# On the Weak Stability of Picard Iteration for Some Contractive Type Mappings and Coincidence Theorems

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## ABSTRACT
Starting from a weak concept of stability, introduced by Berinde [1] and called "weak stability", in [27] we develop a weaker notion, named "w²-stability". Therefore, in this paper we prove some results of this weaker stability concept for certain class of mappings and also we give some examples of w²-stable but not weak stable nor stable iterations.

Because of the restriction of an "approximate" sequence, some fixed point iteration procedures are not weakly stable so if it is used a weaker type of sequence, the stability can be obtained in the meaning of a new concept.

## General Terms
Fixed-point and coincidence theorems.

## Keywords
Coincidence point, fixed point, stable iteration, weak stable iteration

## 1. INTRODUCTION
For the complete metric space *(X, d)*, with $x, y \in X$ and $x \neq y$, Harder [7], [8] presented some mappings $T: X \to X$ satisfying various contraction conditions for which the associated Picard iteration is not stable.

Their corresponding conditions in the case of two mappings *S*, $T: X \to X$ such that $T(X) \subseteq S(X)$, with $x, y \in X$ and $x \neq y$, are in the following form:

(1) $d(Tx,Ty)<\max\{d(Sx,Tx),d(Sy,Ty)\}$
(2) $d(Tx,Ty)<\max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy)\}$
(3) $d(Tx,Ty)<\max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy), d(Sx,Ty), d(Sy,Tx)\}$
(4) $d(Tx,Ty)<\max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy), \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\}$

Berinde [1] introduced the notion of weak stability and in [28] there is a study of weak stability of iterative procedures for some coincidence theorems. Moreover, Timis [27] introduced a weaker concept, called w²-stability, and gave weak stability results of Picard iteration for various contractive mappings defined by Harder [7], [8].

In this paper, we give w²-stability results of Picard iteration for mappings with a coincidence point satisfying conditions (1)-(4).

## 2. WEAK STABILITY OF FIXED POINT ITERATION PROCEDURES
One of the most general contractive definition for which corresponding stability results have been obtained in the case of Kirk, Mann and Ishikawa iteration procedures in arbitrary Banach spaces appears to be the following class of mappings: for *(X, d)* a metric space, $T: X \to X$ is supposed to satisfy the condition

$$d(Tx,Ty) \leq ad(x,y)+Ld(x,Tx),$$

for some $a \in [0,1)$, $L \geq 0$ and for all $x, y \in D \subseteq X$. This condition appears in [15] and other related results may be found in [14], [19], [20].

The concept of stability is not very precise because of the sequence $\{y_n\}_{n=0}^{\infty}$ which is arbitrary taken. So, it would be more natural that $\{y_n\}$ to be an approximate sequence of $\{x_n\}$ and Berinde [1] introduced the notion named "weak stability". Therefore, any stable iteration will be also weakly stable but the reverse is not generally true.

Because some contractive conditions are very strictly and the associated fixed point iteration is not weakly stable, Timis [27] used equivalent sequences in order to introduce the notion of w²-stability.

In the following, we restate the definition of w²-stability for the case of two self-mappings with a coincidence point and using this, we establish stability results for common fixed point iterative procedures in the class of mappings that satisfy the above contraction conditions (1)-(4).

**Definition 2.1.** Let the map $S: X \to X$ and *(X, d)* to be a metric space. Two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are called S-equivalent sequences if $d(Sx_n, Sy_n) \to 0$, as $n \to \infty$.

**Definition 2.2.** Let *(X, d)* be a metric space, $S, T: X \to X$ be two maps such as $T(X) \subseteq S(X)$ and *z* is a coincidence point of *S* and *T*, that is a point for which we have $Sz = Tz = u \in X$. Let $\{Sx_n\}$ be an iteration procedure defined by $x_0 \in X$ and $Sx_{n+1} = f(T, x_n)$, $n \geq 0$.

