Time dependent solution of a Non-Markovian Queue with Triple stages of service having Compulsory vacation and service interruptions

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ABSTRACT
In this paper, single server queue with poisson arrivals, triple stages of service with service interruption & compulsory server vacation is considered. After the completion of first stage and second stage of service, the server must provide the third-stage of service. After the completion of each third stage of service, the server will take compulsory vacation. The vacation time is, exponentially distributed.The time dependent probability generating functions have been obtained in terms of Laplace transforms.

Key Words
Poisson arrival, probability generating functions, time dependent solution, service interruption, server vacation.

1.INTRODUCTION
In this paper we consider a qucueing system with compulsory server vacation and service interruptions. A qucueing system might be interrupted. Due to this, server may not be able to proceed with the process till the interruption is cleared. In this paper,each arriving customer had to undergo three stages of services provided by a single server. The service time follows general distribution. Once the system interrupts, then the customer whose service is interrupted goes back to the head of the queue where the arrivals are poisson. Many authors have discussed about Two stages of services. In this article, we have developed a three stages of services which will be more advantageous in large scale industries. Performance measures will be improved. Char et al. [3] chang and Takine [4] and Igaki [7] have studied queues with generalized vacations. Vacation queens with C servers have been studied by Tian et al [15]

Madan & Marghi [14] have studied batch arrivals having general vacation time. Kulkarni and choi [9] have studied retrial queues, k.c.so [6] formed the optional maintenance policies for single server queue. Vacation queues have been studied extensively by numerous authors Levy and yechiali [10], Doshi [5] and Madan [11], [12], [13] due to their applications in computer network .V.Thangaraj formed a two stage service [2] Aissani and Artalejo [1] made a work on single server retrial qucueing system. Jayewardene and kella [8] have studied M/G/D queues with service interruptions.

2.MATHEMATICAL DESCRIPTION OF THE MODEL

We have the following assumptions:

- Customers arrive at a poison process with arrival rate λ.
- Service time of the three stage follows different general distributions with distribution function Mj (ν),and density function $m_j(\nu)$, j = 1,2,3…
- Let s_i (x) dx be the conditional probability of completion of the i th stage of service during (x, x+dx) given that elapsed time is x, So

$$K_i(x) = \frac{m_i(x)}{1-M_i(x)}, \quad i = 1,2,3$$

$$m_i(\nu) = K_i(\nu) \exp\left(-\int_0^{\nu} K_i(x)\,dx\right), \quad i = 1,2,3$$

- Vacation time follows general (arbitrary) distribution with distribution function V(s) and the density function ν(s).

  The conditional probability distribution γ (x) dx of a completion of a vacation time during (x, x+dx) given that the elapsed vacation time is x, so

$$\gamma(x) = \frac{v(x)}{1-V(x)}$$

$$\nu(s) = \gamma(s) \exp\left[-\left(\int_0^{s} \gamma(x)\,dx\right)\right]$$

  Interruption arrive with mean rate β>o.Let δ be the server rate of attending interruption.

- The mean vacation time is $\frac{1}{\gamma}$

3. DEFINITIONS

We define

- $P_{n,1}^{(1)}(x,t)$ = probability that at time 't' there are 'n' customers in the queue excluding the one being served, the server is active providing first stage of service and the elapsed service time for this customer is x.
- $P_{n,1}^{(1)}(x,t) = \int_0^\infty P_{n,1}^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first stage of service irrespective of the value of x.
- $P_{n,1}^{(2)}(x,t)$ = probability that at time 't' the server is active providing second stage of service and there are n ($\geq$,0) customers in the queue excluding the one being served and the elapsed service time for this customer is x.
- $P_{n,1}^{(2)}(x,t) = \int_0^\infty P_{n,1}^{(2)}(x,t)dx$ denotes the probability that at time 't' there are n customers in the queue excluding the one customer in the second stage of service irrespective of the value of x.
- $P_{n,1}^{(3)}(x,t)$ = probability that at time t, the server is active providing third stage of service & there are n ($\geq$,o) customers in the queue excluding the one customer being served and the elapsed service time foe this customer is x.
- $P_{n,1}^{(3)}(x,t) = \int_0^\infty P_{n,1}^{(3)}(x,t)dx$ denotes the probability that at time t there are n customer in the third stage of service customer in the third stage of service irrespective of the value of x.
- $V_n(x,t)$ = probability that at time 't' the server is under vacation with elapsed vacation time x and there are n ($\geq$,o) customers waiting in the queue for service.
- $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time 't' there are n customers in the queue and the server is under vacation irrespective of the value of x.
- $R_n(t)$ = probability that at time 't' the server is inactive due to arrival of service interruption.
- Q (t) = probability that at time 't' there are no customers in the system and the server is idle but available in the system.

