Complex and Inverse Complex Dynamics of Fractals using Ishikawa Iteration

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ABSTRACT

Complex graphics of dynamical system have been a subject of intense research nowadays. The fractal geometry is the base of these beautiful graphical images. Many researchers and authors have worked to study the complex nature of the two most popular sets in fractal geometry, the Julia set and the Mandelbrot set, and proposed their work in various forms using existing tools and techniques. Still researches are being conducted to study and reveal the new concepts unexplored in the complexities of these two most popular sets of fractal geometry. Recently, Ashish Negi, Rajeshri Rana and Yashwant S. Chauhan are among those researchers who have contributed a lot in the area of Fractal Geometry applications. In this paper we review the recently done work on complex and inverse complex functions for producing beautiful fractal graphics. The reviewed work mainly emphasizes on the study of the nature of complex and inverse complex functional dynamics using Ishikawa iterates and existence of relative superior *Mandel-bar* set.

Keywords
Fractals, Complex dynamics, Inverse Complex dynamics, Relative Superior Mandelbrot Set, Relative Superior Julia Set, Ishikawa Iteration, Relative Superior Mandel-bar Set.

1. INTRODUCTION

In 1918, French mathematician Gaston Julia [9] investigated the iteration process of complex function and attained a Julia set, which is a landmark in the field of fractal theory. The object Mandelbrot set on the other hand was given by Benoit B. Mandelbrot [11] in 1979. The visual complexity, beauty and self similarity of these structures have made these subjects of a wide area of intense research right from its advent. The various extensions and variants of both of these sets have been extensively studied using Picard's iterations. We have applied in this research article a new iteration process called Ishikawa iteration.

The fractals generated from the self-squared function $z \rightarrow z^2 + c$, where z and c are complex quantities, have been studied extensively in literature [5, 6, 7 & 14]. Recently, the generalized transformation function $z \rightarrow z^{-n} + c$ for positive integer value of n has been considered by K.W.Shirriff [14]. The z plane fractal images for the function $z_{n+1} \leftarrow z_n^{\alpha} + c$ for positive and negative, both integer and non-integer values of n have been presented by Gujar et al. along with some conjectures about their visual characteristics [6, 7].

Several programs and papers have used escape-time methods to produce images of fractals based on the complex mapping $z \rightarrow (z^n + c)^{-1}$, where exponent n is a positive integer. On the other hand, Shizuo [15], has presented the various properties of Multicorns and Tricorns for simple complex function, where z and c are complex quantities. Shizuo[16]has also quoted the Multicorns as the generalized Tricorn or Tricorn of higher order. The dynamics of anti-polynomial $z \rightarrow \bar{z}^d + c$ of complex polynomial $z^d + c$, where $d \geq 2$, leads to interesting Tricorns and Multicorns antifractals with respect to function iteration[4, 15, 16]. Multicorns are symmetrical objects. Their symmetry has been studied by Lau and Schieicher [10]. The study of connectedness locus for anti-holomorphic polynomials $\bar{z}^2 + c$ defined as Tricorns coined by Milnor, plays an intermediate role between quadratic and cubic polynomials. Crowe etal.[3] considered as in formal analogy with Mandelbrot set and named it as Mandel-bar set and also brought its features bifurcations along axis rather than at points. Milnor [12] found it as a real slice of cubic connected locus. Winters [17] showed it as boundary along the smooth arc.

In this paper we considered the transformation (for studying complex dynamics) of the functions $z \rightarrow z^2 + c, n \geq 2$ and $z \rightarrow z^{-n} + c, n \geq 2$, and transformation (for studying inverse complex dynamics) of the function $z \rightarrow (z^n + c)^{-1}$, for $n \geq 2$, and analysed the z plane fractal images for the first function and c plane fractal images for the second function, generated from Ishikawa iterations of these functions using Ishikawa iteration procedure. We explored the drastic changes that occur in visual characteristics of the images for different integer and non-integer values of n.

2. ELABORATION OF CONCEPTS INVOLVED
2.1 Mandelbrot Set
Definition 1. The Mandelbrot set M for the quadratic $Q_C(z) = z^2 + c$ is defined as the collection of all

$c \in C$ for which the orbit of point 0 is bounded, that is,

$$M = \{c \in C : \{Q_c^n(0)\}; n = 0,1,2,3\ldots \text{ is bounded}\}$$

An equivalent formulation is

$$M = \{c \in C : \{Q_c^n(0) \text{ does not tends to } \infty \text{ as } n \to \infty\}\}$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

2.2 Julia Set

Definition 2. The set of points K whose orbits are bounded under the iteration function of $Q_c(z)$ is called the Julia set. We choose the initial point 0, as 0 is the only critical point of $Q_c(z)$.

2.3 Ishikawa Iteration

Definition 3. Ishikawa Iterates [8]**:** Let X be a subset of real or complex number and $f : X \to X$ for all $x_0 \in X$, we have the sequence $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n = S'_n f(x_n) + (1 - S'_n)x_n$$

$$x_{n+1} = S_n f(y_n) + (1 - S_n)x_n$$

where $0 \le S'_n \le 1$, $0 \le S_n \le 1$ and S'_n & S_n are both convergent to non-zero number.