Suppose that $\{Sx_n\}$ converges to *u*. If for any equivalent sequence $\{Sy_n\} \in X$ of $\{Sx_n\}$,

$\lim_{n\to\infty} d(Sy_{n+1}, Ty_n) = 0$, implies that $\lim_{n\to\infty} Sy_n = u$,

then we shall say that the iteration procedure is w²-stable with respect to *(S,T)*.

We mention that the concept of *(S,T)*-stability was used in [26] and the transposition to *(S,T)*-weak stability in a metric space was introduced in [28].

## 3. MAIN RESULTS

The basic results of this paper are the following theorems:

**Theorem 3.1.** Let *(X,d)* be a complete metric space and *S, T:* $X \to X$ be two maps such that $T(X) \subseteq S(X)$, satisfying (1), i.e., $d(Tx,Ty)<max\{d(Sx,Tx),d(Sy,Ty)\}$, for all $x, y \in X$ and $x \neq y$.

Let $\{Sx_n\}_{n=0}^{\infty}$ be an iterative procedure defined by $x_0 \in X$ and $Sx_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{Sx_n\}$ converges to *u*, where *u* is a coincidence point of *S* and *T*.

Then, the Picard iteration is w²-stable.

**Proof.** Consider $\{Sy_n\}_{n=0}^{\infty}$ to be an equivalent sequence of $\{Sx_n\}$. Then, according to Definition 2.2., if $\lim_{n\to\infty} d(Sy_{n+1}, Ty_n) = 0$ implies that $\lim_{n\to\infty} Sy_n = u$, then the Picard iteration is w²-stable.

In order to prove this, we suppose that $\lim_{n\to\infty} d(Sy_{n+1}, Ty_n) = 0$. Therefore, for all $\varepsilon>0$, there exists $n_0 = n(\varepsilon)$ such that $d(Sy_{n+1}, Ty_n) < \varepsilon$, for all $n \geq n_0$.

So, $d(Sy_{n+1}, u) \leq d(Sy_{n+1}, Sx_{n+1}) + d(Sx_{n+1}, u) \leq d(Sy_{n+1}, Ty_n) + d(Ty_n, Tx_n) + d(Sx_{n+1}, u) < d(Sy_{n+1}, Ty_n) + max\{d(Sx_n, Tx_n), d(Sy_n, Ty_n)\} + d(Sx_{n+1}, u)$.

From the hypothesis, by $Sx \to u$, we have that $d(Sx_n, Tx_n) = d(Sx_n, Sx_{n+1}) \leq d(Sx_n, u) + d(u, Sx_{n+1}) \to 0$.

If $max\{d(Sx_n, Tx_n), d(Sy_n, Ty_n)\} = d(Sx_n, Tx_n)$, by taking to the limit, we obtain that $d(Sy_{n+1}, u) \to 0$.

If $max\{d(Sx_n, Tx_n), d(Sy_n, Ty_n)\} = d(Sy_n, Ty_n)$, we have that $d(Sy_n, Ty_n) \leq d(Sy_n, Sx_n) + d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1}) + d(Sy_{n+1}, Ty_n)$.

From Definition 2.1., $d(Sy_n, Sx_n) \to 0$ and by taking to the limit, we obtain that $d(Sy_{n+1}, u) \to 0$.

This shows that the Picard iteration is w²-stable with respect to *(S,T)*.

**Theorem 3.2.** Let *(X,d)* be a complete metric space and *S, T:* $X \to X$ such that $T(X) \subseteq S(X)$, satisfying (2), i.e., $d(Tx,Ty)<max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy)\}$, for all $x, y \in X$ and $x \neq y$.

Let $\{Sx_n\}_{n=0}^{\infty}$ be an iterative procedure defined by $x_0 \in X$ and $Sx_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{Sx_n\}$ converges to *u*, where *u* is a coincidence point of *S* and *T*.

Then, the Picard iteration is w²-stable.

