Separation Axioms in Bitopological Spaces

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ABSTRACT
In this paper, we introduced a new type of separation axiom in bitopological spaces called quasi $T_{1/2^*}$ space in terms of the concept of quasi open sets and quasi kernel and investigate some of their fundamental properties. Also we introduced and studied some new notions in bitopological spaces by utilizing quasi open sets.

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Quasi open sets, quasi $T_{1/2^*}$ space, quasi T_D space, weakly quasi separated sets.

1. INTRODUCTION AND PRELIMLNARIES
The concept of a bitopological space was first introduced by Kelly [1]. Further, Mohammed S.Sarsak [2] studied some separation axioms, namely quasi T_i axioms where i $\in$ {0,1/2,1,2}. A bitopological space (X, τ_1, τ_2) is a nonempty set X with two topologies τ_1 and τ_2. A subset A of a space (X, τ_1, τ_2) is said to be quasi open in (X, τ_1, τ_2) if A = U $\cup$ V for some U $\in \tau_1$ and V $\in \tau_2$. The complement of quasi open sets is quasi closed in (X, τ_1, τ_2). The family of all quasi open (respectively quasi closed) sets in (X, τ_1, τ_2) will be denoted by QO(X, τ_1, τ_2) (respectively QC(X, τ_1, τ_2)).

For a subset A of a space (X, τ_1, τ_2), we define the quasi closure of A (briefly qcl(A)) as qcl(A) = $\cap$ {F : F $\in$ (X, τ_1, τ_2), A $\subset$ F}. Obviously, A is quasi closed in (X, τ_1, τ_2) if and only if A = qcl(A) and x $\in$ qcl(A) if and only if every set U $\in$ QO(X, τ_1, τ_2) containing x intersects A. A subset A is called quasi generalized closed in (X, τ_1, τ_2) if qcl(A) $\subset$ A whenever A $\subset$ U, U $\in$ QO(X, τ_1, τ_2). The family of all quasi generalized closed (briefly qg-closed) sets in (X, τ_1, τ_2) is denoted by QGC(X, τ_1, τ_2). For a subset A of a space (X, τ_1, τ_2), quasi kernel of A (briefly qker(A)) is defined as qker(A) = $\cap$ {F : F $\in$ QO(X, τ_1, τ_2), A $\subset$ F}. For any point x of a bitopological space (X, τ_1, τ_2), the quasi shell of a singleton set {x} (briefly qshl({x})) is defined as qshl({x}) = qker({x}) \ {x}.

A space (X, τ_1, τ_2) is said to be quasi T_0 [3] if for any two distinct points x, y of X, there exist A $\in$ QO(X, τ_1, τ_2) such that x $\in$ A, y $\notin$ A or y $\in$ A, x $\notin$ A. A space (X, τ_1, τ_2) is said to be quasi T_1 if for any two distinct points x, y of X, there exist A, B $\in$ QO(X, τ_1, τ_2) such that x $\in$ A, y $\notin$ A and y $\in$ B, x $\notin$ B. A space (X, τ_1, τ_2) is said to be quasi T_2 if for any two distinct points x, y of X, there exist two disjoint sets A, B $\in$ QO(X, τ_1, τ_2) such that x $\in$ A and y $\in$ B. A space (X, τ_1, τ_2) is said to be quasi $T_{1/2}$ if QC(X, τ_1, τ_2) = QGC(X, τ_1, τ_2). It is pointed out in [3], that each of the implications, quasi $T_2 \rightarrow$ quasi $T_1 \rightarrow$ quasi $T_{1/2} \rightarrow$ quasi T_0 is true while none of the reverse implication holds. By a degenerate set, we shall mean a set which contains almost one point, that is, it is either a null set or a singleton set. Throughout the present study, a space means a bitopological space on which no separation axioms are assumed unless otherwise mentioned.

2. QUASI $T_{1/2^*}$ SPACES
Definition 2.1. A bitopological space (X, τ_1, τ_2) is said to be a quasi $T_{1/2^*}$ space if for all x, y in X, x $\neq$ y, qker({x}) $\cap$ qker({y}) is either ϕ or {x} or {y}.