4. EQUATIONS GOVERNING THE SYSTEM

$$\frac{\partial}{\partial x}P_{n,1}^{(1)}(x,t) + \frac{\partial}{\partial t}P_{n,1}^{(1)}(x,t) + (\lambda + K_1(x) + \gamma)P_{n,1}^{(1)}(x,t) = \lambda P_{n-1,1}^{(1)}(x,t) \qquad (1)$$

$$\frac{\partial}{\partial x}P_{0,1}^{(1)}(x,t) + \frac{\partial}{\partial t}P_{0,1}^{(1)}(x,t) + (\lambda + K_1(x) + \gamma)P_{0,1}^{(1)}(x,t) = 0. \qquad (2)$$

$$\frac{\partial}{\partial x}P_{n,1}^{(2)}(x,t) + \frac{\partial}{\partial t}P_{n,1}^{(2)}(x,t) + (\lambda + K_2(x) + \gamma)P_{n,1}^{(2)}(x,t) = \lambda P_{n-1,1}^{(2)}(x,t). \qquad (3)$$

$$\frac{\partial}{\partial x}P_{0,1}^{(2)}(x,t) + \frac{\partial}{\partial t}P_{0,1}^{(2)}(x,t) + (\lambda + K_2(x) + \gamma)P_{0,1}^{(2)}(x,t) = 0. \qquad (4)$$

$$\frac{\partial}{\partial x}P_{n,1}^{(3)}(x,t) + \frac{\partial}{\partial t}P_{n,1}^{(3)}(x,t) + (\lambda + K_3(x) + \gamma)P_{n,1}^{(3)}(x,t) = \lambda P_{n-1,1}^{(3)}(x,t). \qquad (5)$$

$$\frac{\partial}{\partial x}P_{0,1}^{(3)}(x,t) + \frac{\partial}{\partial t}P_{0,1}^{(3)}(x,t) + (\lambda + K_3(x) + \gamma)P_{0,1}^{(3)}(x,t) = 0 \qquad (6)$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + (\lambda + \gamma(x))V_n(x,t) = \lambda V_{n-1}(x,t)\quad n=1,2\ldots\ldots \qquad (7)$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + (\lambda + \gamma(x))V_0(x,t) = 0. \qquad (8)$$

$$\frac{d}{dt}R_n(t) = -(\lambda + \beta)R_n(t) + \lambda R_{n-1}(t) + \beta\int_0^\infty P_{n-1}^{(1)}(x,t)dx + \beta\int_0^\infty P_{n-1}^{(2)}(x,t)dx + \beta\int_0^\infty P_{n-1}^{(3)}(x,t)dx,$$

$$n = 1,2\ldots \qquad (9)$$

$$\frac{d}{dt}R_0(t) = -(\lambda + \beta)R_o(t) \qquad (10)$$

$$\frac{d}{dt}Q(t) = -\lambda\, Q(t) + R_0(t)\beta + \int_0^\infty V_0(x,t)\,\gamma(x)dx \_\_\_\_ (11)$$

By the usage of following boundary conditions, the above equations (1-11) to be solved.

$$P_{0,1}^{(1)}(0,t) = Q(t)\lambda + R_1(t)\beta + \int_0^\infty V_1(x,t)\gamma(x)dx \_\_\_\_\_\_\_\_\_\_\_ (12)$$

$$P_{n,1}^{(1)}(0,t) = R_{n+1}(t)\beta + \int_0^\infty V_{n+1}(x,t)\,\gamma(x)\,dx$$

n=1,2……__________ (13)

$$P_{n,1}^{(2)}(0,t) = \int_0^\infty P_{n,1}^{(1)}(x,t)\,K_1(x)\,dx,\ n=0,1\ldots\ldots$$

.________ (14)

$$P_{n,1}^{(3)}(0,t) = \int_0^\infty P_{n,1}^{(2)}(x,t)\,K_2(x)\,dx,\ n=0,1\ldots\ldots$$