**Proof.** We follow the same assumptions as in Theorem 3.1., by taking $\{Sy_n\}_{n=0}^{\infty}$ to be an equivalent sequence of $\{Sx_n\}$. By Definition 2.2, if $\lim_{n\to\infty} d(Sy_{n+1}, Ty_n) = 0$ implies that $\lim_{n\to\infty} Sy_n = u$, then the Picard iteration is w²-stable.

Theorem 3.1. shows this result if we consider $max\{d(Sx,Tx),d(Sy,Ty)\}$. In this case, there is a new situation, when *max* could be $d(Sx,Sy)$.

Therefore, following the same steps, we get that $max\{d(Sx_n,Tx_n),d(Sy_n,Ty_n),d(Sx_n,Sy_n)\}= d(Sx_n,Sy_n)$. From Definition 2.1., we have that $d(Sy_n, Sx_n) \to 0$ and by taking to the limit as it is shown in the above theorem, we obtain the conclusion.

**Theorem 3.3.** Let *(X,d)* be a complete metric space and *S, T:* $X \to X$ such that $T(X) \subseteq S(X)$, satisfying (3), i.e., $d(Tx,Ty)<max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy), d(Sx,Ty), d(Sy,Tx)\}$, for all $x, y \in X$ and $x \neq y$.

Let $\{Sx_n\}_{n=0}^{\infty}$ be an iterative procedure defined by $x_0 \in X$ and $Sx_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{Sx_n\}$ converges to *u*, where *u* is a coincidence point of *S* and *T*.

Then, the Picard iteration is w²-stable.

**Proof.** We follow the same assumptions as in Theorem 3.2., where is shown this result if we consider $max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy)\}$. In this case, there are new situations, when *max* could be $d(Sx,Ty)$ or $d(Sy,Tx)$. Again, we follow the same steps.

If *max* is $d(Sx_n, Ty_n)$, we have that $d(Sx_n, Ty_n) \leq d(Sx_n, Sy_n) + d(Sy_n, Ty_n)$. From Definition 2.1.,\ref{Def_S_ech}, $d(Sy_n, Sx_n) \to 0$ and the expression of $d(Sy_n, Ty_n)$ was treated in Theorem 3.1.

On the other hand, if *max* is $d(Sy_n, Tx_n)$, then $d(Sy_n, Tx_n) \leq d(Sy_n, Sx_n) + d(Sx_n, Tx_n)$.

By taking to the limit in a same way as in above theorems, we obtain the conclusion.

**Theorem 3.4.** Let *(X,d)* be a complete metric space and *S, T:* $X \to X$ such that $T(X) \subseteq S(X)$, satisfying (4), i.e.,

$$d(Tx,Ty)<max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy), \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\}$$

for all $x, y \in X$ and $x \neq y$.

Let $\{Sx_n\}_{n=0}^{\infty}$ be an iterative procedure defined by $x_0 \in X$ and $Sx_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{Sx_n\}$ converges to *u*, where *u* is a coincidence point of *S* and *T*.

Then, the Picard iteration is w²-stable.

**Proof.** We follow the same assumptions as in Theorem 3.3., where is shown this result if we consider $max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Sy), d(Sx,Ty), d(Sy,Tx)\}$.

In this case, this is a new situation, when *max* could be $\frac{d(Sx,Ty)+d(Sy,Tx)}{2}$. Then, following the same steps as in Theorem 3.3., we obtain that $d(Sx_n, Ty_n) \to 0$ and $d(Sy_n, Tx_n) \to 0$, so, by taking to the limit in the whole expression, we get the result.

From Theorem 3.3., we obtain the following stability result.

**Corollary 3.5.** Let *(X,d)* be a complete metric space and *S, T:* $X \to X$ two mappings such that $T(X) \subseteq S(X)$, satisfying

(5) $d(Tx,Ty)<max\{ d(Sx,Ty), d(Sy,Tx)\}$,

for all $x, y \in X$ and $x \neq y$.