Theorem 2.2. Every quasi T_1 space is quasi $T_{1/2^*}$.

Proof. In a quasi T_1 space, for each x in X, qker({x}) = {x}. Hence qker({x}) $\cap$ qker({y}) = ϕ for x $\neq$ y.

Theorem 2.3. Every quasi $T_{1/2^*}$ space is quasi T_0 .

Proof. Let (X, τ_1, τ_2) be a quasi $T_{1/2^*}$ space. Then, for any x, y in X, x $\neq$ y, qker({x}) $\cap$ qker({y}) is either ϕ or {x} or {y}. Consequently, qker({x}) $\neq$ qker({y}); hence (X, τ_1, τ_2) is quasi T_0 .

Definition 2.4. A point x $\in$ X is said to be a quasi limit point of a subset A of a bitopological space (X, τ_1, τ_2) if qcl(U) $\cap$ A $\neq \phi$, for every quasi open set U of X containing x. The set of all quasi limit points of A is said to be quasi derived set of A and is denoted by qd (A). Also, the quasi derived set of a singleton set {x} is given by qd({x}) = qcl({x}) \ {x}.

Further, for any point x of a bitopological space (X, τ_1, τ_2), we have

(i) qcl({x}) = {y : x $\in$ qker({y}) },
(ii) qker({x}) = {y :x $\in$ qcl({y}) },
(iii) qd({x}) = {y : y $\neq$ x and x $\in$ qker({y}) },
(iv) qshl({x}) = {y : y $\neq$ x and x $\in$ qcl({y}) }.

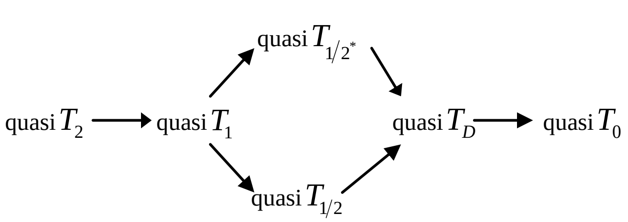
Definition 2.5. A bitopological space (X, τ_1, τ_2) is called quasi T_D space, if for every x in X, qd ({x}) is quasi closed.

Theorem 2.6. Every quasi $T_{1/2^*}$ space is quasi T_D .

Proof. In a quasi $T_{1/2^*}$ space (X, τ_1, τ_2), for any x $\neq$ y, qker({x}) $\cap$ qker({y}) is either ϕ or {x} or {y} and hence qshl({x}) $\cap$ qshl({y}) = ϕ . We claim that, for each x in X, qcl({x}) is degenerate. For, if y, z $\in$ qd({x}) for some x $\in$ X, then for y, z in X, qshl({y}) and qshl({z}) will not be disjoint. It is sufficient to consider the case when qd({x}) = {z}. First we observe that the space (X, τ_1, τ_2) is quasi T_0 and so qcl({x}) $\neq$ qcl({z}). Therefore x $\in$ qcl({z}). Now, if some y other than x, z is such that y $\in$ qcl({z}) (= qcl(qd({x}))), then y $\in$ qcl({x}) and so qd({x}) will not be a singleton set. Therefore, qcl({z}) = qcl(qd({x})) = qd({x}). It follows then that every quasi $T_{1/2^*}$ space is quasi T_D .

Example 2.7. Let X = {a, b, c}, τ_1 = { ϕ , {b} , {b, c} ,X} and τ_2 = { ϕ, {c} , {b, c} ,X}. Then (X, τ_1, τ_2) is quasi $T_{1/2^*}$ and hence quasi T_D .

Remark 2.8. From the above definitions and Theorem, the following implications are obvious.



However, none of the above implications is reversible as the following Example shows.

Example 2.9. Let X = {a, b, c}, τ_1 = { ϕ , {b, c} ,X} and τ_2 = { ϕ , {a, c} ,X}. Then (X, τ_1, τ_2) is quasi T_0 but not quasi T_D .