.________ (15)

$$V_n(0,t) = \int_0^\infty P_{n,1}^{(3)}(x,t)\,K_3(x)dx,\ n=0,1\ldots\ldots \_\_\_\_\_\_\_\_ (16)$$

Assume that initially there is no customer in the system and the server is idle.

$$\left.\begin{array}{l} V_0(0)=0,\ V_n(0)=0,\ Q(0)=1 \\ P^{j}_{n,1}(0) = 0 \text{ for } n=0,1,2\ldots\ldots\ldots\ldots \\ \qquad j=1,2,3\ldots\ldots\ldots\ldots \end{array}\right\} \_\_\_\_\_\_\_\_\_\_\_ (17)$$

Now we shall find the transient solution for the above equations.

First, let us define the probability generating functions.

$$\left.\begin{array}{l} P_{q,1}^{(1)}(x,z,t) = \sum_{n=0}^{1} Z^n\, P_{n,1}^{(1)}(x,t) \\ P_{q,1}^{(1)}(z,t) = \sum_{n=0}^{1} Z^n\, P_{n,1}^{(1)}(t) \\ P_{q,1}^{(2)}(x,z,t) = \sum_{n=0}^{1} Z^n\, P_{n,1}^{(2)}(x,t) \\ P_{q,1}^{(2)}(z,t) = \sum_{n=0}^{1} Z^n\, P_{n,1}^{(2)}(t) \\ P_{q,1}^{(3)}(x,z,t) = \sum_{n=0}^{1} Z^n\, P_{n,1}^{(3)}(x,t) \\ P_{q,1}^{(3)}(z,t) = \sum_{n=0}^{1} Z^n\, P_{n,1}^{(3)}(t) \\ V_q(x,z,t) = \sum_{n=0}^{1} Z^n\, V_n(x,t) \\ V_q(z,t) = \sum_{n=0}^{1} Z^n\, V_n(t) \\ R_q(z,t) = \sum_{n=0}^{1} Z^n\, R_n(t) \end{array}\right\} \_\_\_\_\_\_\_\_\_\_ (18)$$

Define the Laplace transform of f(t) as

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,\ R(s) > o.\ \_\_\_\_\_\_\_\_\_\_ (19)$$

Taking the Laplace transform of equations (1) to (16) & using (17), we obtain

$$\frac{\partial}{\partial x}\bar{p}_{(n,1)}^{(1)}(x,s) + (s+\lambda+K_1(x)+\beta)\,\bar{p}_{(n,1)}^{(1)}(x,s) = \lambda\,\bar{p}_{(n-1,1)}^{(1)}(x,s),\ n = 1,2\ldots\ldots \_\_\_\_\_\_\_\_\_\_ (20)$$

$$\frac{\partial}{\partial x}\bar{p}_{(0,1)}^{(1)}(x,s) + (s+\lambda+K_1(x)+\beta)\,\bar{p}_{(0,1)}^{(1)}(x,s) = 0 \_\_\_\_\_\_\_\_\_\_\_\_ (21)$$

$$\frac{\partial}{\partial x}\bar{p}_{(n,1)}^{(2)}(x,s) + (s+\lambda+K_2(x)+\beta)\,\bar{p}_{(n,1)}^{(2)}(x,s) = \lambda\,\bar{p}_{(n-1,1)}^{(2)}(x,s),\ n = 1,2\ldots\ldots\ldots \_\_\_\_\_\_\_\_\_\_ (22)$$

$$\frac{\partial}{\partial x}\bar{p}_{(n,1)}^{(2)}(x,s) + (s+\lambda+K_2(x)+\beta)\,\bar{p}_{(0,1)}^{(2)}(x,s) = 0.\ \_\_\_\_\_\_\_\_\_ (23)$$

$$\frac{\partial}{\partial x}\bar{p}_{(n,1)}^{(3)}(x,s) + (s+\lambda+K_3(x)+\beta)\,\bar{p}_{(n,1)}^{(3)}(x,s) = \lambda\frac{\partial}{\partial x}\bar{p}_{(n-1,1)}^{(3)}(x,s)\ \ n=1,2\ldots.. \_\_\_\_\_\_\_\_\_\_ (24)$$