Let $\{Sx_n\}_{n=0}^{\infty}$ be an iterative procedure defined by $x_0 \in X$ and $Sx_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{Sx_n\}$ converges to *u*, where *u* is a coincidence point of *S* and *T*.

Then, the Picard iteration is w²-stable.

## 4. EXAMPLES

In the following, we present some examples of mappings that satisfy contraction conditions and for which the associated Picard iteration is not *(S,T)*-stable, it is not *(S,T)*-weakly stable but it is *(S,T)*- w²-stable.

**Example 4.1.** Let *S*, *T: [0,1] → [0,1]* be given by

$Tx=\begin{cases} 0, & x \in \left[0,\frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2},1\right] \end{cases}$ and $Sx=\begin{cases} \frac{1}{2}-x, & x \in \left[0,\frac{1}{2}\right] \\ x-\frac{1}{4}, & x \in \left(\frac{1}{2},1\right] \end{cases}$, where

*[0,1]* is endowed with the usual metric. *S* and *T* are continuous at every point of *[0,1]* except at $\frac{1}{2}$, which is their coincidence point, i.e., $T\left(\frac{1}{2}\right)=S\left(\frac{1}{2}\right)=0=u$ and $T([0,1])=\left\{0,\frac{1}{2}\right\} \subseteq S([0,1])=\left[0,\frac{1}{2}\right] \cup \left(\frac{1}{4},\frac{3}{4}\right]=\left[0,\frac{3}{4}\right]$.

For each *x*, *y* ∈ *[0,1]*, *x ≠ y*, *T* and *S* satisfy the condition (5), i.e., *d(Tx,Ty)<max{ d(Sx,Ty), d(Sy,Tx)}*.

Indeed, first let $x \in \left[0,\frac{1}{2}\right]$, $y \in \left[0,\frac{1}{2}\right]$ and *x ≠ y*.

Then, $|Tx-Ty|=0<\max\left\{\left|\frac{1}{2}-x-0\right|,\left|\frac{1}{2}-x-0\right|\right\}$.

If $x \in \left(\frac{1}{2},1\right]$, $y \in \left(\frac{1}{2},1\right]$ and *x ≠ y*, then

$|Tx-Ty|=0<\max\left\{\left|x-\frac{1}{4}-\frac{1}{2}\right|,\left|y-\frac{1}{4}-\frac{1}{2}\right|\right\}$.

If $x \in \left[0,\frac{1}{2}\right]$ and $y \in \left(\frac{1}{2},1\right]$, then

$|Tx-Ty|=\left|0-\frac{1}{2}\right|=\frac{1}{2}<\max\left\{\left|\frac{1}{2}-x-\frac{1}{2}\right|,\left|y-\frac{1}{4}-0\right|\right\}=\max\left\{|x|,\left|y-\frac{1}{4}\right|\right\}$.

We will show that the Picard iteration is not *(S,T)*-stable, it is not *(S,T)*-weakly stable but it is *(S,T)*- w²-stable.

In order to prove the first claim, let *(*$Sy_n$*)*, with $Sy_n=\frac{n+2}{2n}$, $n \geq 1$. Then $\varepsilon_n=|Sy_{n+1}-Ty_n|=\left|\frac{n+3}{2(n+1)}-\frac{1}{4}-\frac{1}{2}\right|$, because $Sy_n>\frac{1}{2}$, for *n≥1*.

According to *(S,T)*-stability definition of [26], assuming that $\lim_{n\to\infty}\varepsilon_n=0$, we should obtain that $\lim_{n\to\infty} Sy_n=\frac{3}{4}$, but in fact, $\lim_{n\to\infty} Sy_n=\frac{1}{2}$, so the Picard iteration is not *(S,T)*-stable.

In order to study the *(S,T)*-weak stability, from the *(S,T)*-weak stability definition of [28], we take an approximate sequence $\{Sy_n\}$ of $(Sx_n)$. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$, for *n→∞*, such that $|Sx_n-Sy_n| \leq \eta_n,\ n \geq k$.