Example 2.10. Let X = {a, b, c}, τ_1 = { ϕ , {b} ,X} and τ_2 = { ϕ , {b} , {b, c} ,X}. Then (X, τ_1, τ_2) is quasi T_D but not quasi $T_{1/2^*}$ and quasi $T_{1/2}$.

Example 2.11. Let X = {a, b, c}, τ_1 = { ϕ , {a} , {a, b} ,X} and τ_2 = { ϕ , {a, b} ,X}. Then (X, τ_1, τ_2) is quasi T_0 and quasi T_D but not quasi $T_{1/2^*}$.

Example 2.12. Let X = {a, b, c}, τ_1 = { ϕ , {a} , {a, b} ,X} and τ_2 = { ϕ, {b, c}, X}. Then the bitopological space (X, τ_1, τ_2) is quasi $T_{1/2^*}$ but not quasi T_1 .

Example 2.13. Let X = {a, b, c}, τ_1 = { ϕ , {a} , {b} , {a, b} , X} and τ_2 = { ϕ , {a, c} ,X}. Then (X, τ_1, τ_2) is quasi $T_{1/2}$ and quasi $T_{1/2^*}$ but not quasi T_1 .

Example 2.14. Let X = {a, b, c}, τ_1 = { ϕ , {c} , {b, c} ,X} and τ_2 = { ϕ , {a, c} ,X}. Then (X, τ_1, τ_2) is quasi $T_{1/2}$ but not quasi $T_{1/2^*}$.

Example 2.15. Let X = {a, b, c}, τ_1 = { ϕ , {a} , {b, c} ,X} and τ_2 = { ϕ , {a} , {a, c} ,X}. Then (X, τ_1, τ_2) is quasi $T_{1/2^*}$ but not quasi $T_{1/2}$.

Lemma 2.16. In a bitopological space (X, τ_1, τ_2), qker({x}) = qker(qker({x})) for each x in X.

Proof. For this suppose, qker({x}) $\subset$ qker(qker({x})) is clear. Again for the reverse inclusion, suppose y $\in$ qker(qker({x})). Then qker({x}) $\cap$ qcl ({y}) $\neq \phi$, say, some z $\in$ qker({x}) $\cap$ qcl ({y}). Now z $\in$ qcl({y}) implies y $\in$ qker({z}) which together with z $\in$ qker({x}) implies y $\in$ qker({x}).

Lemma 2.17. In a bitopological space (X, τ_1, τ_2) , qshl({x}) = qker(qshl({x})) for each x in (X, τ_1, τ_2).

Proof. Clearly qshl({x}) $\subset$ qker(qshl({x})). To show that qker(qshl({x})) $\subset$ qshl({x}), suppose y $\in$ qker(qshl({x})). Then qshl({x}) $\cap$ qcl({y}) $\neq \phi$ and so there exists z $\in$ qshl({x}) $\cap$ qcl({y}). Therefore, x $\in$ qcl({z}) and z $\in$ qcl({y}).Consequently, x $\in$ qcl({y}) and so y $\in$ qshl({x}). This proves the result.

Theorem 2.18. For a bitopological space (X, τ_1, τ_2), the following statements are equivalent:

(i) (X, τ_1, τ_2) is quasi $T_{1/2^*}$.

(ii) For all x, y in X, x $\neq$ y, either qker({x}) $\cap$ qker({y}) = ϕ or one of the point has an empty quasi shell.

(iii) The bitopological space is quasi T_0 and qshl({x}) $\cap$ qshl ({y}) = ϕ for all x, y in X, x $\neq$ y.

(iv) The bitopological space is quasi T_0 and the quasi kernel of the quasi shells of any two distinct points is disjoint.

Proof. (i) $\Rightarrow$ (ii): In a quasi $T_{1/2^*}$ space (X, τ_1, τ_2), for any x $\neq$ y, qker({x}) $\cap$ qker({y}) is either ϕ or {x} or {y}. If qker({x}) $\cap$ qker({y}) = {x} and so qker({x})= {x} which implies qshl ({x}) = ϕ.

