superior Julia sets for Multibrot set For $F(z) = (-0.0507, 0.1672)$ at d=-6 and s = 0.3



For the negative powers no fixed point are generated. The images of superior Julia set for Multibrot set are similar like a wheel and as we increase the power, we see (1-d) fold rotational symmetry. There are some amazing fractals noted for third degree Julia sets. One of them shows a human backbone like structure and the other exhibits a spiral flower like structure.

Case 3. For d = fractional:

The study for Fractional orders of d is given as follows:

Fig 23. Superior Multibrot set of third degree *d* = 3.3 and *s*= 0.1

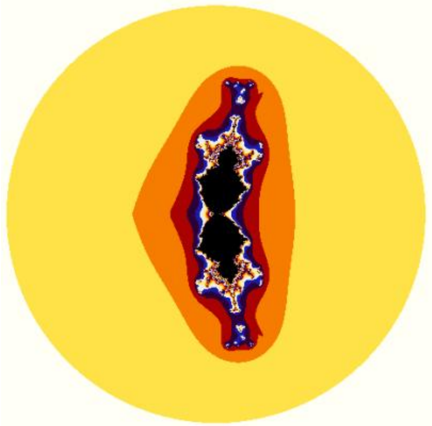


Fig 24. Superior Multibrort set of forth degree *d* = 4.5 and *s*= 0.1

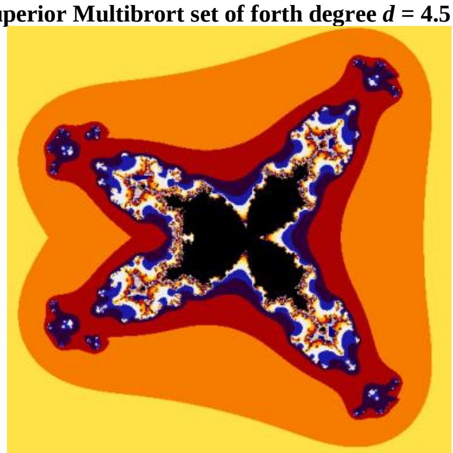


Fig 25. Superior Multibrot set of higher degree *d* = 6.7 and *s*= 0.1

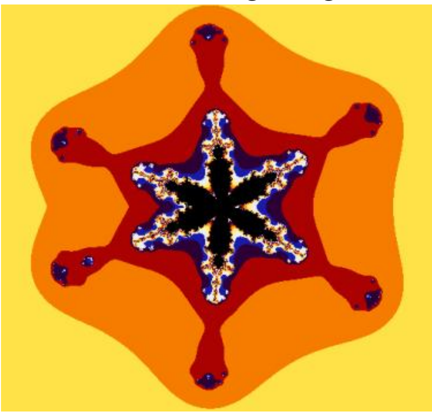
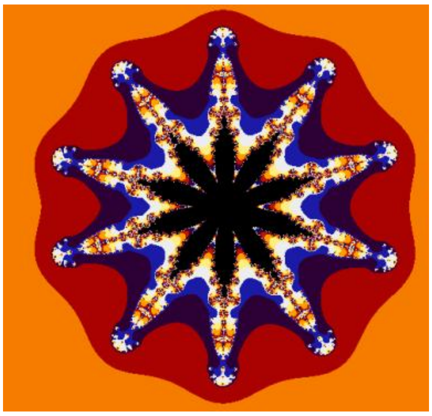


Fig 26. Superior Multibrot set of higher degree *d* = 10.9 and *s*= 0.1



As we increase the power of superior multibrot function, we get *d-1* bulb in each image. At higher power we get images like a star. It is also Observed that, as we increase the mann iteration we can see the birth of new bulb. Further, we have analysis the generation of Superior Julia sets for Multibrot fractals for $d \geq 2$ and calculated their fixed points given as below.