2.4 Relative Superior Orbit

Definition 4. [13] The sequence x_n and y_n constructed above is called Ishikawa sequence of iteration or relative superior sequence of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$.

Notice that $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1$ is $SO(x_0, s_n, t)$ i.e. mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduce to $O(x_0, t)$. We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is relative superior orbit.

2.5 Relative Superior Mandelbrot Set

Now we define Mandelbrot set for the function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

Definition 5. [13] Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where n = 1,2,3,… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded i.e.
$RSM = \{c \in C : Q_c^k(0) : k = 0,1,2,3\ldots\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number.

2.6 Relative Superior Julia Set

Definition 6. [2] The set of points RSK whose orbits are bounded under relative superior iteration of function Q(z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK.

2.7 Mandel-bar Set

Definition 7. [2] The Mandel-bar set A_c, for the quadratic $A_c(z) = z^m + c$ is defined as the collection of all $c \in C$ for which the orbit of point 0 is bounded, that is,
$A_c = \{c \in C : A_c(0)_{n=0,1,2,3,\ldots} \text{ is bounded}\}$.

An equivalent formulation is
$A_c = \{c \in C : A_c(0) \text{ not tends to } \infty \text{ as n} \to \infty\}$.

2.8 Relative Superior Mandel-bar Set

Definition 8. [13] Relative superior Mandel-bar set RSMB for the function of the form $Q_c(z) = z^n + c$, where n=1,2,3,4,… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded i.e.
$RSMB = \{c \in C : Q_c^k(0) : k = 0,1,2,3,\ldots\}$ is bounded.

3. GENERATING PROCESS

The basic principle of generating fractals employs the iterative formula: $z_{n+1} \leftarrow f(z_n)$ where z_0 = the initial valueof z, and z_i = the value of complex quantity z at the ith iteration [6, 7]. For example, the Mandelbrot's self-squared function for generating fractal is: f(z) = z^2+c , where z and c are both complex quantities. We propose the use of transformation function $z \to z^n + c, n \ge 2$ and $z \to (z^n + c)^{-1}$ for generating fractal images with respect to Ishikawa iterates, where z and c are the complex quantities and n is a real number. Each of these fractal images is constructed as two-dimensional array of pixel. Each pixel is represented by a pair of (x,y) coordinates. The complex quantities z and c can be represented as:

$$z = z_x + iz_y$$

$$c = c_x + ic_y$$

where $i = \sqrt{(-1)}$ and z_x , c_x are the real parts and z_y , c_y are the imaginary parts of z and c respectively. The pixel coordinates (x,y) may be associated with (c_x, c_y) or (z_x, z_y).

Based on this concept, the fractal images can be classified as follows:

(a) **z-Plane** fractals, wherein (x,y) is a function of (z_x, z_y).
(b) **c-Plane** fractals, wherein (x,y) is a function of (c_x, c_y).

In the literature, the fractals for n=2 in z plane are termed as the Mandelbrot set while the fractals for n=2 in c plane are known as Julia sets [14]

4. ESCAPE CRITERION FOR RELATIVE SUPERIOR JULIA AND MANDELBROT SETS

4.1 Escape Criterion for Quadratics

[13] Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then

$|z_n| > (1+\lambda)^n |z|$ and $|z| \to \infty$ as $n \to \infty$. So,

$|z| \geq |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

4.2 Escape Criterion for Cubics

[13] Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$

then $|z_n| \to \infty$ as $n \to \infty$. This gives the escape criterion

for cubic polynomials.

4.3 General Escape Criterion

[13] Consider $|z| > \max\{|c|, (2/s)^{1/2}, (2/s')^{1/2}\}$ then

$|z_n| \to \infty$ as $n \to \infty$ is the escape criterion.

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z \to (z^n + c)^{-1}$. However, instead of starting with $z_0 =$ infinity, it is simpler to start with $z_1 = c$, which yields the same result. A critical point of $z \to F(z) + c$ is a point where $F'(z) = 0$.