(ii) $\Rightarrow$ (i): Straightforward.

(i) $\Rightarrow$ (iii): In a quasi $T_{1/2^*}$ space (X, τ_1, τ_2), for any x $\neq$ y, qker({x}) $\cap$ qker({y}) is either ϕ or {x} or {y}. Then clearly, qshl({x}) $\cap$ qshl ({y}) = ϕ and so qker({x}) $\neq$ qker({y}) for all x $\neq$ y. That is, the bitopological space (X, τ_1, τ_2) is quasi T_0.

(iii) $\Rightarrow$ (iv): Follows from Lemma 2.17.

(iv)) $\Rightarrow$ (i): qshl({x}) $\cap$ qshl ({y}) = ϕ implies that qker({x}) $\cap$ qker({y}) is either ϕ or {x} or {y} or {x, y}. But since the space is quasi T_0, x, y cannot be both in qker({x}) $\cap$ qker({y}).

3. WEAKLY QUASI SEPARATED SETS

Definition 3.1. Let (X, τ_1, τ_2) be a bitopological space and A $\subset$ X. Then A is said to be weakly quasi separated from set B if there exists a quasi open set G such that A $\subset$ G and G $\cap$ B = ϕ or A $\cap$ qcl(B) = ϕ.

Example 3.2. Let X = {a, b, c}, τ_1 = { ϕ, {c} , {a, b} ,X} and τ_2 = { ϕ, {a, c} ,X}. Let A = {a}, B = {b}, C = {c}. Then A is weakly quasi separated from B, B is weakly quasi separated from C and C is weakly quasi separated from A.

Remark 3.3. In view of Definition 3.1, we have the following for x, y $\in$ X.

(i) qcl({x}) = {y : y is not weakly quasi separated from x }

(ii) qker({x}) = {y : y is not weakly quasi separated from x }

(iii) qd({x}) = {y : y $\neq$ x and x $\in$ qker({y})} = {y : y $\neq$ x and y is not weakly quasi separated from x}

(iv) qshl({x}) = {y : y $\neq$ x and x $\in$ qcl({y})} = { y : y $\neq$ x and x is not weakly quasi separated from y}.

Definition 3.4. Let (X, τ_1, τ_2) be a bitopological space. Then we define

(i) q-N-D = {x : x $\in$ X and qd({x}) = ϕ }

(ii) q-N-shl = {x : x $\in$ X and qshl({x}) = ϕ }

(iii) q - $\langle$ x $\rangle$ = qcl({x}) $\cap$ qker({x}).

Theorem 3.5. Let x, y $\in$ X. Then the following conditions hold:

(i) y $\in$ qker({x}) if and only if x $\in$ qcl({y})

(ii) y $\in$ qshl({x}) if and only if x $\in$ qd({y})

(iii) y $\in$ qcl({x}) implies qcl({y}) $\subseteq$ qcl({x}) and

(iv) y $\in$ qker({x}) implies qker({y}) $\subseteq$ qker ({x})

Proof. The proof of (i) and (ii) are obvious from Remark 3.3.

(iii) Let z $\in$ qcl({y}). Then z is not weakly quasi separated from y. So there exists a quasi open set G containing z such that G $\cap$ {y} $\neq$ ϕ .Hence y $\in$ G and by assumption G $\cap$ {x} $\neq$ ϕ. Hence z is not weakly quasi separated from x. So z $\in$ qcl({x}). Therefore, qcl({y}) $\subseteq$ qcl({x}).

(iv) Let z $\in$ qker ({y}). Then y is not weakly quasi separated from z. So y $\in$ qcl({z}). Hence qcl({y}) $\subseteq$ qcl({z}). By assumption y $\in$ qker({x}) and then x $\in$ qcl({y}). So qcl({x}) $\subseteq$ qcl({y}). Ultimately qcl({x}) $\subseteq$ qcl({z}). Hence x $\in$ qcl({z}), that is z $\in$ qker ({x}). This shows that qker({y}) $\subseteq$ qker({x}).