Then, $-\eta_n \leq Sx_n-Sy_n \leq \eta_n$ and it results that $0 \leq Sy_n \leq Sx_n+\eta_n,\ n \geq k$.

If $x_0 \in \left[0,\frac{1}{2}\right]$, $Sx_1=Tx_0=0$, therefore $Sx_n=0$, *n≥1*. On the other hand, if $x_0 \in \left(\frac{1}{2},1\right]$, then $Sx_1=Tx_0=\frac{1}{2}$ and $Sx_2=Tx_1=0$, so $Sx_n=0$, *n≥2*.

If $x_n \in \left[0,\frac{1}{2}\right]$, then $Sx_n=\frac{1}{2}-x_n$. So, $0 \leq x_n \leq \frac{1}{2} \Leftrightarrow 0 \geq -x_n \geq -\frac{1}{2} \Leftrightarrow \frac{1}{2} \geq \frac{1}{2}-x_n \geq 0 \Leftrightarrow 0 \leq \frac{1}{2}-x_n=Sx_n \leq \frac{1}{2}$. Hence, in this situation, $Sx_n$ can have the value of *0*.

If $x_n \in \left(\frac{1}{2},1\right]$, then $Sx_n=x_n-\frac{1}{4}$. So, $\frac{1}{2}<x_n \leq 1 \Leftrightarrow \frac{1}{4}<x_n-\frac{1}{4}=Sx_n \leq \frac{3}{4}$. In this case, $Sx_n$ can not be *0*.

Therefore, $x_n \in \left[0,\frac{1}{2}\right]$ and then, $Tx_n=0$.

Since $Sx_n=0$, for *n≥2*, we obtain that $0 \leq Sy_n \leq \eta_n,\ n \geq k_1=\max\{2,k\}$. We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{2}$, n≥ $k_1$ and therefore $0 \leq Sy_n \leq \frac{1}{2}, \forall\, n \geq k_1$.

If $y_n \in \left[0,\frac{1}{2}\right]$, then $Sy_n=\frac{1}{2}-y_n$, so, $0 \leq y_n \leq \frac{1}{2} \Leftrightarrow 0 \geq -y_n \geq -\frac{1}{2} \Leftrightarrow -\frac{1}{2} \leq -y_n \leq 0 \Leftrightarrow 0 \leq \frac{1}{2}-y_n=Sy_n \leq \frac{1}{2}$, situation that can be possible. In this case, we have that $Ty_n=0$.

If $y_n \in \left(\frac{1}{4},\frac{3}{4}\right] \cap \left(\frac{1}{2},1\right]=\left(\frac{1}{2},\frac{3}{4}\right]$, then $Sy_n=y_n-\frac{1}{4}$, so, $\frac{1}{2}<y_n \leq \frac{3}{4} \Leftrightarrow \frac{1}{4}<y_n-\frac{1}{4}=Sy_n \leq \frac{1}{2}$,

and this can be possible, too. Hence, for $y_n \in \left(\frac{1}{2},\frac{3}{4}\right]$, we have that $Ty_n=\frac{1}{2}$.

If $d(Sy_{n+1},Ty_n) \to 0$ implies that $d(Sy_n,u) \to 0$, for $n \to \infty$, the *(S,T)*-weak stability should be obtained.

If $y_n \in \left[0,\frac{1}{2}\right]$, then from $d(Sy_{n+1},Ty_n)=d(Sy_{n+1},0) \to 0$, we obtain that $Sy_{n+1} \to 0$ but if $y_n \in \left(\frac{1}{2},\frac{3}{4}\right]$, then from $d(Sy_{n+1},Ty_n)=d\left(Sy_{n+1},\frac{1}{2}\right) \to 0$, we obtain that $Sy_{n+1} \to \frac{1}{2}$, so $Sy_n \to \frac{1}{2}$. Therefore, the Picard iteration is not *(S,T)*-weakly stable.