5. SIMULATIONS AND RESULTS

Generation of Relative Superior Mandelbrot set

Fig.1: For cubic function [1]: s=0.8, s'=0.3, n=3.2

Fig.2: For cubic function: s=0.8, s'=0.3, n=3.8
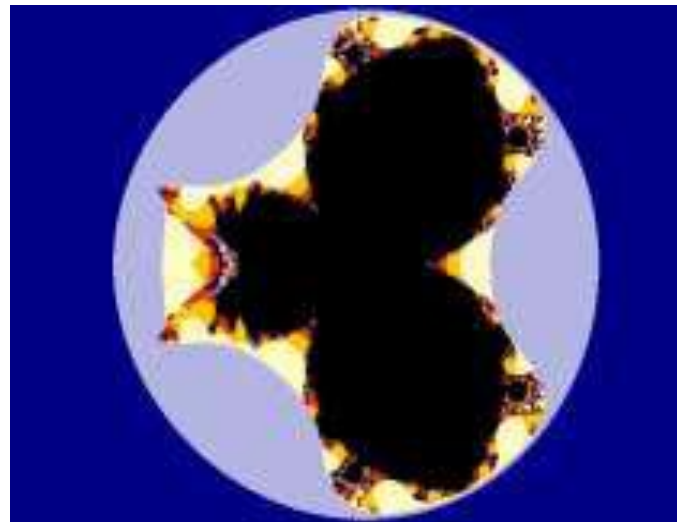
Fig.3: For bi-quadratic function: s=0.5, s'=0.4, n=4.2
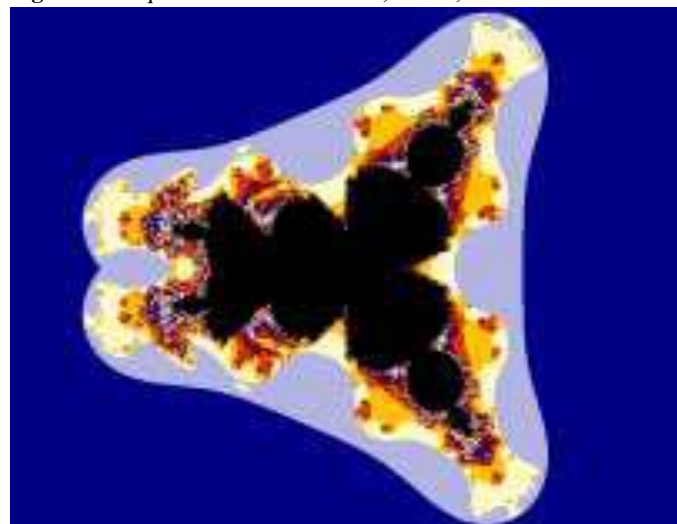
Fig.4: For bi-quadratic function: s=0.5, s'=0.4, n=4.8
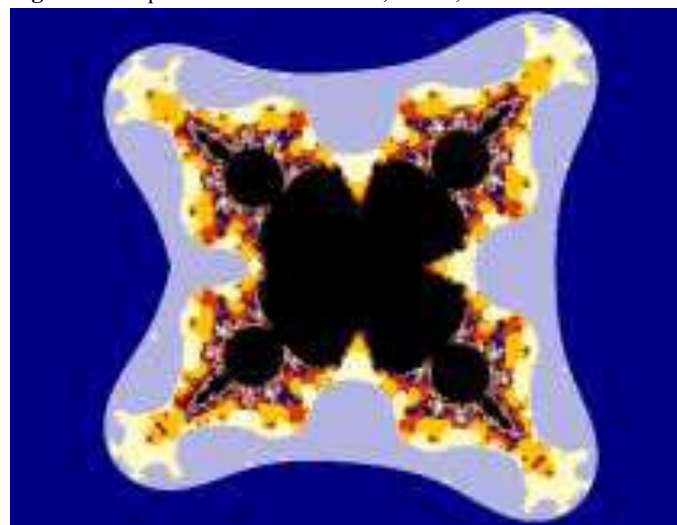
Generalization of Relative Superior Mandelbrot Set

Fig.5: s=0.4, s'=0.3, n=15.8

Fig.6: s=0.5, s'=0.4, n=16.8

Generation of Relative Superior Julia Sets:

(for cubic function)

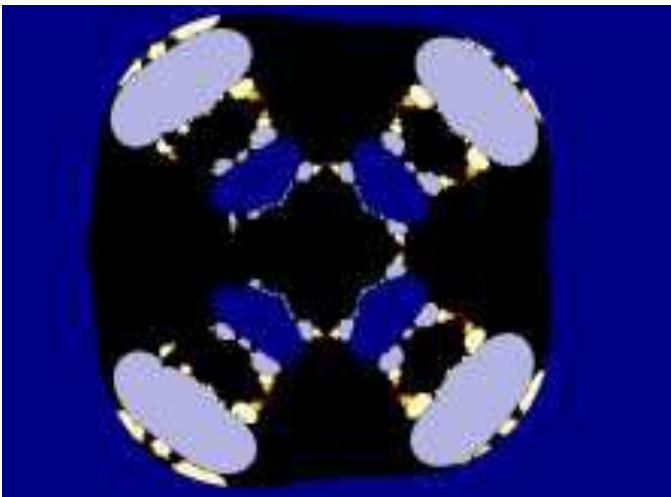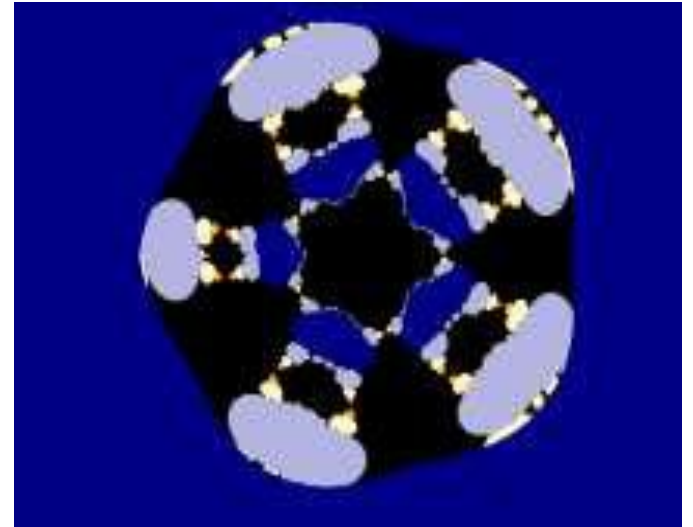**Fig.7:** s=0.5, s'=0.4, n=3.2
c=0.03488180321-0.01680604855i