Theorem 3.6. Let (X, τ_1, τ_2) be a bitopological space and x, y $\in$ X . Then,

(i) for every x $\in$ X, qshl ({x}) is degenerate if and only if for all x, y $\in$ X, x $\neq$ y, qd({x}) $\cap$ qd({y}) = ϕ.

(ii) for every x $\in$ X, qd ({x}) is degenerate if and only if for all x, y $\in$ X, x $\neq$ y, qshl({x}) $\cap$ qshl({y}) = ϕ.

Proof. (i) Let qd({x}) $\cap$ qd({y}) $\neq$ ϕ. Then there exists z $\in$ X such that z $\in$ qd({x}) and z $\in$ qd({y}). Then x $\neq$ y $\neq$ z and z $\in$ qcl({x}) and z $\in$ qcl({y}), that is, x, y $\in$ qker({z}). Hence, qker({z}) and so qshl({z}) is not a degenerate set. Conversely, let x, y $\in$ qshl({z}). Then we get x $\neq$ z, x $\in$ qker({z}) and y $\neq$ z, y $\in$ qker({z}) and hence z is an element of both qcl({x}) and qcl({y}), which is a contradiction. The proof of (ii) is similar and hence omitted.

Theorem 3.7. If y $\in$ q - $\langle$ x $\rangle$, then q - $\langle$ x $\rangle$ = q - $\langle$ y $\rangle$.

Proof. If y $\in$ q - $\langle$ x $\rangle$, then by definition, y $\in$ qcl({x}) $\cap$ qker({x}). Hence y $\in$ qcl({x}) and y $\in$ qker({x}). So we have qcl({y}) $\subset$ qcl({x}) and qker({y}) $\subset$ qker({x}). Then qcl({y}) $\cap$ qker({y}) $\subset$ qcl({x}) $\cap$ qker({x}). Hence q- $\langle$ y $\rangle$ $\subset$ q- $\langle$ x $\rangle$ .From the fact that y $\in$ qcl({x}) implies x $\in$ qker({y}) and y $\in$ qker({x}) implies x $\in$ qcl({y}) we have q - $\langle$ x $\rangle$ $\subset$ q - $\langle$ y $\rangle$. Hence q - $\langle$ x $\rangle$ = q - $\langle$ y $\rangle$.

Theorem 3.8. For all x, y $\in$ X, q - $\langle$ x $\rangle$ $\cap$ q - $\langle$ y $\rangle$ = ϕ or q - $\langle$ x $\rangle$ = q - $\langle$ y $\rangle$.

Proof. If q - $\langle$ x $\rangle$ $\cap$ q - $\langle$ y $\rangle$ $\neq$ ϕ, then there exists z $\in$ X such that z $\in$ q - $\langle$ x $\rangle$ and z $\in$ q - $\langle$ y $\rangle$. So by Theorem 3.7, q - $\langle$ z $\rangle$ = q - $\langle$ x $\rangle$ = q - $\langle$ y $\rangle$.

Theorem 3.9. A bitopological space (X, τ_1, τ_2) is quasi T_0 if and only if any of the following conditions hold:
(i) For arbitrary x, y $\in$ X, x $\neq$ y either x is weakly quasi separated from y or y is weakly quasi separated from x.
(ii) y $\in$ qcl({x}) implies x $\notin$ qcl({y}).
Proof. (i) Obvious from the definitions.
(ii) By hypothesis, y $\in$ qcl({x}) and so y is not weakly quasi separated from x. Since X is quasi T_0 , x should be weakly quasi separated from y, that is x $\notin$ qcl({y}).

Theorem 3.10. A bitopological space (X, τ_1, τ_2) is quasi T_0 if and only if (qcl({x}) $\cap$ {y}) $\bigcup$ (qcl({y}) $\cap$ {x}) is degenerate.