In order to study the w²-stability with respect to *(S,T)*, from Definition 2.2., we should have that $\lim_{n\to\infty} d(Sy_{n+1},Ty_n)=0$, implies that $\lim_{n\to\infty} Sy_n=u$.

Let an equivalent sequence *{*$Sy_n$*}* of $Sx_n$ and by Definition 2.1., *d(*$Sx_n$*,* $Sy_n$*) → 0,* as *n → ∞*.

So, *d(*$Sy_n$*, u)≤ d(*$Sy_n$*,* $Sx_n$*)+ d(*$Sx_n$*, u)= d(*$Sy_n$*,* $Sx_n$*) → 0* and this proves the w²-stability with respect to *(S,T)*.

**Example 4.2.** Let *S*, *T: [0,1] → [0,1]* be given by

$Tx=\begin{cases} \frac{x+1}{2}, & x \in \left[0,\frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2},1\right] \end{cases}$ and $Sx=\begin{cases} \frac{1}{2}-x, & x \in \left[0,\frac{1}{2}\right] \\ x-\frac{1}{4}, & x \in \left(\frac{1}{2},1\right] \end{cases}$, where

*[0,1]* is endowed with the usual metric.

*S* and *T* have two coincidence points, i.e., $T(0)=S(0)=T\left(\frac{3}{4}\right)=S\left(\frac{3}{4}\right)=\frac{1}{2}=u$ and $T([0,1])=\left[\frac{1}{2},\frac{\frac{1}{2}+1}{2}\right] \cup \left\{\frac{1}{2}\right\}=\left[\frac{1}{2},\frac{3}{4}\right] \subseteq S([0,1])=\left[0,\frac{1}{2}\right] \cup \left(\frac{1}{4},\frac{3}{4}\right]=\left[0,\frac{3}{4}\right]$.

For each *x*, *y* ∈ *[0,1]*, *x ≠ y*, *T* and *S* satisfy the condition (5), i.e., *d(Tx,Ty)<max{ d(Sx,Ty), d(Sy,Tx)}*.

Indeed, first let $x \in \left[0,\frac{1}{2}\right]$, $y \in \left[0,\frac{1}{2}\right]$ and *x ≠ y*.

Then, $|Tx-Ty|=\left|\frac{x}{2}+\frac{1}{2}-\frac{y}{2}-\frac{1}{2}\right|=\frac{1}{2}|x-y|=\left[0,\frac{1}{4}\right]<\max\left\{\left|\frac{1}{2}-x-\frac{y}{2}-\frac{1}{2}\right|,\left|\frac{1}{2}-y-\frac{x}{2}-\frac{1}{2}\right|\right\}=\max\left\{\left|x+\frac{y}{2}\right|,\left|y+\frac{x}{2}\right|\right\}=\max\left\{\left[0,\frac{1}{2}\right]+\left[0,\frac{1}{4}\right],\left[0,\frac{1}{2}\right]+\left[0,\frac{1}{4}\right]\right\}=\max\left\{\left[0,\frac{3}{4}\right],\left[0,\frac{3}{4}\right]\right\}=\left[0,\frac{3}{4}\right]$.

If $x \in \left(\frac{1}{2},1\right]$, $y \in \left(\frac{1}{2},1\right]$ and *x ≠ y*, then

$|Tx-Ty|=0<\max\left\{\left|x-\frac{1}{4}-\frac{1}{2}\right|,\left|y-\frac{1}{4}-\frac{1}{2}\right|\right\}=\max\left\{\left|x-\frac{3}{4}\right|,\left|y-\frac{3}{4}\right|\right\}=\left[0,\frac{1}{4}\right]$.