Fig.8: s=0.5, s'=0.4, n=3.8
c=-0.0442117701+0.03592300032i

(for bi-quadratic function)

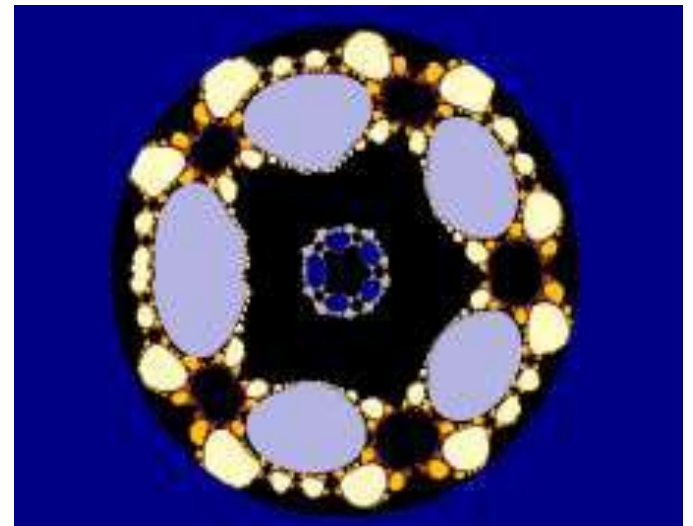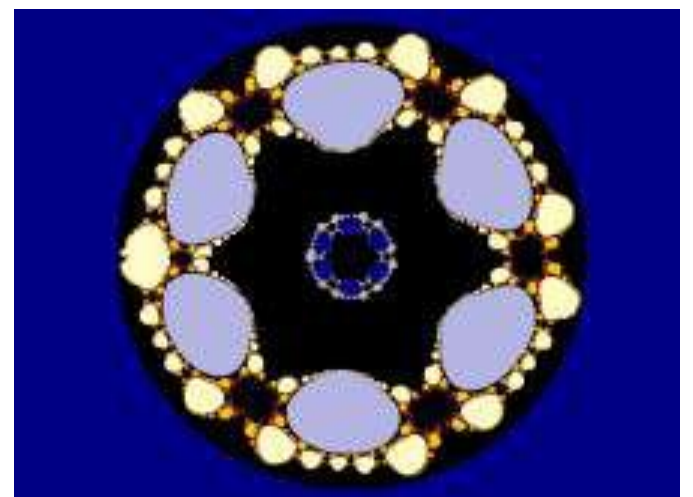**Fig.9:** s=0.8, s'=0.2, n=4.2
c=0.04015470789+0.03592299963i

Fig.10: s=0.8, s'=0.2, n=4.8
c=-0.01784724577+0.03065009474i

Generalization of Relative Superior Julia Sets:

Fig.11: s=0.5, s'=0.4, n=15.8
c=-0.001391094496+0.005718433219i
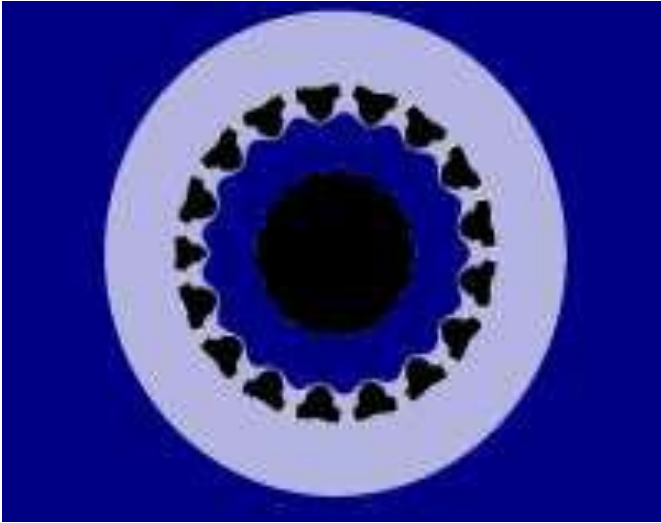
**Fig.12**: s=0.5, s'=0.4, n=16.8
c=-0.06389131715+0.02134348888i
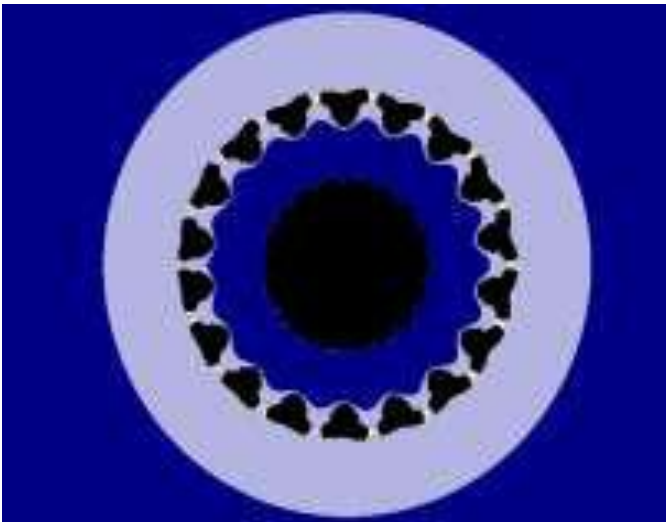
Fixed points :