$$\frac{\partial}{\partial x}\,\overline{p}_{(0,1)}^{(3)}(x,s) + (s + \lambda + K_3(x) + \beta)\,\overline{p}_{(0,1)}^{(3)}(x,s) = 0. \quad\text{\_\_\_\_\_\_\_\_\_\_(25)}$$

$$\frac{\partial}{\partial x}\,\overline{V_n}(x,s) + (s + \lambda + \gamma(x))\,\overline{V_n}(x,s) = \lambda\,\overline{V_{n-1}}(x,s),$$
n =1,2……__________(26)

$$\frac{\partial}{\partial x}\,\overline{V_0}(x,s) + (s + \lambda + \gamma(x))\,\overline{V_0}(x,s) = 0.\quad\text{\_\_\_\_\_\_\_\_\_\_(27)}$$

$$(s+\lambda+\delta)\,\overline{R_n}(s) = \lambda\,\overline{R_{n-1}}(s) + \beta\int_0^\infty \overline{p}_{(n-1,1)}^{(1)}(x,s)dx + \beta\int_0^\infty \overline{p}_{(n-1,1)}^{(2)}(x,s)dx + \beta\int_0^\infty \overline{p}_{(n-1,1)}^{(3)}(x,s)dx$$
n = 1,2…….__________(28)

$$(s+\lambda+\delta)\,\overline{R_0}(s) = 0\quad\text{\_\_\_\_\_\_\_\_\_\_(29)}$$

$$(s+\lambda)\,\overline{Q}(s) = 1+\overline{R_0}(s) + \int_0^\infty \overline{V_0}(x,s)\,\gamma(x)dx \quad\text{\_\_(30)}$$

$$\overline{p}_{(0,1)}^{(1)}(0,s) = \lambda\overline{Q}(s) + \overline{R_1}(s)\,\delta + \int_0^\infty \overline{V_1}(x,s)\,\gamma(x)dx$$
__________(31)

$$\overline{p}_{(n,1)}^{(1)}(0,s) = \overline{R_{n+1}}(s)\,\delta + \int_0^\infty \overline{V_{n+1}}(x,s)\,\gamma(x)\,dx$$

n = 1,2……..__________(32)

$$\overline{p}_{(n,1)}^{(2)}(o,s) = \int_0^\infty \overline{p}_{(n,1)}^{(1)}(x,s)\,K_1(x)\,dx$$
n =0,1…..__________(33)

$$\overline{p}_{(n,1)}^{(3)}(o,s) = \int_0^\infty \overline{p}_{(n,1)}^{(2)}(x,s)\,K_2(x)\,dx$$
n =0,1…..__________(34)

$$\overline{V_n}(o,s) = \int_0^\infty \overline{p}_{(n,1)}^{(3)}(x,s)\,K_3(x)\,dx$$
n =0,1…..__________(35)

Multiply equation (20) by z^n & summing over n from 1 to ∞, adding to equation 21 & using the generating function, we have

$$\frac{\partial}{\partial x}\,\overline{p}_{(q,1)}^{(1)}(x,z,s) + (s + \lambda - \lambda z + K_1(x) + \beta)\,\overline{p}_{(q,1)}^{(1)}(x,z,s) = 0.\quad\text{\_\_\_\_\_\_\_\_\_\_(36)}$$

Similar operations are to be carried for the equations (22) to (29)

$$\frac{\partial}{\partial x}\,\overline{p}_{(q,1)}^{(2)}(x,z,s) + (s + \lambda - \lambda z + K_2(x) + \beta)\,\overline{p}_{(q,1)}^{(2)}(x,z,s) = 0.\quad\text{\_\_\_\_\_\_\_\_\_\_(37)}$$

$$\frac{\partial}{\partial x}\,\overline{p}_{(q,1)}^{(3)}(x,z,s) + (s + \lambda - \lambda z + K_3(x) + \beta)\,\overline{p}_{(q,1)}^{(3)}(x,z,s) = 0.\quad\text{\_\_\_\_\_\_\_\_\_\_(38)}$$

$$\frac{\partial}{\partial x}\,\overline{V_q}(x,z,s) + (s + \lambda - \lambda z + \gamma(x))\,\overline{V_q}(x,z,s) = 0.$$
__________(39)

$$(s+\lambda-\lambda z+\delta)\,\overline{R_q}(z,s) = \alpha z\left(\int_0^\infty \overline{p}_{(q,1)}^{(1)}(x,z,s)dx + \int_0^\infty \overline{p}_{(q,1)}^{(2)}(x,z,s)\,dx + \int_0^\infty \overline{p}_{(q,1)}^{(3)}(x,z,s)\,dx\right)$$
__________ (40)