If $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$, then

$|Tx - Ty| = \left|\frac{x}{2} + \frac{1}{2} - \frac{1}{2}\right| = \frac{1}{2}|x| = \left[0, \frac{1}{4}\right] < \max\left\{\left|\frac{1}{2} - x - \frac{1}{2}\right|, \left|y - \frac{1}{4} - \frac{x}{2} - \frac{1}{2}\right|\right\} = \max\left\{|x|, \left|y - \frac{x}{2} - \frac{3}{4}\right|\right\} = \max\left\{\left[0, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right] - \left[0, \frac{1}{4}\right] - \frac{3}{4} = \left[0, \frac{1}{4}\right]\right\} = \left[0, \frac{1}{2}\right].$

We will show that the Picard iteration is not *(S,T)*-stable, it is not *(S,T)*-weakly stable but it is *(S,T)*- w²-stable.

In order to prove the first claim, let *($Sy_n$)*, with $Sy_n = \frac{n+2}{2n}$, $n \geq 1$. Then $\varepsilon_n = |Sy_{n+1} - Ty_n| = \left|\frac{n+3}{2(n+1)} - \frac{1}{4} - \frac{1}{2}\right|$, because $Sy_n > \frac{1}{2}$, for *n≥1*.

According to *(S,T)*-stability definition of [26], assuming that $\lim_{n\to\infty} \varepsilon_n = 0$, we should obtain that $\lim_{n\to\infty} Sy_n = \frac{3}{4}$, but in fact, $\lim_{n\to\infty} Sy_n = \frac{1}{2}$, so the Picard iteration is not *(S,T)*-stable.

For the *(S,T)*-weak stability, from the *(S,T)*-weak stability definition of [28], for any $x_0 \in [0,1]$, the sequence $\{Sx_n\}$ generated by the iterative procedure $Sx_{n+1} = Tx_n,\ n > 0$, converges to $u = \frac{1}{2}$.

Indeed, if $x_0 \in \left[0, \frac{1}{2}\right]$, then $Sx_1 = Tx_0 = \frac{x_0+1}{2} \in \frac{\left[0,\frac{1}{2}\right]+1}{2} = \frac{\left[1,\frac{3}{2}\right]}{2} = \left[\frac{1}{2}, \frac{3}{4}\right]$. Now, if $x_1 \in \left[0, \frac{1}{2}\right]$, then $Sx_1 = \frac{1}{2} - x_1 \in \frac{1}{2} - \left[0, \frac{1}{2}\right] = \left[0, \frac{1}{2}\right]$. Only for $x_1 = 0$, we have that $Sx_2 = Tx_1 = \frac{1}{2}$, so, $Sx_n = Tx_n = \frac{1}{2}$, ∀ *n≥2*.

On the other hand, if $x_1 \in \left(\frac{1}{2}, 1\right]$, then $Sx_1 = x_1 - \frac{1}{4} \in \left(\frac{1}{2}, 1\right] - \frac{1}{4} = \left(\frac{1}{4}, \frac{3}{4}\right]$. Only for $x_1 \in \left(\frac{3}{4}, 1\right]$, we have that $Sx_1 \in \left(\frac{1}{2}, \frac{3}{4}\right]$. Hence, $Sx_2 = Tx_1 = \frac{1}{2}$, so, $Sx_n = Tx_n = \frac{1}{2}$, ∀ *n≥2*.

If $x_0 \in \left(\frac{1}{2}, 1\right]$, then $Sx_1 = Tx_0 = \frac{1}{2}$, so, $Sx_n = Tx_n = \frac{1}{2}$, ∀ *n≥1*.

We take an approximate sequence *{$Sy_n$}* of $Sx_n$. Then, there exists a decreasing sequence of nonnegative numbers *{$\eta_n$}* converging to some $\eta \geq 0$, for $n\to\infty$, such that $|Sx_n - Sy_n| \leq \eta_n,\ n \geq k$.

Then, $-\eta_n \leq Sx_n - Sy_n \leq \eta_n$ and it results that $0 \leq Sy_n \leq Sx_n + \eta_n,\ n \geq k$.