Fixed points of Quadratic Polynomial [13] :

Table 1: Orbit of F(z) at s=0.5 and s'=0.1
for (z0=-0.01192288639+0.01042379668i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.015837	6	0.85943
2	0.98458	7	0.85943
3	0.86429	8	0.85942
4	0.85883	9	0.85942
5	0.85933	10	0.85942

**Fig.13:** Observation : the value converges to a fixed point after 08 iterations
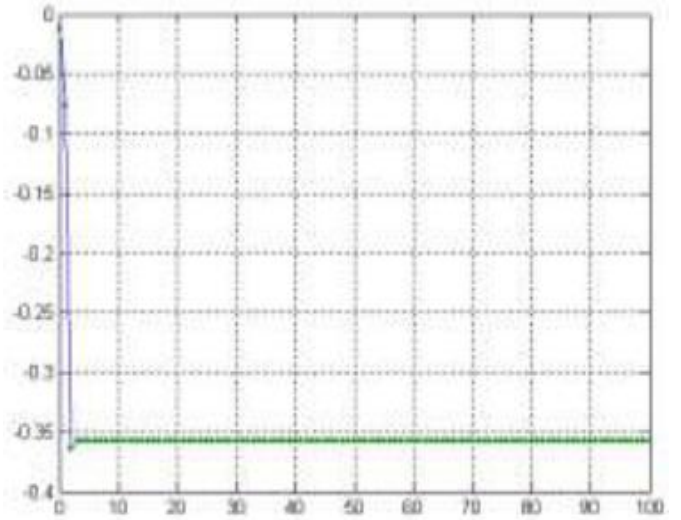
Fixed points of Cubic polynomial [13]

Table 2 : Orbit of F(z) at s=0.5 and s'=0.1
for (z0=-0.00888346751+0.01650347336i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.018742	6	0.86749
2	0.97928	7	0.86747
3	0.85738	8	0.86747
4	0.86871	9	0.86747
5	0.86732	10	0.86747

Fig.14: Observation : the value converges to a fixed point after 07 iterations
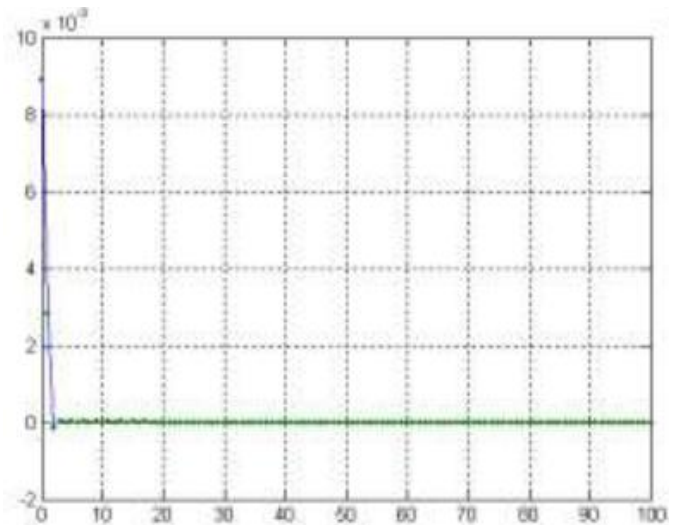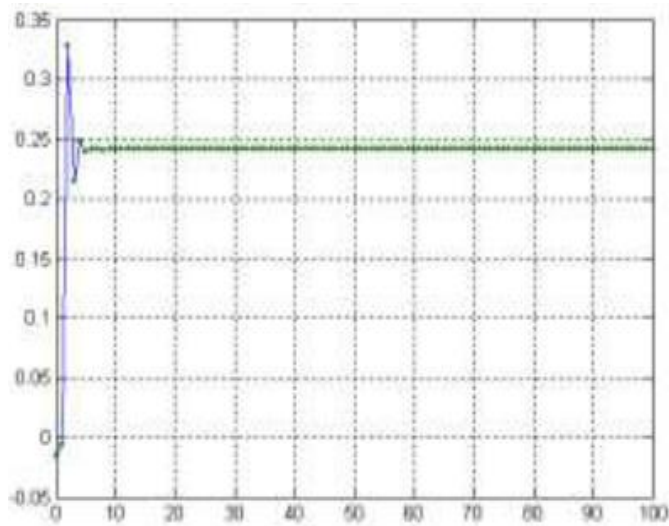
Fixed points of Bi-quadratic polynomial [13]

Table 3 : Orbit of F(z) at s=0.5 and s'=0.1
for (z0=-0.01573769494+0.03678871897i)

Number of iteration i	$|F(z)|$	Number of iteration i	$|F(z)|$
1	0.040014	8	0.8968
2	0.98215	9	0.89704
3	0.90556	10	0.89699
4	0.88426	11	0.89699
5	0.90308	12	0.89699
6	0.89476	13	0.89699
7	0.8977	14	0.89699