Fourth degree superior Julia sets for Multibrot set:

Fig 27.For *F* (*z*) = (-5.6138, 4.8791) at *d*=4.5 and *s* = 0.1

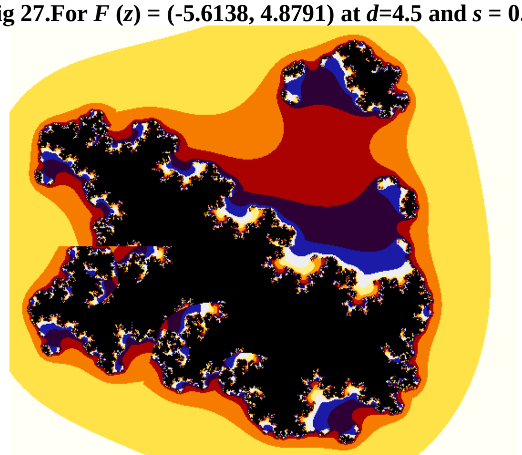


Table 3: *F* (*z*) for (z_0 = -5.6138, 4.8791) at *d*=4.5 and *s* = 0.1

Number of iteration i	I F(z)l
221	1.4883
222	1.4883
223	1.4883
224	1.4883
225	1.4883
226	1.4883
227	1.4883
228	1.4883
229	1.4883
230	1.4883

(We skipped 218 iterations and after 219 iterations value converges to a fixed point.)

Fig 28. Orbit of Superior Julia set for *F* (*z*) = (-5.6138, 4.8791) at *d*=4.5 and *s* = 0.1

Number of iteration i	I F(z)I
35	1.0968
36	1.0138
37	1.0968
38	1.0138
39	1.0968
40	1.0138
41	1.0968
42	1.0138
43	1.0968
44	1.0138

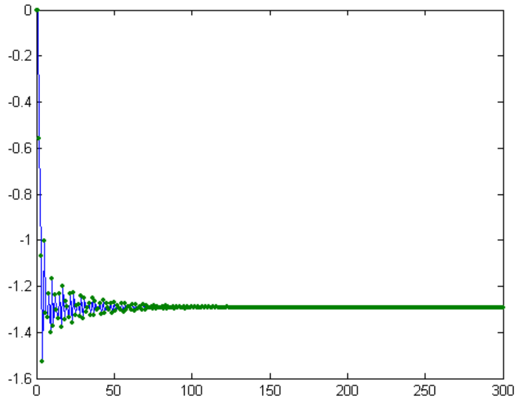
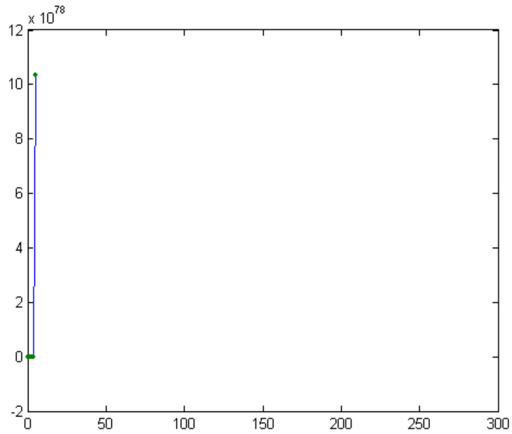


Fig 29. Orbit of Superior Julia set for *F* (*z*) = (-5.6138, 4.8791) at *d*=4.5 and *s* = 1



Higher degree superior Julia sets for Multibrot set:

Fig 30.For *F* (*z*) = (-0.0409, 3.0050) at *d*=10.9 and *s* = 0.1

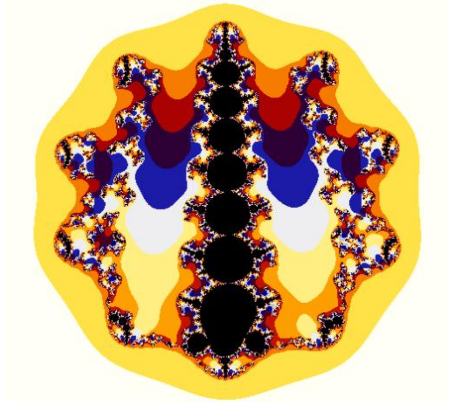


Table 4: *F* (*z*) for (z_0 = (-0.0409, 3.0050) at *d*=10.9 and *s* = 0.1

(We skipped 32 iterations and after 33 iterations value converges to two fixed point.)