Multiply both sides of equation (31) by z, multiply both sides of equation (32) by z^{n+1}, sum over n from 1 to ∞, add the two results, then using (18) we get.

$$Z\,\overline{p}_{(q,1)}^{(1)}(o,z,s) = \lambda z\,\overline{Q}(s) + \delta\,\overline{R_q}(z,s) - \delta\,\overline{R_0}(s) + \int_0^\infty \overline{V_q}(x,z,s)\,\gamma(x)\,dx - \int_0^\infty \overline{V_0}(x,s)\,\gamma(x)\,dx.$$
__________(41)

Performing similar operations on equations (33), (34) and (35), we get

$$\overline{p}_{(q,1)}^{(2)}(o,z,s) = \int_0^\infty \overline{p}_{(q,1)}^{(1)}(x,z,s)\,K_1(x)\,dx. \quad (42)$$

$$\overline{p}_{(q,1)}^{(3)}(o,z,s) = \int_0^\infty \overline{p}_{(q,1)}^{(2)}(x,z,s)\,K_2(x)\,dx.$$
__________ (43)

$$\overline{V_q}\ (o,z,s) = \int_0^\infty \overline{p}_{(q,1)}^{(3)}\ (x,z,s)\, K_3(x)\, dx. \_\_\_\_\_ \quad (44)$$

Using equation (30), equation (41) becomes,

$$Z\ \overline{p}_{(q,1)}^{(1)}\ (o,z,s) = (1\text{-}S\,\overline{Q}(s) + \lambda (z\text{-}1)\ \overline{Q}\ (s) + \delta\, \overline{R_q}\ (z,s) + \int_0^\infty \overline{V_q}\ (x,z,s)\, \gamma(x)\, dx. \_\_\_\_\_\_\_\_\_\_(45)$$

Integrating equation (36) from o to x, we get

$$\overline{p}_{(q,1)}^{(1)}\ (x,z,s) = \overline{p}_{(q,1)}^{(1)}\ (0,z,s) \exp\{-\ (s+\lambda-\lambda z+\beta)\, x - \int_0^\infty k_1(t)\, dt\ \} \quad \_\_\_\_\_\_\_\_\_\_(46)$$

Where $\overline{p}_{(q,1)}^{(1)}$ (0,z,s)is given by equation (45). Again integrating (46) by parts with respect to x,

$$\overline{p}_{(q,1)}^{(1)}\ (z,s) = \overline{p}_{(q,1)}^{(1)}\ (0,z,s) \left(\frac{1-\overline{M}_1(s+\lambda-\lambda z+\beta)}{s+\lambda-\lambda z+\beta} \right) \_\_\_\_\_\_\_\_\_\_\ (47)$$

Where

$$\overline{M_1}\ (s+\lambda-\lambda z+\beta) = \int_0^\infty \exp(-\ (s+\lambda-\lambda z+\beta)x)\, dM_1(x)$$

is the Laplace stieltjes transform of the first stage service time M_1 (x).

Now multiply both sides of equation (46) by K_1 (x) & integrating over x, we obtain

$$\int_0^\infty \overline{p}_{(q,1)}^{(1)}\ (x,z,s) K_1(x)\, dx = \overline{p}_{(q,1)}^{(1)}\ (0,z,s)\ \overline{M_1}\ (s+\lambda-\lambda z+\beta)\_\_\_\_\_\_\_(48)$$

Similary integrating the equations (37) & (39) from 0 to x, we obtain.

$$\overline{p}_{(q,1)}^{(2)}\ (x,z,s) = \overline{p}_{(q,1)}^{(2)}\ (0,z,s) \exp\{-(s+\lambda-\lambda z+\beta)x - \int_0^\infty K_2(t)\, dt\ \}\_\_\_\_\_\_\_\_\_\_(49)$$

$$\overline{p}_{(q,1)}^{(3)}\ (x,z,s) = \overline{p}_{(q,1)}^{(3)}\ (0,z,s) \exp\{-\ (s+\lambda-\lambda z+\beta)\, x - \int_0^\infty K_3(t)\, dt\}\ \_\_\_\_\_\_\_\_\_\_(50)$$

$$\overline{V_q}\ (x,z,s) = \overline{V_q}\ (0,z,s) \exp\{-(s+\lambda-\lambda z+\beta)x - \int_0^\infty \gamma(t)\, dt\ \}\_\_\_\_\_\_\_\_\_\_(51)$$

Where $\overline{p}_{(q,1)}^{(2)}$ (0,z,s), $\overline{p}_{(q,1)}^{(3)}$ (0,z,s) and $\overline{V_q}$ (0,z,s) are given by equations (42), (43) and (44).