Proof. ***Necessity:*** Let X be quasi T_0 . Then we have any one of the two cases viz , x is weakly quasi separated from y or y is weakly quasi separated from x.
Case (i): If x is weakly quasi separated from y, then we have {x} $\cap$ qcl{y} = ϕ and {y} $\cap$ qcl({x}) is a degenerate set.
Case (ii): If y is weakly quasi separated from x, then we have {y} $\cap$ qcl({x}) = ϕ and {x} $\cap$ qcl({y}) is a degenerate set. Hence (qcl({x}) $\cap$ {y}) $\bigcup$ (qcl({x}) $\cap$ {y}) is a degenerate set.

Sufficiency: Suppose that (qcl({y}) $\cap$ {x}) $\bigcup$ (qcl({x}) $\cap$ {y}) is a degenerate set. Then it is either an empty set or a singleton set. If it is an empty set, then there is nothing to prove. If it is a singleton set, its value is either {x} or {y}. If it is {x}, then y is weakly quasi separated from x. If it is {y}, then x is weakly quasi separated from y. This shows that (X, τ_1, τ_2) is quasi T_0 .

Corollary 3.11. A bitopological space (X, τ_1, τ_2) is quasi T_0 if and only if (qcl({x}) $\cap$ {y}) $\bigcap$ (qcl({y}) $\cap$ {x}) is degenerate.
Proof. Obvious.

Theorem 3.12. A bitopological space (X, τ_1, τ_2) is quasi T_0 if and only if qd({x}) $\cap$ qshl({x}) = ϕ .

Proof. ***Necessity:*** Suppose that qd({x}) $\cap$ qshl({x}) $\neq \phi$. Then let z $\in$ qd({x}) and z $\in$ qshl({x}). Then there exists z $\neq$ x and z $\in$ qcl({x}) and z $\in$ qker({x}). Then z is not weakly quasi separated from x and also x is not weakly quasi separated from z, which is a contradiction.

Sufficiency: Let qd({x}) $\cap$ qshl({x}) = ϕ . Then there exists z $\neq$ x and z $\in$ qcl({x}) and z $\notin$ qker({x}). Hence if we have, z is not weakly quasi separated from x, then x is weakly quasi separated from z.

Corollary 3.13. If (X, τ_1, τ_2) is quasi T_0 , then for any x $\in$ X, q - $\langle$ x $\rangle$ = {x}.

Theorem 3.14. A bitopological space (X, τ_1, τ_2) is quasi T_1 if and only if one of the following conditions holds:
(i) For arbitrary x, y $\in$ X, x $\neq$ y, x is weakly quasi separated from y.
(ii) For every x $\in$ X, qd({x}) = ϕ or q-N-D({x}) = X.
(iii) For every x $\in$ X, qker({x}) = {x}.
(iv) For every x $\in$ X, qshl({x}) = ϕ or q-N-shl({x}) = X.
(v) For every x, y $\in$ X, x $\neq$ y, qcl({x}) $\cap$ qcl({y}) = ϕ .
(vi) For every arbitrary x, y $\in$ X, x $\neq$ y, we have qker({x}) $\cap$ qker({y}) = ϕ .
Proof. (i), (ii) and (iii) are clear.
(iv) If x is weakly quasi separated from y, then for y $\neq$ x, we have y $\notin$ qcl({x}) and hence x $\notin$ qker({y}). Therefore qker({y}) = {y}. The proof of converse is similar.
(v) As (X, τ_1, τ_2) is quasi T_1 ,qcl({x}) = {x} and qcl({y}) = {y}. So, when x $\neq$ y, qcl({x}) $\cap$ qcl({y}) = ϕ .
(vi) Follows from (v).

4. CONCLUSION

In this paper we have introduced two new spaces called quasi $T_{1/2^*}$ and quasi T_D spaces and studied the relationship between them and with other spaces like quasi T_0 , quasi T_1, quasi $T_{1/2}$ and quasi T_2 . Also we have introduced and studied the properties of weakly quasi separated sets in connection with quasi kernel, quasi shell and quasi closure and its relation with quasi T_0 and quasi T_1 spaces.

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