Fig.15: Observation : the value converges to a fixed point
after 10 iterations

Generation of Relative Superior Mandelbrot Set:

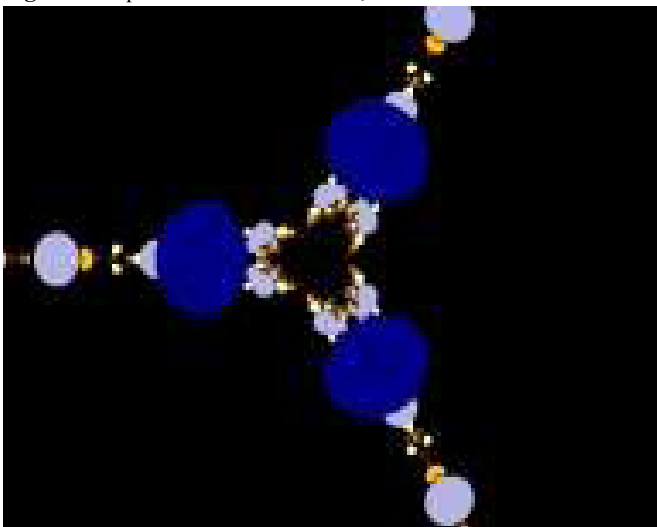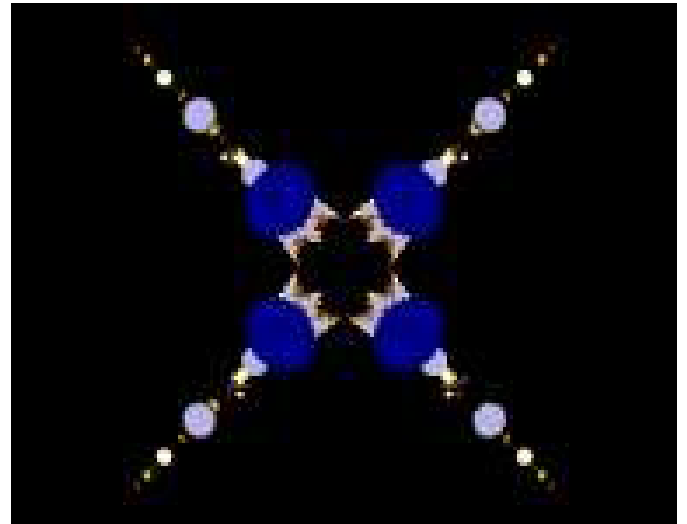**Fig.16:** For quadratic function: s=0.8, s'=0.3

Fig.17: For cubic function: s=0.8, s'=0.3

Generation of Relative Superior Julia Sets:

Fig.18: For quadratic function: s=0.5, s'=0.4
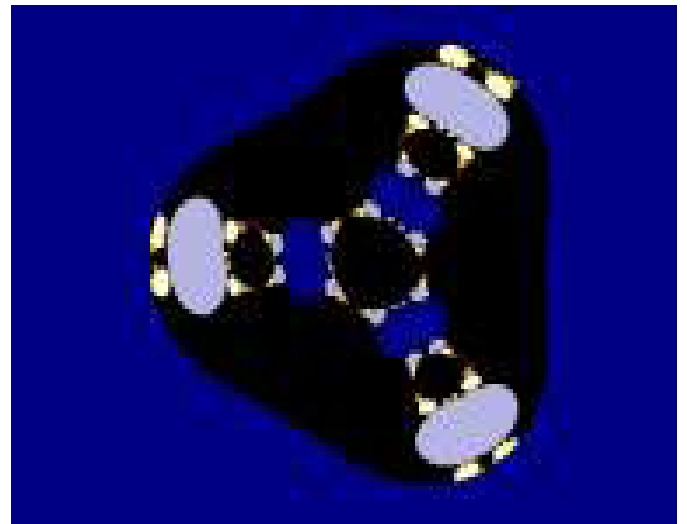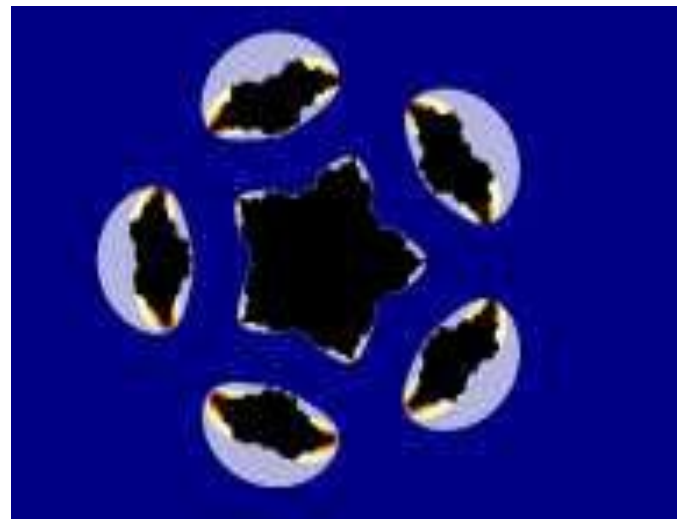c=0.002169194079+0.465750756i