Again integrating (49),(50) and (51) by parts with respect to x, we get

$$\overline{p}_{(q,1)}^{(2)}\ (z,s) = \overline{p}_{(q,1)}^{(2)}\ (0,z,s) \left(\frac{1-\overline{M}_2(s+\lambda-\lambda z+\beta)}{s+\lambda-\lambda z+\beta} \right) \_\_\_\_\_\_\_\_\_\_(52)$$

$$\overline{p}_{(q,1)}^{(3)}\ (z,s) = \overline{p}_{(q,1)}^{(3)}\ (0,z,s) \left(\frac{1-\overline{M}_3(s+\lambda-\lambda z+\beta)}{s+\lambda-\lambda z+\beta} \right) \_\_\_\_\_\_\_\_\_(53)$$

$$\overline{V_q}\ (z,s) = \overline{V_q}\ (0,z,s) \left(\frac{1-\overline{V}\ (s+\lambda-\lambda z)}{s+\lambda-\lambda z} \right) \_\_\_\_\_\_\_\_\_\_ \quad (54)$$

Where

$$\overline{M_2}\ (s+\lambda-\lambda z+\beta) = \int_0^\infty \exp(-\ (s+\lambda-\lambda z+\beta)x)\, dM_2(x)\ \_\_\_\_(55)$$

$$\overline{M_3}\ (s+\lambda-\lambda z+\beta) = \int_0^\infty \exp(-\ (s+\lambda-\lambda z+\beta)x\ )\, dM_3(x)\ \_\_\_\_\_(56)$$

is the Laplace - stieltjes transform of the second stage service time M_2 (x) and third stage service time M_3(x).

Multiply both sides of equation (50) by K_2 (x) and integrating over x & also multiply both sides of equation (51) by K_3 (x) and integrating over x, we get

$$\int_0^\infty \overline{p}_{(q,1)}^{(2)}\ (x,z,s) K_2(x)\, dx = \overline{p}_{(q,1)}^{(2)}\ (0,z,s)\ \overline{M_2}\ (s+\lambda-\lambda z+\beta)\_\_\_\_\_\_\_\_\_\_(57)$$

$$\int_0^\infty \overline{p}_{(q,1)}^{(3)}\ (x,z,s)\, K_3(x)\, dx = \overline{p}_{(q,1)}^{(3)}\ (0,z,s)\ \overline{M_3}\ (s+\lambda-\lambda z+\beta)\ \_\_\_\_\_\_\_\_\_\_(58)$$

$$\overline{V}\ (s+\lambda-\lambda z) = \int_0^\infty \exp(-(s+\lambda-\lambda z)x)\, dV(x)\ \_\_\_\_\_\_\_\_\_\_(59)$$

is the Laplace stieltjes transform of V (x).

Multiply both sides of equation (51) by γ (x) & integrating over x, we get

$$\int_0^\infty \overline{V_q}\ (x,z,s)\,\gamma\ (x)\ dx = \overline{V_q}\ (0,z,s)\ \overline{V}\ \ (s+\lambda-\lambda z)$$

__________(60)

Now using equations (42), (48), (57), (58) we can write equation (44) as,

$$\overline{V_q}\ (0,z,s) = \int_0^\infty \overline{p}_{(q,1)}^{(3)}\ (x,z,s)\ K_3\ (x)\ dx$$

$$= \overline{p}_{(q,1)}^{(3)}\ (0,z,s)\ \overline{M}_3\ (s+\lambda-\lambda z+\beta)$$

$$= \int_0^\infty \overline{p}_{(q,1)}^{(2)}\ (x,z,s)\ K_2\ (x)\ dx\ [\overline{M}_3\ (s+\lambda-\lambda z+\beta)]$$

$$= \overline{p}_{(q,1)}^{(2)}\ (0,z,s)\ \overline{M}_2\ (s+\lambda-\lambda z+\beta)\ \overline{M}_3\ (s+\lambda-\lambda z+\beta)$$