Fig 31. Orbit of Superior Julia set for *F* (*z*) = (-0.0409, 3.0050) at *d*=10.9 and *s* = 0.1

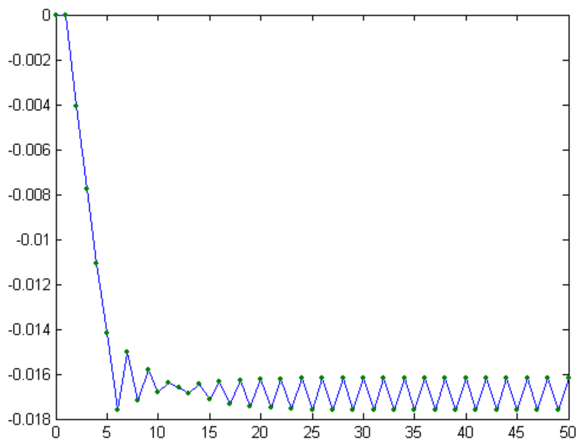


Fig 32. Orbit of Superior Julia set for *F* (*z*) = (-0.0409, 3.0050) at *d*=10.9 and *s* = 0.1

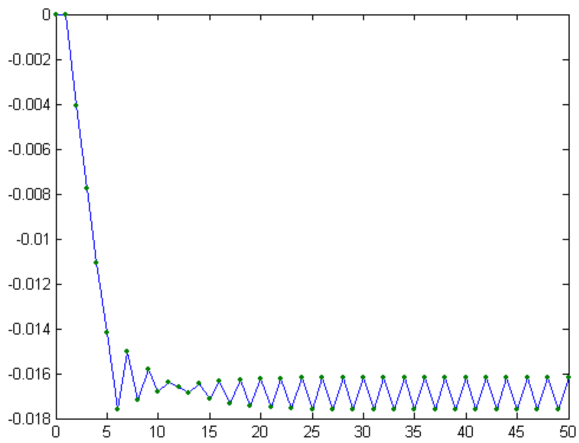


Fig 33.For *F* (*z*) = (0.3123, 0.0499) at *d*=7.5 and *s* = 0.1

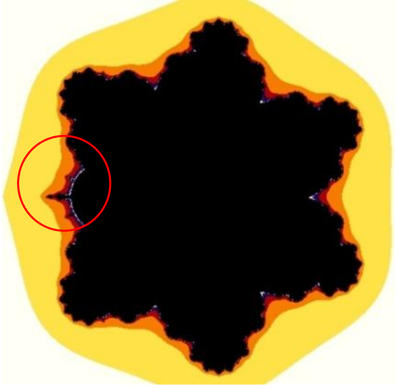
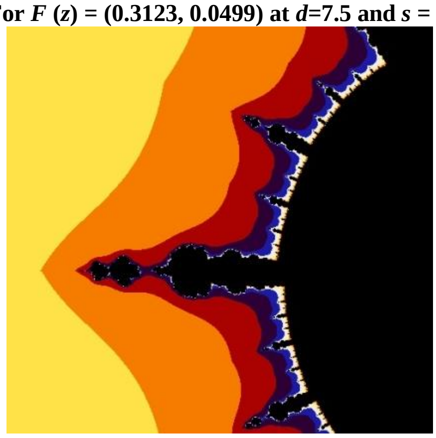


Fig 34.For *F* (*z*) = (0.3123, 0.0499) at *d*=7.5 and *s* = 0.1



In the seventh degree superior Julia sets for Multibrot set we can easily see the birth of new bulb Fig.[34] and the zoom is shown in the corresponding figure. Fourth degree superior Julia sets for Multibrot set having distortion along *x* axis and after 219 iterations value converges to a fixed point. In Higher degree superior Julia sets for Multibrot set, we also get symmetry along ***y*** axis and after 30 and 33 iterations value converges to one and two fixed point respectively.

CONCLUSION

In this paper we have presented the dynamics and fixed point analysis of Mandelbrot set by using Superior Iterates for higher, negative and fractional power. In the above discussion we studied the symmetry of the Multibrot sets along axis and noticed the distortion in the main bulb of Multibrot set. It is observed that for some values the fixed points converges to one, two and infinity. Further, we also presented some remarkable spiral shaped Julia sets of Multibrot fractals (see Fig 19-22).

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