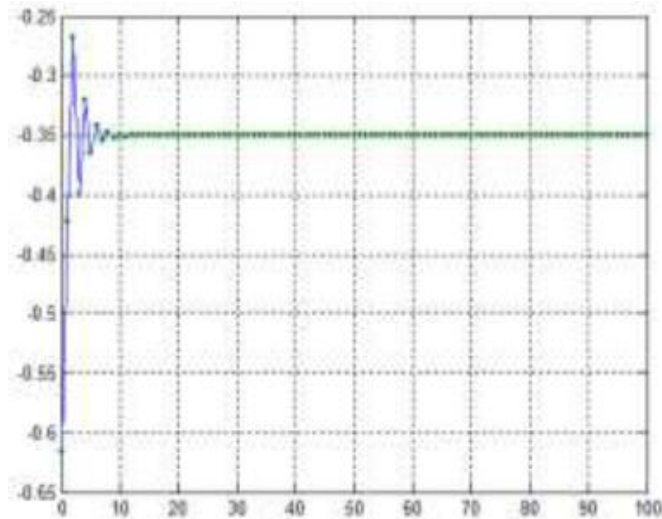**Fig.19:** For bi-quadratic function: s=0.8, s'=0.3,
c=-0.0227144337+0.04376545773i

Fixed points :

Fixed points for quadratic polynomial [2]:

Table 4 : Orbit of F(z) at s=0.5 and s'=0.7
for (z0=-0.6160374839+0.0135629073i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.61619	14	0.35866
2	0.5189	15	0.35835
3	0.288	16	0.35852
4	0.43079	17	0.35842
5	0.32218	18	0.35848
6	0.37886	19	0.35845
7	0.34703	20	0.35846
8	0.36492	21	0.35845
9	0.35484	22	0.35846
10	0.36049	23	0.35846
11	0.35732	24	0.35846
12	0.3591	25	0.35846
13	0.3581	26	0.35846

21	0.4871	31	0.50236
22	0.51134	32	0.50245
23	0.49733	33	0.5024
24	0.50536	34	0.50243
25	0.50073	35	0.50241
26	0.50339	36	0.50242
27	0.50186	37	0.50242

Fig.21: Observation : we skipped 17 iteration and the value converges to a fixed point after 36 iterations

Generation of Relative Superior Mandel-bar Set:

Fig.22: For quadratic function: s=0.6, s'=0.2

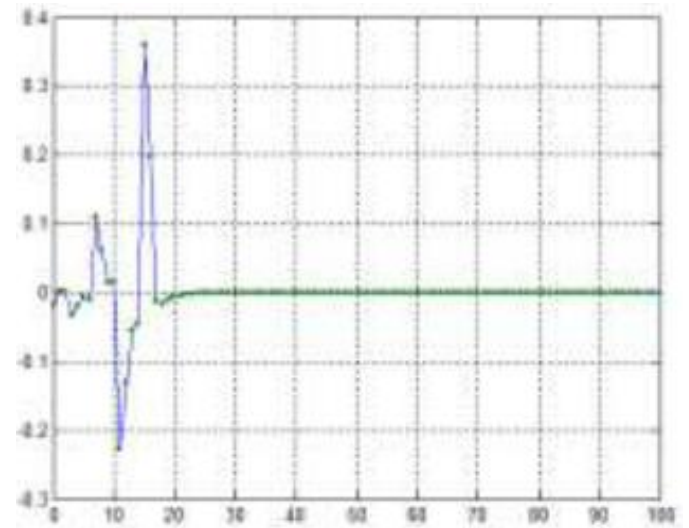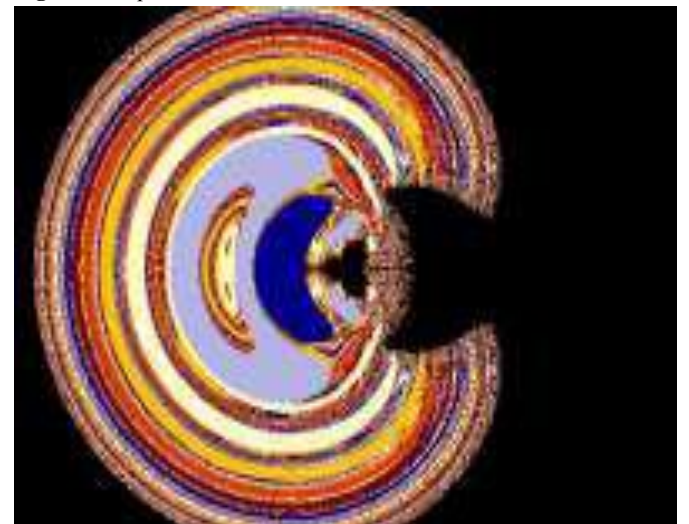**Fig.20:** Observation : the value converges to a fixed point after 22 iterations

Fixed points for cubic polynomial [2]:

Table 5: Orbit of F(z) at s=0.4 and s'=0.2
for (z0=-0.0189704705+0.028867852789i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
18	0.6098	28	0.50274
19	0.45643	29	0.50223
20	0.52977	30	0.50252