$$= \int_0^\infty \overline{p}_{(q,1)}^{(1)}\ (x,z,s)\ K_1\ (x)\ dx\ [\overline{M}_2\ (s+\lambda-\lambda z+\beta)]\ [\overline{M}_3\ (s+\lambda-\lambda z+\beta)]$$

$$\overline{V_q}\ (0,z,s) = \overline{p}_{(q,1)}^{(1)}\ (0,z,s)\ \overline{M}_1\ (s+\lambda-\lambda z+\beta)\ \overline{M}_2\ (s+\lambda-\lambda z+\beta)\ \overline{M}_3\ (s+\lambda-\lambda z+\beta)$$ __________(61)

Using above equations, equation (54) becomes,

$$\overline{V_q}\ (z,s) = \overline{p}_{(q,1)}^{(1)}\ (0,z,s)\ [\overline{M}_1\ (s+\lambda-\lambda z+\beta)\ ][\overline{M}_2\ (s+\lambda-\lambda z+\beta)][\ \overline{M}_3\ (s+\lambda-\lambda z+\beta)]\ \left(\frac{1-\overline{V}\ (s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right)$$ _______(62)

By using (61), equation (60) becomes,

$$\int_0^\infty \overline{V_q}\ (x,z,s)\ \gamma\ (x)\ dx = \overline{p}_{(q,1)}^{(1)}\ (0,z,s)[\overline{M}_1\ (s+\lambda-\lambda z+\beta)]\ [\overline{M}_2\ (s+\lambda-\lambda z+\beta)][\ \overline{M}_3\ (s+\lambda-\lambda z+\beta)][\ \overline{V}\ \ (s+\lambda-\lambda z]$$ ___(63)

Now using (48) & (49) equation (42) & (43) reduces to

$$\overline{p}_{(q,1)}^{(2)}\ (0,z,s) = \overline{p}_{(q,1)}^{(1)}\ (0,z,s)\ \overline{M}_1\ (s+\lambda-\lambda z+\beta)$$ ____(64)

$$\overline{p}_{(q,1)}^{(3)}\ (0,z,s) = \overline{p}_{(q,1)}^{(2)}\ (0,z,s)\ \overline{M}_2\ (s+\lambda-\lambda z+\beta)$$ ___(65)

Using equations (48), (49), (64) & (65) equations (40) becomes,

$$(s+\lambda-\lambda z+\delta)\ R_q\ (z,s) = \beta z\ [\int_0^\infty \overline{p}_{(q,1)}^{(1)}\ (x,z,s)dx + \int_0^\infty \overline{p}_{(q,1)}^{(2)}\ (x,z,s)\ dx+\int_0^\infty \overline{p}_{(q,1)}^{(3)}\ (x,z,s)\ dx\ ]$$

$$= \beta z\ [\ \overline{p}_{(q,1)}^{(1)}\ (0,z,s)\ \left(\frac{1-\overline{M}_1(s+\lambda-\lambda z+\beta)}{s+\lambda-\lambda z+\beta}\right)+ \overline{p}_{(q,1)}^{(2)}\ (0,z,s)\ \left(\frac{1-\overline{M}_2(s+\lambda-\lambda z+\beta)}{s+\lambda-\lambda z+\beta}\right) + \overline{p}_{(q,1)}^{(3)}\ (0,z,s)\ \left(\frac{1-\overline{M}_3(s+\lambda-\lambda z+\beta)}{s+\lambda-\lambda z+\beta}\right)\ ]$$

$$= \beta z \left(\frac{\overline{p}_{q,1}^{(1)}(0,z,s)}{s+\lambda-\lambda z+\beta}\right)\Big\{[1-\overline{M}_1\ (s+\lambda-\lambda z+\beta)]+\ \overline{M}_1\ (s+\lambda-\lambda z+\beta)\ [1-\overline{M}_2\ (s+\lambda-\lambda z+\beta)]+[1-\overline{M}_3\ (s+\lambda-\lambda z+\beta)]\ \overline{M}_2\ (s+\lambda-\lambda z+\beta)\ [\ \overline{M}_1\ (s+\lambda-\lambda z+\beta)]\Big\}$$