Since $Sx_n = \frac{1}{2}$, for *n≥2*, we obtain that $0 \leq Sy_n \leq \frac{1}{2} + \eta_n$, $n \geq k_1 = \max\{2, k\}$. We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{4}$, $n \geq k_1$ and therefore $0 \leq Sy_n \leq \frac{3}{4}$, $\forall\, n \geq k_1$.

If $d(Sy_{n+1}, Ty_n) \to 0$ implies that $d(Sy_n, u) \to 0$, for $n \to \infty$, the *(S,T)*-weak stability should be obtained.

If $y_n \in \left(\frac{1}{2}, 1\right] \cap \left(\frac{1}{4}, \frac{3}{4}\right] = \left(\frac{1}{2}, \frac{3}{4}\right]$, then $Sy_n = y_n - \frac{1}{4} \in \left(\frac{1}{2}, \frac{3}{4}\right] - \frac{1}{4} = \left(\frac{1}{4}, \frac{1}{2}\right] \in \left[0, \frac{3}{4}\right]$ and $Ty_n = \frac{1}{2}$. From $d(Sy_{n+1}, Ty_n) = d\left(Sy_{n+1}, \frac{1}{2}\right) \to 0$, we obtain that $Sy_{n+1} \to \frac{1}{2}$, so $Sy_n \to \frac{1}{2} = u$, but if $y_n \in \left[0, \frac{1}{2}\right]$, then $Sy_n = \frac{1}{2} - y_n = \frac{1}{2} - \left[0, \frac{1}{2}\right] = \left[0, \frac{1}{2}\right] \in \left[0, \frac{3}{4}\right]$ and $Ty_n = \frac{y_n+1}{2} \in \frac{1}{2}\left[1, \frac{3}{2}\right] = \left[\frac{1}{2}, \frac{3}{4}\right]$.

Therefore, $d(Sy_{n+1}, Ty_n) = \left|\left[0, \frac{1}{2}\right] - \left[\frac{1}{2}, \frac{3}{4}\right]\right| = \left[\frac{1}{4}, \frac{1}{2}\right]$ and then $\lim_{n\to\infty} d(Sy_{n+1}, Ty_n)$ can not be 0. Therefore, the Picard iteration is not *(S,T)*-weakly stable.

In order to study the w²-stability with respect to *(S,T)*, from Definition 2.2., we should have that $\lim_{n\to\infty} d(Sy_{n+1}, Ty_n) = 0$, implies that $\lim_{n\to\infty} Sy_n = u$.

Let an equivalent sequence *{$Sy_n$}* of $Sx_n$ and by Definition 2.1., *$d(Sx_n, Sy_n) \to 0$*, as *$n \to \infty$*.

So, *$d(Sy_n, u) \leq d(Sy_n, Sx_n) + d(Sx_n, u) = d(Sy_n, Sx_n) \to 0$* and this proves the w²-stability with respect to *(S,T)*.

## 5. CONCLUSIONS

The concept of stability is slightly not very precise because of the sequence $\{Sy_n\}_{n=0}^{\infty}$ which is arbitrary taken. From a numerical point of view, $\{Sy_n\}_{n=0}^{\infty}$ must be an approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$.

By adopting a concept of such kind of approximate sequences, Berinde [1] introduced a weaker and more natural concept of stability, called weak stability. So, any stable iteration will be also weakly stable but the reverse is not generally true.

But using an approximate sequence in the definition of weak stability, some fixed point iteration procedures are not weakly stable but if it is used a weaker type of sequence, the stability can be obtained in the meaning of a new concept, named w²-stability.

Therefore, here we proved that for the class of mappings which satisfy some contraction conditions presented by Harder and Hicks [8] and for the class of mappings with a coincidence point, the associated Picard iterations are w²-stable.

We also gave some illustrative examples of mappings that satisfy contraction conditions and for which the associated Picard iteration is not *(S,T)*-stable, it is not *(S,T)*-weakly stable but it is *(S,T)*- w²-stable.

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