Fig.23: For cubic function: s=0.4, s'=0.2

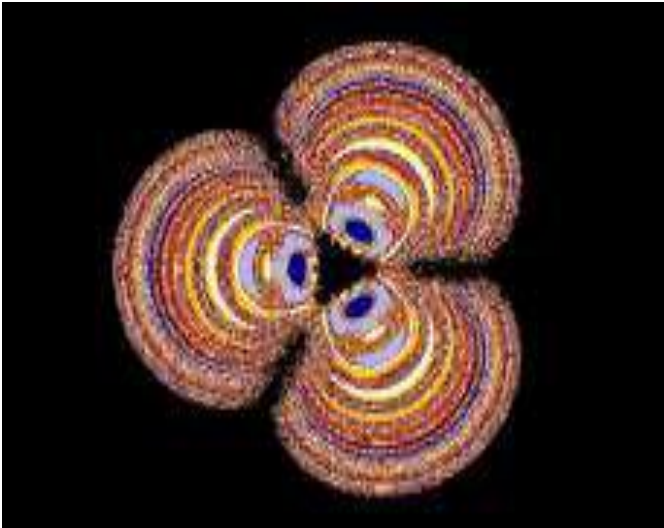**Fig.24:** For bi-quadratic function: s=0.6, s'=0.2

Fig.25: Generalization of RSMB : s=0.5, s'=0.2, n=19

Generation of Relative Superior Julia Sets for Mandel-bar set:

Fig.26: For quadratic function: s=0.6, s'=0.2, c=0.08166620257+0.00739899807i
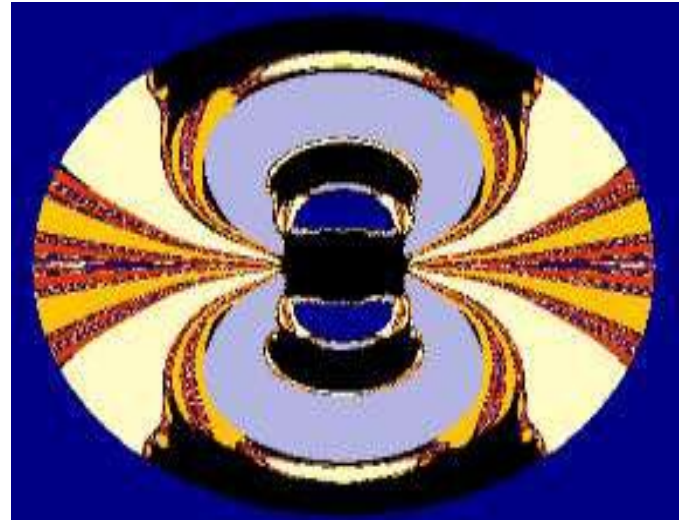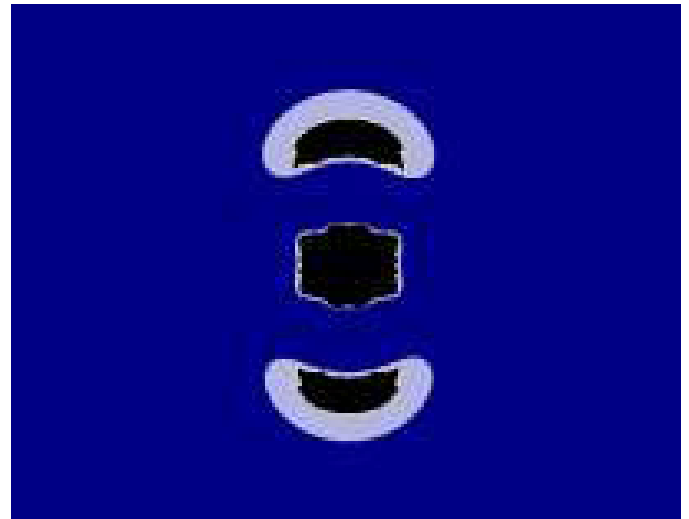

Fig.27: For cubic function: s=0.8, s'=0.3, c=-0.003854849909+0.01666833389i

6. CONCLUSION

- The z plane fractal image for the function $z \rightarrow z^2 + c$, $n \geq 2$ showed that the stable region is bounded by unstable region. Besides this, the non integer value change brought the embryonic structure in the form of lobe.

- On the other hand the c plane geometrical analysis of the function $z \rightarrow (z^n + c)^{-1}$, $n \geq 2$ represented the planetary type structure comprising of central planet with satellites. Here non integer value change showed the embryonic self similar growth in satellite pattern.

- Relative Superior Mandelbrot of inverse function showed lace like structure with multi-coloured small circles. Geometrical analysis of Relative Superior Julia sets of inverse function shows that the boundary of fixed point region forms a (n+1) hypocycloid.

- The geometry of Relative Superior Mandelbrot set and Relative Superior Julia sets of inverse function showed their rotational as well as reflection symmetry. One of the most fascinating results is the central planet with satellite like structure obtained for biquadratic Relative Superior Julia sets.

- Further, for the odd value of n, all the Relative Superior Mandelbar sets are symmetrical objects, and for the even values of n, all the relative superior Mandelbar sets are symmetrical about x-axis. Besides this, our antifractals are different from the normal Tricorns and Multicorns as they have (n-1) wings.

7. ACKNOWLEDGMENTS

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