$$= \beta z \left(\frac{\overline{p}_{q,1}^{(1)}(0,z,s)}{s+\lambda-\lambda z+\beta}\right)\ [1-\overline{M}_1\ (s+\lambda-\lambda z+\beta\ )\ \overline{M}_3\ (s+\lambda-\lambda - z+\beta)\ \overline{M}_2\ (s+\lambda-\lambda z+\beta)\ ]$$

$$\overline{R_q}\ (z,s) = \beta z \left(\frac{\overline{p}_{q,1}^{(1)}(0,z,s)}{s+\lambda-\lambda z+\beta}\right)\frac{1-\overline{M}_1(s+\lambda-\lambda z+\beta)\overline{M}_2(s+\lambda-\lambda z+\beta)\overline{M}_3(s+\lambda-\lambda z+\beta)}{s+\lambda-\lambda z+\delta}$$

(66)

Using equations (60) & (66) in eq (45), solving for

$\overline{p}_{(q,1)}^{(1)}\ (0,z,s)$, we get

$$\overline{p}_{(q,1)}^{(1)}\ (0,z,s) = \frac{f_1(z)\,f_2(z)[1-s\overline{Q}(s)+\lambda(z-1)\overline{Q}(s)]}{f_1(z)\,f_2(z)[z-\overline{M}_1 f_1(z)\overline{M}_2 f_1(z)\overline{M}_3 f_1(z)]\ [\overline{V}\ (s+\lambda\text{ - }\lambda z)]\text{- }\delta\beta z[1\text{- }\overline{M}_1 f_1(z)\overline{M}_2 f_1(z)\overline{M}_3 f_1(z)]}$$

__________ (67)

Where $f_1\ (z) = s+\lambda-\lambda z + \beta$, $f_2\ (z) = s + \lambda - \lambda z + \delta$

Substituting the value of $\overline{p}_{(q,1)}^{(1)}$ (0,z,s) in equations (47), (49), (50), (62) & (66) we get

$$\overline{p}_{(q,1)}^{(1)}(z,s) = \frac{f_2(z)[1-s\overline{Q}(s)+\lambda(z-1)\overline{Q}(s)][1-\overline{M_1}f_1(z)]}{DR} \quad (68)$$

$$\overline{p}_{(q,1)}^{(2)}(z,s) = \frac{f_2(z)[1-s\overline{Q}(s)+\lambda(z-1)\overline{Q}(s)][1-\overline{M_2}f_1(z)]}{DR} \quad (69)$$

$$\overline{p}_{(q,1)}^{(3)}(z,s) = \frac{f_2(z)[1-s\overline{Q}(s)+\lambda(z-1)\overline{Q}(s)][1-\overline{M_3}f_1(z)]}{DR} \quad (70)$$

$$\overline{V_q}(z,s) = \frac{f_1(z)f_2(z)[1-s\overline{Q}(s)+\lambda(z-1)\overline{Q}(s)][\overline{M_1}f_1(z)\overline{M_2}f_1(z)\overline{M_3}f_1(z)]}{DR}\left(\frac{1-\overline{V}(s+\lambda-\lambda z)}{s+\lambda-\lambda z)}\right) \quad (71)$$

$$\overline{R_q}(z,s) = \frac{\beta z[1-s\overline{Q}(s)+\lambda(z-1)\overline{Q}(s)][1-\overline{M_1}f_1(z)\overline{M_2}f_1(z)\overline{M_3}f_1(z)]}{DR} \quad (72)$$

Where DR =

$$f_1(z)f_2(z)z-\overline{M_1}f_1(z)\overline{M_2}f_1(z)\overline{M_3}f_1(z)$$
$$[\overline{V}(s+\lambda-\lambda z)]-\delta\beta z[1-\overline{M_1}f_1(z)\overline{M_2}f_1(z)\overline{M_3}f_1(z)]$$

Thus $\overline{p}_{(q,1)}^{(1)}$ (z,s), $\overline{p}_{(q,1)}^{(2)}$ (z,s), $\overline{p}_{(q,1)}^{(3)}$, $\overline{V_q}$ (z,s) and $\overline{R_q}$ (z,s) are completely determined. Thus the transient solution is obtained.

5.CONCLUSION

This paper clearly analyses the transient solution of the queueing system with three stages of service, server provides essential service in all the stages to the arriving customers.This model can be utilized in Large scale manufacturing industries. As a future work, the steady state behaviour of the system and the other performance measures can be derived.

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